MONASTIR MINI-COURSE:
THE SELBERG CLASS OF ZETA- AND L-FUNCTIONS

JÖRN STEUDING

"What is a zeta-function (an L-function)? We know one when we see one!"

This quotation is attributed to M.N. Huxley and reflects the number-theoretical problem to classify those generating functions which carry arithmetically relevant data. In 1989, Selberg introduced a general class $\mathcal{S}$ of Dirichlet series having an Euler product representation, analytic continuation and a functional equation of Riemann-type, and formulated some fundamental conjectures. In only twenty years this so-called Selberg class of L-functions became an important branch of research in Analytic Number Theory with important arithmetical applications. It is widely expected that the Selberg class contains all aritmetically important L-functions, moreover, that it consists of exactly all automorphic L-functions. In this mini-course, consisting of six lectures, we give an introduction to this topic. Our focus is on general methods, e.g., Hecke's approach to functional equations and analytic continuation, Tauberian theorems to deduce information about prime numbers, as well as linear and non-linear twists, a powerful tool recently invented by Kaczorowski & Perelli to investigate the structure of the Selberg class. The course is mainly based on the excellent surveys of Kaczorowski [20] and Perelli [42, 43], and the monographs [38, 54] of M.R. Murty & V.K. Murty and the author, respectively. Basic knowledge in Real and Complex Analysis is expected, and a background from Number Theory is useful, although we have added several appendices in order to present a self-contained introduction.

Contents

Date: Monastir, April 11-16, 2011.

1. Classical Theory of Zeta- and L-Functions

Since Dirichlet's proof of the prime number theorem for arithmetic progressions from 1837 and Riemann's famous path-breaking paper in 1859, zeta and L-functions play a central role in Analytic Number Theory. Being generating functions formed out of local data associated with either an arithmetic object or with an automorphic form, these functions possess a Dirichlet series and an Euler product representation (if the underlying object is of multiplicative nature). The famous Riemann zeta-function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}\qquad(\mathrm{Re}\, s>1) \tag{1}$$

may be regarded as the prototype. Here the product is taken over all prime numbers and the identity between this product and the series is an analytic version of the unique prime factorization of the integers (which becomes obvious by expanding each factor). According to their inventors, any product over primes is called an Euler product and any series of the above type is called a Dirichlet series. L-functions encode in their value-distribution information about the underlying arithmetical or algebraic structure, e.g. the infinitude of prime numbers follows immediately from (1) and the divergence of the harmonic series; a more advanced study yields the celebrated prime number theorem (which we discuss more detailed in §3). Another example is Dirichlet's analytic class number formula which measures the deviation from unique prime factorization in the ring of integers of quadratic number fields. Actually, two of the seven millennium problems are questions about L-functions: the famous Riemann hypothesis on the zeros of $\zeta(s)$ and the conjecture of Birch & Swinnerton-Dyer that the rank of the Mordell-Weil group of an elliptic curve is equal to the order of the zero of the associated L-function $L_E(s)$ at $s=1$.

In order to deduce information on the value-distribution of zeta- and L-functions, first analytic continuation beyond the abscissa of convergence of the defining Dirichlet series is needed. In many arithmetically interesting examples this can be realized by a functional equation. In the case of the Riemann zeta-function this functional equation takes the form of a point symmetry with respect to $s=\frac{1}{2}$:

Theorem 1 (Riemann's Functional Equation). *The Riemann zeta-function $\zeta(s)$ has an analytic continuation to the whole complex plane except for a simple pole at $s=1$ with residue 1, and satisfies the identity*

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Proof. The Gamma-function plays an important part in the theory of the zeta-function (see [56], §1.86 and §4.41, for a collection of its most important properties); for $\mathrm{Re}\, z>0$, it is defined by Euler's integral

$$\Gamma(z)=\int_0^{\infty}u^{z-1}\exp(-u)\,\mathrm{d}u.$$

Substituting $u=\pi n^2x$, leads to

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\frac{1}{n^s}=\int_0^{\infty}x^{\frac{s}{2}-1}\exp(-\pi n^2x)\,\mathrm{d}x.$$

Summing up over all $n\in\mathbb{N}$, yields

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^s}=\sum_{n=1}^{\infty}\int_0^{\infty}x^{\frac{s}{2}-1}\exp(-\pi n^2x)\,\mathrm{d}x.$$

On the left-hand side we find the Dirichlet series defining $\zeta(s)$; hence the latter formula is valid only for $\mathrm{Re}\, s > 1$. On the right-hand side we may interchange summation and integration, which is allowed by absolute convergence. Thus we obtain

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^\infty \exp(-\pi n^2 x)\,\mathrm{d}x.$$

We split the integral at $x = 1$ to get

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \left\{\int_0^1 + \int_1^\infty\right\} x^{\frac{s}{2}-1}\omega(x)\,\mathrm{d}x, \tag{2}$$

where

$$\omega(x) := \sum_{n=1}^\infty \exp(-\pi n^2 x) = \frac{1}{2}\left(\theta(x) - 1\right)$$

can be expressed in terms of Jacobi's theta-function $\theta(x) := \sum_{n=-\infty}^\infty \exp(-\pi n^2 x)$. Next we shall use the functional equation for the theta-function,

$$\theta(x) = \frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right), \tag{3}$$

valid for any $x > 0$; this formula will be proved in Appendix B as an application of Poisson's summation formula. Hence, we find

$$\omega\left(\frac{1}{x}\right) = \frac{1}{2}\left(\theta\left(\frac{1}{x}\right) - 1\right) = \sqrt{x}\omega(x) + \frac{1}{2}(\sqrt{x} - 1).$$

By the substitution $x \mapsto \frac{1}{x}$ the first integral in (2) equals

$$\int_1^\infty x^{-\frac{s}{2}-1}\omega\left(\frac{1}{x}\right)\,\mathrm{d}x = \int_1^\infty x^{-\frac{s+1}{2}}\omega(x)\,\mathrm{d}x + \frac{1}{s-1} - \frac{1}{s}.$$

Inserting this in (2) yields

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \left(x^{-\frac{s+1}{2}} + x^{\frac{s}{2}-1}\right)\omega(x)\,\mathrm{d}x. \tag{4}$$

Since $\omega(x) \ll \exp(-\pi x)$, the last integral converges for all values of s, and thus (4) holds, by analytic continuation, throughout the complex plane.* The right-hand side remains unchanged by $s \mapsto 1 - s$ which proves the functional equation for zeta. It easily follows from (2) that $\zeta(s) - \frac{1}{s-1}$ is an entire function; we leave the details to the reader. •

The given proof is one of the two proofs Riemann found. It relies heavily on the functional equation (3) of the theta-function, which is an easy consequence of the Poisson summation formula (Theorem 30 in Appendix B). In the sequel we shall see how Riemann's approach via the theta-function allows interesting generalizations. Other proofs of the functional equation for $\zeta(s)$ can be found in [56].

Before we continue to consider further examples of L-functions we shall discuss the impact of the functional equation on the values of $\zeta(s)$ at the integers and on the distribution of zeros. By the Euler product representation (1) the zeta-function does not vanish in the half-plane of convergence $\mathrm{Re}\, s > 1$ of its Dirichlet series. In view of the functional equation the only zeros in the left half-plane $\mathrm{Re}\, s < 0$ occur at the poles of the Gamma-factor $\Gamma(\frac{s}{2})$; those zeros lie at $s = -2n$, $n \in \mathbb{N}$, and are said to be trivial. All other zeros are called nontrivial and, consequently, they appear inside the so-called critical strip $0 \leq \mathrm{Re}\, s \leq 1$,

*We write $f(x) \ll g(x)$ or $f(x) = O(g(x))$ with a positive function g if $\limsup_{x\to\infty} |f(x)|/g(x) < \infty$; in this case there exists a positive constant C such that $|f(x)| \leq Cg(x)$ for all sufficiently large x.

symmetrically distributed with respect to the so-called critical line $\frac{1}{2}+i\mathbb{R}$ and the real axis. In §3, we shall show that there are no zeros on the boundary of the critical strip.

Exercise 1. *Show that, for* $\mathrm{Re}\, s>0$,

$$\zeta(s)=(1-2^{1-s})^{-1}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}, \tag{5}$$

and deduce that $\zeta(s)$ *does not vanish for* $s\in(0,1)$.

It was Riemann's great contribution to number theory to point out the relevance of the zeta zeros to the distribution of prime numbers. In this context he formulated that very likely all nontrivial zeros lie on the critical line $\frac{1}{2}+i\mathbb{R}$ which is now widely known as the

Riemann Hypothesis. $\zeta(s)\neq 0$ *for* $\mathrm{Re}\, s>\frac{1}{2}$.

It is amazing that already in the 18th century Euler had partial results toward the functional equation for $\zeta(s)$, namely, formulae for the values of $\zeta(s)$ for integral s (and even for half-integral s) relating s with $1-s$:

Theorem 2 (Euler's Theorem). *For* $k,n\in\mathbb{N}$,

$$\zeta(2k)=(-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k},\quad \zeta(0)=-\tfrac{1}{2},\qquad \textit{and}\qquad \zeta(-n)=-\frac{B_{n+1}}{n+1},$$

where the Bernoulli numbers B_n *are defined by the identity*

$$\frac{z}{\exp z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!}=1-\frac{1}{2}z+\frac{1}{12}z^2\mp\ldots.$$

Nearly nothing is known about the values of zeta at the positive odd integers; in 1979, Apéry proved that $\zeta(3)$ is irrational but the arithmetic character of $\zeta(5)$ is still unknown.

Exercise 2. *Prove Theorem 2 as follows: First show*

$$\sum_{k=1}^{\infty}(-1)^k\frac{(2\pi)^{2k}}{(2k)!}B_{2k}z^{2k}=\pi z\cot(\pi z)-1=z\frac{\mathrm{d}}{\mathrm{d}z}\log\frac{\sin(\pi z)}{\pi z},$$

then use the product representation for $\frac{\sin(\pi z)}{\pi z}$ *to evaluate* $\zeta(2k)$. *Next apply the functional equation to obtain the values* $\zeta(1-n)$. *Why is it impossible to deduce information about* $\zeta(5)$ *this way? How did Euler obtain his formulae? For this and advice see* [55].

Next we consider Dirichlet L-functions which are defined by

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1}\qquad(\mathrm{Re}\, s>1); \tag{6}$$

here χ denotes a Dirichlet character, i.e., a group homomorphism from the group of prime residue classes modulo some positive integer q into $\mathbb{C}^*$; see Appendix A for a short introduction to these useful tools in Number Theory. Since characters are completely multiplicative, $L(s,\chi)$ may be represented both, as Dirichlet series and as Euler product (similarly to (1)). In fact, the Riemann zeta-function may be considered as the Dirichlet L-function to the unique character χ_0 mod 1. By analytic continuation, $L(s,\chi)$ extends to a meromorphic function in the complex plane with a single pole at $s=1$ if χ is a principle character (i.e., $\chi(n)=1$ for all n coprime with some q). Of special interest are Dirichlet L-functions associated with primitive characters χ (that means χ is not induced by a character of smaller modulus), since any non-principle character is induced by a uniquely determined primitive character and the corresponding L-functions differ from one another only by a finite Euler product which extends to an entire function with a very regular value-distribution. In a similar manner as above one can prove the functional equation for Dirichlet L-functions $L(s,\chi)$ with a primitive character $\chi \bmod q$. Here we have to distinguish the cases $\chi(-1)=+1$ and $\chi(-1)=-1$. In the first case we find

$$\theta(x,\chi) := \sum_{n\in\mathbb{Z}} \chi(n)\exp(-\pi n^2 x/q) = \frac{\tau(\overline{\chi})}{\sqrt{qx}}\theta\left(\frac{1}{x},\overline{\chi}\right), \tag{7}$$

where $\tau(\chi) := \sum_{a \bmod q} \chi(a)\exp(2\pi i \frac{a}{q})$ is the Gaussian sum which satisfies

$$\tau(\chi)\tau(\overline{\chi}) = \chi(-1)|\tau(\chi)|^2 = \chi(-1)q.$$

The second case, $\chi(-1)=-1$, is slightly more difficult. Here we make use of

$$\tilde{\theta}(x,\chi) := \sum_{n\in\mathbb{Z}} \chi(n) n\exp(-\pi n^2 x/q) = \frac{\tau(\overline{\chi})}{i\sqrt{q}x^{\frac{3}{2}}}\tilde{\theta}\left(\frac{1}{x},\overline{\chi}\right), \tag{8}$$

Also the proofs of these functional equations rely on the Poisson summation formula and basic facts from character theory. Formulae (7) and (8) lead by more or less the same method as for the zeta-function to

Theorem 3 (**Functional Equation for Dirichlet L-Functions**). *Let χ be a primitive character* mod q. *Then, $L(s,\chi)$ extends to an entire function and satisfies*

$$\left(\frac{q}{\pi}\right)^{\frac{s+\delta}{2}}\Gamma\left(\frac{s+\delta}{2}\right)L(s,\chi) = \frac{\tau(\chi)}{i^{\delta}\sqrt{q}}\left(\frac{q}{\pi}\right)^{\frac{1+\delta-s}{2}}\Gamma\left(\frac{1+\delta-s}{2}\right)L(1-s,\overline{\chi}),$$

where $\delta := \frac{1}{2}(1-\chi(-1))$.

Exercise 3. *Prove the latter theorem as well as all identities for the involved theta-functions, that are (3), (7) and (8). What can be said about zeros and zero-free regions?*

It should be noticed that, previous to Riemann, already in 1846 Malmstén and a little later Schlömilch obtained among other identities

$$L(1-s,\chi_{-4}) = \left(\frac{\pi}{2}\right)^{-s}\sin\frac{\pi s}{2}\Gamma(s)L(s,\chi_{-4})$$

with the unique character χ_{-4} modulo 4 defined by $\chi_{-4}(-1)=-1$. In fact, this functional equation is a special case of the latter theorem (as follows from some properties of the Gamma-function). In 1849, Eisenstein derived an even more general functional equation (cf. Bombieri [2]; see also Weil's treatise [60] for details).

Next we shall briefly investigate an interesting example. The Davenport-Heilbronn zeta-function is given by

$$L(s) = \frac{1-i\kappa}{2}L(s,\chi) + \frac{1+i\kappa}{2}L(s,\overline{\chi}),$$

where $\kappa := \frac{\sqrt{10-2\sqrt{5}}-2}{\sqrt{5}-1}$ and χ is the character mod 5 with $\chi(2)=i$. It is an easy consequence of Theorem 3 that the Davenport-Heilbronn zeta-function satisfies the functional equation

$$\left(\frac{5}{\pi}\right)^{\frac{s}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s) = \left(\frac{5}{\pi}\right)^{\frac{1-s}{2}}\Gamma\left(1-\frac{s}{2}\right)L(1-s).$$

Davenport & Heilbronn introduced this function as an example for a Dirichlet series having infinitely many zeros on the critical line and also infinitely many zeros in the half-plane $\operatorname{Re} s > 1$ in spite of satisfying a Riemann-type functional equation. The localization of these zeros is not too easy (see also [57]). Following Balanzario [1], we give another example: consider of a Dirichlet series satisfying a Riemann-type functional equation for which the analogue of the Riemann hypothesis does not hold. Consider the following functions with 5-periodic Dirichlet coeffcients:

$$\begin{aligned}(1+5^{\frac{1}{2}-s})\zeta(s) &= 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1+\sqrt{5}}{5^s}+\ldots,\\ L(s,\chi) &= 1-\frac{1}{2^s}-\frac{1}{3^s}+\frac{1}{4^s}+\frac{0}{5^s}+\ldots,\end{aligned}$$

where χ is the unique character mod 5 with $\chi(2)=-1$. Both functions satisfy the same functional equation,

$$F(s)=5^{\frac{1}{2}-s}2(2\pi)^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2}F(1-s). \tag{9}$$

Now let z be an arbitrary complex number, then the function

$$L(z,\chi)(1+5^{\frac{1}{2}-s})\zeta(s)-L(s,\chi)(1+5^{\frac{1}{2}-z})\zeta(z)$$

vanishes for $s=z$, satisfies the functional equation (9) which can be rewritten as an identity with a point symmetry with respect to $s=\frac{1}{2}$, and possesses a Dirichlet series expansion for $\mathrm{Re}\, s>1$. Thus, we observe that a functional equation of Riemann type is not sufficient for having all complex zeros on a straight line! It is expected that the Euler product is responsible for the location of the nontrivial zeros on the critical line although the Euler product does not converge inside the critical strip.

Now we shall prove Hecke's important correspondence between Dirichlet series with a Riemann-type functional equation and modular forms of the upper half-plane. Roughly speaking, these modular forms are analytic functions defined on the upper half-plane which satisfy a bunch of functional equations similar to (3) for the theta-functions above. A precise definition of modular forms will be given in the next section; for the first just notice that the following theorem is not about the empty set; actually, with Ogg's monograph [41] there is a whole book on this topic, and the proof reflects some ideas from Riemann's proof of the functional equation for zeta as well.

Theorem 4 (**Hecke's Converse Theorem).** *Let λ and k be fixed positive real numbers. Given two sequences $\{a(n)\}_{n\in\mathbb{N}_0}$ and $\{b(n)\}_{n\in\mathbb{N}_0}$ of complex numbers satisfying $a(n),b(n)\ll n^c$ for some positive constant c, define $\phi(s)=\sum_{n=1}^\infty a(n)n^{-s}$ and $\psi(s)=\sum_{n=1}^\infty b(n)n^{-s}$, as well as*

$$\Phi(s)=\left(\frac{\lambda}{2\pi}\right)^s\Gamma(s)\phi(s)\qquad\textit{and}\qquad\Psi(s)=\left(\frac{\lambda}{2\pi}\right)^s\Gamma(s)\psi(s).$$

The functions $\phi(s)$ and $\psi(s)$ are analytic in the half-plane $\mathrm{Re}\, s>c+1$, while the functions given by

$$f(z)=\sum_{n=0}^\infty a(n)\exp(2\pi i nz/\lambda)\qquad\textit{and}\qquad g(z)=\sum_{n=0}^\infty b(n)\exp(2\pi i nz/\lambda)$$

are analytic in the upper half-plane $\mathbb{H}:=\{z:=x+iy\in\mathbb{C}\,:\,y>0\}$ satisfying the boundary condition $f(x+iy),g(x+iy)\ll y^{-c-1}$ as $y\to 0+$. Furthermore, the following statements are equivalent:

(i) *The function $\Phi(s)+\frac{a(0)}{s}+\frac{\epsilon b(0)}{k-s}$ is entire and bounded on every vertical strip and satisfies the functional equation*

$$\Phi(s)=\epsilon\Psi(k-s);$$

(ii) *for any* $z \in \mathbb{H}$,

$$f(z) = \epsilon \left(\frac{i}{z}\right)^k g\left(\frac{-1}{z}\right).$$

This correspondence provides plenty of examples of L-functions associated with automorphic forms. An important generalization is due to Weil [59].

Proof. First of all, we observe that the statement concerning the convergence of the Dirichlet series is trivial. In order to derive the holomorphy and the boundary condition it suffices to consider the function $f(z)$ only. By Stirling's formula,

$$(-1)^n \binom{-c-1}{n} = \frac{\Gamma(c+1+n)}{\Gamma(c+1)\Gamma(n+1)} \sim c_1 n^c$$

with some positive constant c_1.[†] Hence, the series for $f(x+iy)$ is dominated term-by-term by

$$\sum_{n=0}^{\infty} (-1)^n \binom{-c-1}{n} \exp(-2\pi n y/\lambda) = (1-\exp(-2\pi y/\lambda))^{-(c+1)} \ll y^{-c-1}.$$

By the way, given the boundary condition, we can conversely bound the coefficients $a(n)$ by using their integral representation

$$a(n) = \int_0^1 f(x+iy) \exp(-2\pi i n(x+iy)/\lambda) \,\mathrm{d}x;$$

using this with $y = \frac{1}{n}$, we get

$$a(n) = \int_0^1 f\left(x + \frac{i}{n}\right) \exp\left(-2\pi i n \left(x + \frac{i}{n}\right)/\lambda\right) \mathrm{d}x \ll n^c.$$

It remains to show the equivalence of (i) and (ii). We start with the implication (ii)⇒(i). We note that, for sufficiently large $\mathrm{Re}\, s$,

$$\Phi(s) = \sum_{n=1}^{\infty} a(n) \int_0^{\infty} \left(\frac{\lambda}{2\pi}\right)^s x^{s-1} \exp(-nx) \,\mathrm{d}x = \sum_{n=1}^{\infty} \int_0^{\infty} a(n) y^{s-1} \exp(-2\pi n y/\lambda) \,\mathrm{d}y$$

as in the proof of the functional equation for $\zeta(s)$. Now interchanging summation and integration (justified by absolute convergence), we get

$$\Phi(s) = \int_0^{\infty} \sum_{n=1}^{\infty} a(n) y^{s-1} \exp(-2\pi n y/\lambda) \,\mathrm{d}y = \int_0^{\infty} y^{s-1}(f(iy) - a(0)) \,\mathrm{d}y.$$

The integral is improper for $y \to 0+$ and $y \to \infty$; we consider the contributions of the intervals $(0,1)$ and $(1,\infty)$ separately. Since $f(iy) - a(0) \ll \exp(-cy)$ as $y \to \infty$ for some positive constant c, it follows that

$$\int_1^{\infty} y^{s-1}(f(iy) - a(0)) \,\mathrm{d}y$$

converges uniformly on vertical strips, and so it defines an entire function which is bounded on vertical strips. For the integral taken over $(0,1)$ we make use of (ii). We have

$$\int_0^1 y^{s-1}(f(iy) - a(0)) \,\mathrm{d}y = -\frac{a(0)y^s}{s}\Big|_{y=0}^{1} + \int_1^{\infty} y^{1-s} f\left(\frac{i}{y}\right) \frac{\mathrm{d}y}{y^2}.$$

[†] We write $f(x) \sim g(x)$ with a positive function g if the limit $\lim_{x\to\infty} f(x)/g(x)$ exists and is equal to 1. Sometimes this notation is also used for other limiting processes than $x \to \infty$.

Now by (ii) we get

$$\int_0^1 y^{s-1}(f(iy)-a(0))\,\mathrm{d}y = -\frac{a(0)}{s} + \epsilon\int_1^\infty y^{k-s-1}(g(iy)-b(0))\,\mathrm{d}y - \epsilon\frac{b(0)}{k-s}.$$

(In view of (3) we may understand Jacobi's theta-function appearing in Riemann's proof of Theorem reffunctional as a modular form of weight $k=1/2$.) Hence,

$$\Phi(s) + \frac{a(0)}{s} + \frac{\epsilon b(0)}{k-s} = \int_1^\infty \left\{ y^{s-1}(f(iy)-a(0)) + \epsilon y^{k-s-1}(g(iy)-b(0))\right\}\,\mathrm{d}y$$

is an entire function bounded on vertical strips. Furthermore, we observe that (i) holds.

Now we assume (i) and deduce (ii). We shall use the formula

$$\exp(-x) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} x^{-s}\Gamma(s)\,\mathrm{d}s,$$

where $\alpha>0$ and $x>0$. The latter formula is the Mellin inversion of Euler's integral representation of the Gamma-function; Mellin transforms are important tools in zeta- and L-function theory and will appear several times later on. It follows that

$$f(iy) - a(0) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} y^{-s}\Phi(s)\,\mathrm{d}s; \tag{10}$$

however, here we have to choose the abscissa $\alpha>k$ such that the path of integration lies inside the half-plane of absolute convergence for $\phi(s)$. We shall move the path of integration over the origin to the left. Incorporating the residues at $s=0$ and $s=k$, we obtain

$$f(iy) - a(0) = \frac{1}{2\pi i}\int_{-\alpha-i\infty}^{-\alpha+i\infty} y^{-s}\Phi(s)\,\mathrm{d}s + \{\mathrm{Res}_{s=0} + \mathrm{Res}_{s=k}\}\, y^{-s}\Phi(s). \tag{11}$$

In view of (i) we find $\mathrm{Res}_{s=0} y^{-s}\Phi(s) = -a(0)$ and $\mathrm{Res}_{s=k} y^{-s}\Phi(s) = \epsilon b(0)y^{-k}$. Thus, we may replace (11) by

$$f(iy) - \epsilon b(0)y^{-k} = \frac{1}{2\pi i}\int_{-\alpha-i\infty}^{-\alpha+i\infty} y^{-s}\Phi(s)\,\mathrm{d}s.$$

Taking into account (i), we get

$$f(iy) - \epsilon b(0)y^{-k} = \frac{1}{2\pi i}\int_{-\alpha-i\infty}^{-\alpha+i\infty} y^{-s}\epsilon\Psi(k-s)\,\mathrm{d}s = \frac{\epsilon}{2\pi i}\int_{k+\alpha-i\infty}^{k+\alpha+i\infty} y^{-(k-s)}\Psi(s)\,\mathrm{d}s,$$

by substituting s by $k-s$. The right-hand side above is equal to

$$\epsilon y^{-k}\left(g\left(\frac{i}{y}\right) - b(0)\right)$$

(by the same argument as for (10)). This gives (ii) and the theorem is proved. •

There are only a few methods known to obtain analytic or meromorphic continuation for a Dirichlet series. The proof of a functional equation is probably the most elegant way to extend an L-function to the whole complex plane. However, for so-called symmetric power L-functions such functional equations are only conjectured (see [38, 54]). Moreover, from a functional equation we can sometimes only deduce that the underlying L-function is meromorphic, for control of possible poles more subtle information is needed. Here Artin L-functions are examples (and more information concerning them can be found in Appendix D). A good reading on other techniques for analytic continuation is the survey [13] of Gelbart & Miller. On the contrary, many functions defined as Dirichlet series or Euler products in some half-plane have somewhere a vertical line consisting of densely packed singularities such that there is no analytic continuation beyond this natural boundary possible.

---

2. The Selberg Class: Axioms, Examples, and Structure

In 1989, Selberg [47] defined what is now widely known as the Selberg class. His aim was to study the value-distribution of linear combinations of L-functions. In the meantime this class became an important object of research for various reasons. In this section we give the precise definition, discuss important examples, and do first investigations of its structure.

The Selberg class $\mathcal{S}$ consists of Dirichlet series

$$\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \tag{12}$$

satisfying the following hypotheses:

(i) Ramanujan Hypothesis: $a(n) \ll n^{\epsilon}$ for any $\epsilon > 0$, where the implicit constant may depend on ϵ.

(ii) Analytic Continuation: there exists an integer $k \geq 0$ such that $(s-1)^k \mathcal{L}(s)$ is an entire function of finite order.

(iii) Functional Equation: $\mathcal{L}(s)$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s) = \omega \overline{\Lambda_{\mathcal{L}}(1-\overline{s})}, \qquad \text{where} \quad \Lambda_{\mathcal{L}}(s) := \mathcal{L}(s) Q^s \prod_{j=1}^{f} \Gamma(\lambda_j s + \mu_j)$$

with positive real numbers Q, λ_j, and complex numbers μ_j, ω with $\operatorname{Re} \mu_j \geq 0$ and $|\omega| = 1$.

(iv) Euler Product: $\mathcal{L}(s)$ has a product representation

$$\mathcal{L}(s) = \prod_p \mathcal{L}_p(s), \qquad \text{where} \quad \mathcal{L}_p(s) = \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}} \right)$$

with suitable coefficients $b(p^k)$ satisfying $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$.

Recall the notion of the order of a meromorphic function. Assume that $\mathcal{L}(s)$ is analytic in some strip $\sigma_1 \leq \operatorname{Re} s \leq \sigma_2$ except for at most a finite number of poles. Then $\mathcal{L}(s)$ is said to be of finite order in this strip if there exists a positive constant c such that the estimate

$$\mathcal{L}(\sigma + it) \ll |t|^c \qquad \text{as} \quad |t| \to \infty \tag{13}$$

holds uniformly for $\sigma_1 \leq \operatorname{Re} s \leq \sigma_2$; here and in the sequel we shall sometimes write the complex variable as $s = \sigma + it$ (which is tradition since Landau). Similarly, one defines the notion of finite order for half-planes $\operatorname{Re} s \geq \sigma_1$. Clearly, a function given by a Dirichlet series is of finite order in its half-plane of convergence. Given $\operatorname{Re} s$, define $\mu_{\mathcal{L}}(\sigma)$ to be the lower bound of all c for which (13) holds:

$$\mu_{\mathcal{L}}(\sigma) = \limsup_{t \to \pm\infty} \frac{\log |\mathcal{L}(\sigma + it)|}{\log |t|};$$

this quantity is called the order of $\mathcal{L}(s)$ on the vertical line $\sigma + i\mathbb{R}$. One can show that the function $\mu_{\mathcal{L}}(\sigma)$ is convex downwards (in particular, it is continuous). Moreover, $\mu_{\mathcal{L}}(\sigma) = 0$

for any σ strictly greater than the abscissa of absolute convergence. Conversely, the Dirichlet series is convergent in the half-plane where $\mathcal{L}(s)$ is regular and $\mu_{\mathcal{L}}(\sigma)=0$. Exploiting the functional equation for $\mathcal{L}\in\mathcal{S}$, it is not difficult to prove that

$$\mu_{\mathcal{L}}(\sigma)\leq\begin{cases}0 & \text{if}\quad \sigma>1,\\ \frac12\,\mathrm{d}_{\mathcal{L}}(1-\sigma) & \text{if}\quad 0\leq\sigma\leq 1,\\ (\frac12-\sigma)\,\mathrm{d}_{\mathcal{L}} & \text{if}\quad \sigma<0.\end{cases}\tag{14}$$

Exercise 4. *Prove Formula (14). For this purpose rewrite the functional equation as*

$$\mathcal{L}(s)=\Delta_{\mathcal{L}}(s)\overline{\mathcal{L}(1-\overline{s})},\qquad \textit{where}\quad \Delta_{\mathcal{L}}(s):=\omega Q^{1-2s}\prod_{j=1}^{f}\frac{\Gamma(\lambda_j(1-s)+\overline{\mu_j})}{\Gamma(\lambda_j s+\mu_j)}.$$

Applying Stirling's formula and using the so-called Phragmén-Lindelöf principle (for both see Appendix B), show that

$$\mathcal{L}(\sigma+it)\asymp|t|^{(\frac12-\sigma)\,\mathrm{d}_{\mathcal{L}}}|\mathcal{L}(1-\sigma+it)|,\tag{15}$$

uniformly in σ, as $|t|\to\infty$.[‡] *Finally, deduce (14).*

In view of the functional equation, resp. the convexity of $\mu_{\mathcal{L}}(\sigma)$, the value for $\sigma=\frac12$ is essential. In particular, we obtain $\mu_{\mathcal{L}}(\frac12)\leq\frac14\,\mathrm{d}_{\mathcal{L}}$ or, equivalently,

$$\mathcal{L}\left(\frac12+it\right)\ll|t|^{\frac14\,\mathrm{d}_{\mathcal{L}}+\epsilon}\tag{16}$$

for $|t|\geq 1$; this bound is known as the convexity bound. The best known upper bound for the Riemann zeta-function is $\mu_\zeta(\frac12)\leq\frac{32}{205}$, due to Huxley [16].

The most simple examples of elements of the Selberg class are the Riemann zeta-function and shifts $L(s+i\theta,\chi)$ of Dirichlet L-functions attached to primitive characters χ with $\theta\in\mathbb{R}$. To verify this one just needs to recall Theorem 1 and 3 from the previous section. More advanced examples are L-functions associated with certain modular forms which we introduce now.

Denote by $\mathbb{H}$ the upper half-plane $\{z:=x+iy\in\mathbb{C}\,:\,y>0\}$, and let k and N be positive integers, k being even. Recall that the modular group $\mathsf{SL}_2(\mathbb{Z})$ is the set of all 2×2-matrices with integer entries and determinant 1; this group is generated by the matrices $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. The subgroup

$$\Gamma_0(N):=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathsf{SL}_2(\mathbb{Z})\,:\,c\equiv 0\bmod N\right\}$$

of $\mathsf{SL}_2(\mathbb{Z})$ is called Hecke subgroup of level N or congruence subgroup $\bmod N$. A holomorphic function $f(z)$ on $\mathbb{H}$ is said to be a cusp form of weight k and level N, if

$$f\left(\frac{az+b}{cz+d}\right)=(cz+d)^k f(z)$$

for all $z\in\mathbb{H}$ and all matrices $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(N)$, and if f vanishes at all cusps. The vanishing of f at the cusps is equivalent with

$$z:=x+iy\;\mapsto\; y^k|f(z)|^2$$

is bounded on $\mathbb{H}$. Then f has for $z\in\mathbb{H}$ a Fourier expansion

$$f(z)=\sum_{n=1}^{\infty}c(n)\exp(2\pi inz).\tag{17}$$

[‡]We write $f(x)\asymp g(x)$ with some positive function g if both, $f(x)\ll g(x)$ and $g(x)\ll|f(x)|$.

The cusp forms on $\Gamma_0(N)$ of weight k form a finite dimensional complex vector space, denoted by $S_k(\Gamma_0(N))$, with the Petersson inner product, defined by

$$\langle f, g\rangle = \int_{\mathbb{H}/\Gamma_0(N)} f(z)\overline{g(z)} y^k \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

for $f, g \in S_k(\Gamma_0(N)$. Suppose that $M|N$. If $f \in S_k(\Gamma_0(M))$ and $dM|N$, then $z \mapsto f(dz)$ is a cusp form on $\Gamma_0(N)$ of weight k too. The forms which may be obtained in this way from divisors M of the level N with $M \neq N$ span a subspace $S_k^{\mathrm{old}}(\Gamma_0(N))$, called the space of oldforms. Its orthogonal complement with respect to the Petersson inner product is denoted $S_k^{\mathrm{new}}(\Gamma_0(N))$. For $n \in \mathbb{N}$ we define the Hecke operator $T(n)$ by

$$T(n)f = \frac{1}{n}\sum_{ad=n} a^k \sum_{0\leq b<d} f\left(\frac{az+b}{d}\right)$$

for $f \in S_k(\Gamma_0(N))$. The operators $T(n)$ are multiplicative, i.e., $T(mn) = T(m)T(n)$ for coprime m, n, and they encode plenty of arithmetic information about modular forms. The theory of Hecke operators implies the existence of an orthogonal basis of $S_k^{\mathrm{new}}(\Gamma_0(N))$ made of eigenfunctions of the operators $T(n)$ for n coprime with N. By the multiplicity-one principle of Atkin & Lehner, the elements f of this basis are in fact eigenfunctions of all $T(n)$, i.e., there exist complex numbers $\lambda_f(n)$ for which $T(n)f = \lambda_f(n)f$ and $c(n) = \lambda_f(n)c(1)$ for all $n \in \mathbb{N}$. Furthermore, it follows that the first Fourier coefficient $c(1)$ of such an f is non-zero. Such a simultaneous eigenfunction is said to be an eigenform. A newform is defined to be an eigenform that does not come from a space of lower level and is normalized to have $c(1) = 1$. The newforms form a finite set which is an orthogonal basis of the space $S_k^{\mathrm{new}}(\Gamma_0(N))$. For instance, Ramanujan's cusp form

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)\exp(2\pi inz) := \exp(2\pi iz)\prod_{n=1}^{\infty}(1-\exp(2\pi inz))^{24} \tag{18}$$

is a normalized eigenform of weight 12 to the full modular group, and hence a newform of level 1.

To see that we fix a positive even integer $k > 2$ and define the Eisenstein series of weight k by

$$G_k(z) = \frac{(k-1)!}{2(2\pi i)^k} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}$$

(the condition $k > 2$ is needed to guarantee absolute convergence). The action of $M = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathsf{SL}_2(\mathbb{Z})$ on this function replaces (m, n) by $(am + cn, bm + dn)$ and therefore permutes the terms of the sum. We obtain

$$G_k\left(\frac{az+b}{cz+d}\right) = (cz+d)^k G_k(z). \tag{19}$$

Using Lipschitz' formula,

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{d=1}^{\infty} d^{k-1}\exp(2\pi idz),$$

we find by splitting the G_k-defining sum into terms with $m=0$ and the terms with $m\neq 0$ that

$$\begin{aligned} G_k(z) &= \frac{(k-1)!}{(2\pi i)^k}\sum_{n=1}^\infty \frac{1}{n^k}+\sum_{m=1}^\infty\left(\frac{(k-1)!}{(2\pi i)^k}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^k}\right)\\ &= (-1)^{\frac{k}{2}}\frac{(k-1)!}{(2\pi)^k}\zeta(k)+\sum_{m=1}^\infty\sum_{d=1}^\infty d^{k-1}\exp(2\pi i dmz). \end{aligned}$$

In view of the values of the zeta-function at the integers in terms of the Bernoulli numbers B_k, Theorem 2 from §1, we get the Fourier series expansion

$$G_k(z)=-\frac{B_k}{2k}+\sum_{n=1}^\infty \sigma_{k-1}(n)\exp(2\pi i nz);$$

here $\sigma_{k-1}(n)$ denotes the sum of divisors of n in the power $k-1$. The so-called discriminant is defined by

$$\Delta(z)=\frac{(2\pi)^{12}}{1728}\left((240G_4(z))^3-(504G_6(z))^2\right)$$

(the name discriminant comes from the theory of elliptic curves). In view of (19) it follows that

$$\begin{aligned} \Delta\left(\frac{az+b}{cz+d}\right) &= \frac{(2\pi)^{12}}{1728}\left\{\left(240G_4\left(\frac{az+b}{cz+d}\right)\right)^3-\left(504G_6\left(\frac{az+b}{cz+d}\right)\right)^2\right\}\\ &= (cz+d)^{12}\frac{(2\pi)^{12}}{1728}\left\{(240G_4(z))^3-(504G_6(z))^2\right\}\\ &= (cz+d)^{12}\Delta(z) \end{aligned}$$

for all $M=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathsf{SL}_2(\mathbb{Z})$. The proof of the product representation (18) can be found in Koblitz [26]). The Fourier series expansion for $\Delta(z)$ in (18) defines Ramanunajan's τ-function for which he conjectured multiplicativity and that they satisfy the estimate $|\tau(p)|\leq 2p^{\frac{11}{2}}$ for every prime number p. The multiplicativity was proved by Mordell, in particular by the beautiful formula

$$\tau(m)\tau(n)=\sum_{d|(m,n)} d^{11}\tau\left(\frac{mn}{d^2}\right).$$

The estimate was shown by Deligne who proved for the coefficients of any newform f of weight k the estimate

$$|c(n)|\leq n^{\frac{k-1}{2}}d(n), \tag{20}$$

where $d(n):=\sum_{d|n}1$ is the divisor function.

In the 1930s, Hecke started investigations on modular forms and Dirichlet series with a Riemann-type functional equation (see his Converse Theorem 4). His studies were completed by Atkin & Lehner for newforms. Here we shall focus on newforms. Given a newform f with Fourier expansion (17), we define the associated L-function by

$$L(s,f)=\sum_{n=1}^\infty \frac{c(n)}{n^s}. \tag{21}$$

In view of the classic bound $d(n)\ll n^\epsilon$ it follows from Deligne's estimate (20) that the series (21) converges absolutely for $\operatorname{Re} s>\frac{k+1}{2}$. By the theory of Hecke operators, the

Fourier coefficients of newforms are multiplicative. Hence, in the half-plane of absolute convergence, there is an Euler product representation and it is given by

$$L(s,f)=\prod_{p|N}\left(1-\frac{c(p)}{p^s}\right)^{-1}\prod_{p\not|N}\left(1-\frac{c(p)}{p^s}+\frac{1}{p^{2s+1-k}}\right)^{-1}. \tag{22}$$

Hecke, resp. Atkin & Lehner, proved that $L(s,f)$ has an analytic continuation to an entire function and satisfies the functional equation

$$N^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L(s,f)=\omega(-1)^{\frac{k}{2}}N^{\frac{k-s}{2}}(2\pi)^{s-k}\Gamma(k-s)L(k-s,f),$$

where $\omega=\pm1$ is the eigenvalue of the Atkin-Lehner involution $\left(\begin{smallmatrix}0&-N\\1&0\end{smallmatrix}\right)$ on $S_k(\Gamma_0(N))$. This follows more or less from the Converse Theorem 4 with $f=g$ and $\lambda=1$. For congruence subgroups $\Gamma_0(N)$ a similar converse theorem is due to Weil; however, for identifying an L-function associated with a modular form of $\Gamma_0(N)$ one has to consider sufficiently many twists with primitive characters and their analytic behaviour (see Iwaniec [17] for details).

In the context of the Selberg class we shall normalize these L-functions as follows. Suppose that f is a newform of weight k to some congruence subgroup $\Gamma_0(N)$ with Fourier expansion (17). Writing

$$a(n)=c(n)n^{\frac{1-k}{2}}$$

we find via (22) the Euler product representation

$$L\left(s+\frac{k-1}{2},f\right)=\prod_p\left(1-\frac{a(p)}{p^s}+\frac{\chi(p)}{p^{2s}}\right)^{-1},$$

where $\chi(p)=0$ if $p\mid N$, and $\chi(p)=1$ otherwise. In the latter case, i.e., $p\mid N$, the corresponding Euler factor can be rewritten as

$$\left(1-\frac{a(p)}{p^s}+\frac{1}{p^{2s}}\right)^{-1}=\left(1-\frac{\alpha_1(p)}{p^s}\right)^{-1}\left(1-\frac{\alpha_2(p)}{p^s}\right)^{-1},$$

where $\alpha_1(p),\alpha_2(p)$ are complex numbers satisfying

$$\alpha_1(p)+\alpha_2(p)=a(p)\qquad\text{and}\qquad\alpha_1(p)\alpha_2(p)=1;$$

Deligne's estimate (20) translates to

$$|\alpha_1(p)|=|\alpha_2(p)|=1,\qquad\text{i.e.,}\qquad\alpha_1(p)=\overline{\alpha_2(p)}.$$

Thanks to the transformation $s\mapsto s+\frac{k-1}{2}$ the critical strip is normalized to $0\le\operatorname{Re}s\le1$ (as for $\zeta(s)$, independent of the weight). In the sequel we shall assume that L-functions to modular forms are normalized in this way, and we denote them again by $L(s,f)$. Further examples of $\mathcal{S}$ of similar type are Rankin-Selberg convolution and L-functions and symmetric power L-functions, however, we do not give their definition here but refer to the monographs of Iwaniec & Kowalski [18], M.R. Murty & V.K. Murty [38], and Kaczorowski's survey [20]. The elements in the Selberg class are automorphic or at least conjecturally automorphic L-functions, and it is conjectured that $\mathcal{S}$ consists of all automorphic L-functions.

In view of the Euler product representation it is clear that any element $\mathcal{L}(s)$ of the Selberg class does not vanish in the half-plane of absolute convergence $\operatorname{Re}s>1$. This gives rise to the notions of critical strip and critical line $s=\frac{1}{2}+i\mathbb{R}$ (as in the theory of the Riemann zeta-function). The zeros of $\mathcal{L}(s)$ located at the poles of gamma-factors appearing in the functional equation are called trivial and they all lie in $\operatorname{Re}s\le0$. All other zeros are said to be nontrivial. It is expected that for every function in the Selberg class the analogue of the Riemann hypothesis holds:

Grand Riemann Hypothesis. *If $\mathcal{L} \in \mathcal{S}$, then $\mathcal{L}(s) \neq 0$ for $\sigma > \frac{1}{2}$.*

There are only a few attempts towards the Riemann Hypothesis and its generalizations. An old idea due to Hilbert and Polyá is to find a self-adjoint operator in some Hilbert space whose spectrum coincides with the zeros of an L-function. Weil found generalizations and expressed the truth of the Riemann hypothesis in the positivity of a certain hermitian form (cf. [2]). Recently, Mazhouda & Omar [31, 32] extended an approach of Li [29] for the Riemann zeta-function (which is not unrelated to Weil's criterion) to the whole Selberg class.[§]

The zero-distribution is essential for the Selberg class which is also manifested in each of the defining axioms. Following Conrey & Ghosh [7] we motivate the axioms defining $\mathcal{S}$. We have already seen that the Ramanujan hypothesis implies the regularity of $\mathcal{L}(s)$ in $\mathrm{Re}\, s > 1$. The assumption that there be at most one pole, and that this one is located at $s = 1$, is natural in the theory of L-functions. It seems that the point $s = 1$ is the only possible pole for an automorphic L-function and that such a pole is always related to the simple pole of the Riemann zeta-function in the sense that the quotient with an appropriate power of $\zeta(s)$ is another L-function which is entire (examples for this scenario are Dedekind zeta-functions). The restriction $\mathrm{Re}\,\mu_j \geq 0$ in the functional equation comes from the theory of Maass waveforms. If one assumes the existence of an arithmetic subgroup of $\mathsf{SL}_2(\mathbb{R})$ together with such a non-analytic cusp form that corresponds to an exceptional eigenvalue, and if one further supposes that all local roots are sufficiently small (more precisely, that the Ramanujan-Petersson conjecture holds), then the L-function associated with the Maass cusp form has a functional equation where the μ_j satisfy $\mathrm{Re}\,\mu_j < 0$, but this L-function violates the analogue of Riemann's hypothesis; see Kaczorowski [20] for more details. The axiom on the Euler product representation will be discussed later in this section.

Of special interest is the structure of the Selberg class. The degree of $\mathcal{L} \in \mathcal{S}$ is defined by

$$\mathrm{d}_{\mathcal{L}} = 2\sum_{j=1}^{f} \lambda_j.$$

Although the data of the functional equation is not unique, the degree is well-defined as follows from an asymptotic formula for the number of zeros in analogy to the classical Riemann-von Mangoldt formula for Riemann's zeta-function which we state as

Theorem 5 (**Riemann–von Mangoldt Formula**). *If $N_{\mathcal{L}}(T)$ counts the number of zeros of $\mathcal{L} \in \mathcal{S}$ in the rectangle $0 \leq \mathrm{Re}\, s \leq 1, |\mathrm{Im}\, s| \leq T$ (according to multiplicities), then*

$$N_{\mathcal{L}}(T) = \frac{\mathrm{d}_{\mathcal{L}}}{\pi} T \log T + O(T),$$

Sketch of Proof. We apply the principle of the argument from the Theory of Functions. For this aim we integrate the logarithmic derivative of the left-hand of the functional equation,

$$\Lambda_{\mathcal{L}}(s) := \mathcal{L}(s) Q^s \prod_{j=1}^{f} \Gamma(\lambda_j s + \mu_j),$$

[§]It is legend that Hardy said that if the Riemann Hypothesis for the zeta-function will be proved some day, the analogue for Dirichlet L-functions will be shown the following day at latest.

along the rectangular contour $\mathcal{R}$ given by the vertices $-1 \pm iT, 2 \pm iT$ in counterclockwise direction; here we may assume that T is not the ordinate of a zero of $\mathcal{L}(s)$ so there are no zeros on the contour. Then

$$N_{\mathcal{L}}(T) = \frac{1}{2\pi i}\int_{\mathcal{R}} \frac{\Lambda_{\mathcal{L}}'}{\Lambda_{\mathcal{L}}}(s)\,\mathrm{d}s + O(1),$$

where the error term results from the possible pole at $s = 1$. Taking into account the principle of the argument, we may compute $N(T)$ as the variation of the argument of $\Lambda_{\mathcal{L}}(s)$ along $\mathcal{R}$. Using the functional equation, we may replace that part of $\mathcal{R}$ which is lying to the left of the critical line by another change in the argument such that

$$N_{\mathcal{L}}(T) = \frac{2}{\pi}\Delta\arg\Lambda_{\mathcal{L}}(s) + O(1),$$

where Δ denotes the variation along the line segments from 2 to $2+iT$ and from $2+iT$ to $\frac{1}{2}+iT$. We observe that $\Delta\arg$ is additive and that the main contribution comes from the Gamma-factors which is computed by Stirling's formula (Theorem 32 from Appendix B); this explains the appearance of the degree $\mathrm{d}_{\mathcal{L}}$ as factor in the main term. The estimate of the argument of $\mathcal{L}(\frac{1}{2}+iT)$ is the most difficult part and can be made via Jensen's formula (as in the case of the zeta-function; see [57, 55]). A precise evaluation leads to the asymptotic formula of the theorem (or even a more precise one with another main term cT, where the constant c contains data from the functional euqation, and an error term $O(\log T)$. •

A different proof of Theorem 5 is given in Chapter 7 of [54] (based on a method of Levinson).

Exercise 5. *Fill all gaps left in the sketch of proof of Theorem 5. See* [57] *for the special case of the zeta-function;* [18] *might offer some help for the case that all weights in the axiom on the functional equation are* $\lambda_j = \frac{1}{2}$. *Study the method of proof in* [54]. *What are advantages and disadvantages of either approach?*

We shall give an example: the following identity is equivalent to the form of the functional equation we obtained in §1:

$$\left(\frac{\pi}{2}\right)^{-\frac{s}{2}}\Gamma\left(\frac{s}{4}\right)\Gamma\left(\frac{s}{4}+\frac{1}{2}\right)\zeta(s) = \left(\frac{\pi}{2}\right)^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{4}\right)\Gamma\left(\frac{1-s}{4}+\frac{1}{2}\right)\zeta(1-s).$$

This functional equation can be transformed into the one of Theorem 1 by the duplication formula for the Gamma-function. From both functional equations we deduce the degree $\mathrm{d}_\zeta = 1$. As a matter of fact, the study of invariants defined in terms of the data of $\mathcal{L} \in \mathcal{S}$ is an important tool in deeper studies of the Selberg class. For further reading on invariants we refer to [43].

Another class of examples of elements of the Selberg class are L-functions of number fields $\mathbb{K}$ (i.e., finite algebraic extensions of $\mathbb{Q}$. For example, the Dedekind zeta-function of a number field $\mathbb{K}$ over $\mathbb{Q}$ is given by

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a}} \frac{1}{\mathrm{N}(\mathfrak{a})^s} = \prod_{\mathfrak{p}}\left(1 - \frac{1}{\mathrm{N}(\mathfrak{p})^s}\right)^{-1} \qquad (\mathrm{Re}\, s > 1), \tag{23}$$

where the sum is taken over all non-zero integral ideals, the product is taken over all prime ideals of the ring of integers of $\mathbb{K}$, and $\mathrm{N}(\mathfrak{a})$ is the norm of the ideal $\mathfrak{a}$. The Riemann zeta-function may be regarded as the Dedekind zeta-function for $\mathbb{Q}$. Note that, by the splitting of primes, the above Euler product representation over prime ideals can be rewritten in the form of axiom (4) (as we shall indicate below in the case of quadratic number fields). The functional equation for $\zeta_{\mathbb{K}}(s)$ was found by Hecke in 1917 and takes the form

$$A^s\Gamma\left(\frac{s}{2}\right)^{r_1}\Gamma(s)^{r_2}\zeta_{\mathbb{K}}(s)=A^{1-s}\Gamma\left(\frac{1-s}{2}\right)^{r_1}\Gamma(1-s)^{r_2}\zeta_{\mathbb{K}}(1-s)$$

with $A:=2^{-r_2}\pi^{-n}|\Delta_{\mathbb{K}}|^{\frac12}$, where r_1 is the number of real embeddings, r_2 is the number of pairs of conjugate complex embeddings, $n=r_1+2r_2=[\mathbb{K}:\mathbb{Q}]$ is the degree of the field extension, and $\Delta_{\mathbb{K}}$ is the discriminant of $\mathbb{K}$. It follows that the analytic degree of $\zeta_{\mathbb{K}}(s)$ coincides with the algebraic degree of the field extension $\mathbb{K}/\mathbb{Q}$, that is, $\mathrm{d}_{\zeta_{\mathbb{K}}}=[\mathbb{K}:\mathbb{Q}]$. In particular, we see that there exist elements in $\mathcal{S}$ of arbitrary large degree.

We shall briefly discuss the example $\mathbb{K}=\mathbb{Q}(\sqrt{D})$. We write for short $\mathrm{d}:=\Delta_{\mathbb{Q}(\sqrt{D})}$ (since now there is no confusion with the degree), which is equal to D if $D\equiv 1 \bmod 4$, and equal to $4D$ if $D\equiv 2,3 \bmod 4$. In view of the splitting of the primes one easily finds

$$\begin{aligned}\zeta_{\mathbb{Q}(\sqrt{D})}(s) &= \prod_{(\frac{\mathrm{d}}{p})=+1}\left(1-\frac{1}{p^s}\right)^{-2}\prod_{(\frac{\mathrm{d}}{p})=0}\left(1-\frac{1}{p^s}\right)^{-1}\prod_{(\frac{\mathrm{d}}{p})=-1}\left(1-\frac{1}{p^s}\right)^{-1}\left(1+\frac{1}{p^s}\right)^{-1}\\ &= \zeta(s)L(s,\chi_{\mathrm{d}}), \qquad (24)\end{aligned}$$

with the Jacobi symbol, defined by

$$\chi_{\mathrm{d}}:\mathbb{N}\to\mathbb{C},\quad n\mapsto\left(\frac{\mathrm{d}}{n}\right)=\prod_{j=1}^{\nu}\left(\frac{\mathrm{d}}{p_j}\right),$$

where $n=p_1\cdot\ldots\cdot p_\nu$ is the factorization of the integer n into prime factors (not necessarily distinct). An excellent reading in Algebraic Number Theory is Narkiewicz [40].

The following important conjecture is based on the above and further examples of elements of the Selberg class and its solution would provide a lot of structure:

Degree Conjecture. *All $\mathcal{L}\in\mathcal{S}$ have integral degree. Moreover, all λ_j appearing in the Gamma-factors of the functional equation can be chosen to be equal to $\frac12$.*

Recently, Kaczorowski & Perelli proved the degree conjecture for the range $(0,2)$. Here, we shall prove

Theorem 6 (Structure Theorem – Small Degrees). *If $\mathcal{L}\in\mathcal{S}$ has degree $0\le \mathrm{d}_{\mathcal{L}}<1$, then $\mathrm{d}_{\mathcal{L}}=0$ and $\mathcal{L}(s)\equiv 1$.*

It should be noted that the statement proved above is implicitly contained in the works of Bochner, Richert, and Vignéras on classifying the solutions of Riemann-type functional equations. In some sense Theorem 6 may be considered as a converse theorem similar to Hecke's Theorem 4. In the sequel we shall present further results of this type; the case of degree $[1,2)$ will be considered in Sections §4 and §5.

Proof. We consider the coefficients $a(n)$ of the Dirichlet series representation of $\mathcal{L}$. Let B be a constant such that $a(n)\ll n^B$. By Perron's formula (Theorem 31),

$$\sum_{n\le x}a(n)=\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\mathcal{L}(s)\frac{x^s}{s}\,\mathrm{d}s+O\left(\frac{x^{c+B}}{T}\right),$$

where $c>1$ is a constant. Shifting the path of integration to the left, yields, by the Phragmén-Lindelöf principle (Theorem 33), the asymptotic formula

$$\sum_{n\le x}a(n)=xP(\log x)+O\left(x^{(1+B)\frac{\mathrm{d}_{\mathcal{L}}-1}{\mathrm{d}_{\mathcal{L}}+1}+\epsilon}\right),$$

where $P(x)$ is a computable polynomial according to the principal part of the Laurent expansion of $\mathcal{L}(s)$ at $s=1$. By subtraction, this implies

$$a(n) \ll n^{(1+B)\frac{\mathrm{d}_{\mathcal{L}}-1}{\mathrm{d}_{\mathcal{L}}+1}+\epsilon}, \tag{25}$$

where the implicit constant depends on B. For $\mathrm{d}_{\mathcal{L}} < 1$ the exponent is negative, and we may choose B arbitrarily large. Then $\mathcal{L}(s)$ is uniformly bounded in every right half-plane. This is a contradiction for $\mathcal{L} \in \mathcal{S}$ with positive degree since the functional equation implies a certain order of growth (see (15)). This shows that $\mathcal{S}$ is free of elements having degree $0 < \mathrm{d} < 1$.

It remains to consider the case that $\mathrm{d}_{\mathcal{L}} = 0$. Then the functional equation takes the form:

$$Q^s\mathcal{L}(s) = \omega Q^{1-s}\overline{\mathcal{L}(1-\overline{s})}$$

(there are no Gamma-factors). By (25) the $a(n)$ are so small that the Dirichlet series for $\mathcal{L}(s)$ converges in the whole complex plane. Thus we may rewrite the functional equation as

$$\sum_{n=1}^{\infty} a(n)\left(\frac{Q^2}{n}\right)^s = \sum_{n=1}^{\infty} \omega Q\frac{\overline{a(n)}}{n}n^s. \tag{26}$$

We may regard this as an identity between absolutely convergent Dirichlet series. Thus, if $a(n) \neq 0$, then Q^2/n is an integer. In particular, $q := Q^2 \in \mathbb{N}$. Moreover, since q has only finitely many divisors, it follows that $\mathcal{L}(s)$ is a Dirichlet polynomial. If $q=1$, then $\mathcal{L}(s) \equiv 1$ and we are done. Hence, we may assume $q > 1$ from now on.

Since the Dirichlet coefficients $a(n)$ are multiplicative, we have $a(1)=1$ and via (26)

$$a(1)Q^{2s} = \omega Q^{-1}\overline{a(Q^2)}Q^{2s};$$

thus, $|a(q)| = Q$. In particular, there exists a prime p such that the exponent ν of p in the prime factorization of q is positive and, by the multiplicativity of the $a(n)$'s,

$$|a(p^\nu)| \geq p^{\frac{\nu}{2}}.$$

Now consider the logarithm of the corresponding Euler factor:

$$\log\left(1+\sum_{m=1}^{\nu}\frac{a(p^m)}{p^{ms}}\right) = \sum_{k=1}^{\infty}\frac{b(p^k)}{p^{ks}}.$$

Viewing this as a power series in $X = p^{-s}$, we write

$$\log P(X) = \sum_{k=1}^{\infty} B_k X^k \qquad \text{with} \quad B_k = b(p^k).$$

Since $a(1)=1$, we find

$$P(X) = 1 + \sum_{m=1}^{\nu} a(p^m)X^m = \prod_{j=1}^{\nu}(1-C_jX) \qquad \text{with} \quad B_k = -\frac{1}{k}\sum_{j=1}^{\nu} C_j^k.$$

Now

$$\prod_{j=1}^{\nu}|C_j| = |a(p^\nu)| \geq p^{\frac{\nu}{2}},$$

and thus the maximum of the values $|C_j|$ is greater than or equal to $p^{\frac{1}{2}}$. We have

$$\lim_{k\to\infty}|b(p^k)|^{\frac{1}{k}} = \lim_{k\to\infty}\left|\frac{1}{k}\sum_{j=1}^{\nu}C_j^k\right|^{\frac{1}{k}} = \max_{1\leq j\leq\nu}|C_j|;$$

by our foregoing observations the right-hand side is greater than or equal to $p^{\frac{1}{2}}$. This is a contradiction to the condition $b(p^k) \ll p^{k\theta}$ with some $\theta < \frac{1}{2}$ in the axiom on the Euler product. Hence, $q = 1$ and $\mathcal{L}(s) \equiv 1$. This proves the theorem. •

Now let us consider the axiom concerning the Euler product. It is well-known that the existence of an Euler product is a necessary (but not sufficient) condition for Riemann's hypothesis. On first sight the condition $\theta < \frac{1}{2}$ seems to be a little bit unnatural. However, for $\theta = \frac{1}{2}$, there are examples violating the Riemann hypothesis: the function

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

has zeros off the critical line; the latter representation follows from (5). As we have seen in the proof of Theorem 6, the bound for θ rules out non-trivial Dirichlet polynomials from $\mathcal{S}$ as for example

$$(1 - 2^{a-s})(1 - 2^{b-s}) \qquad \text{with} \quad a + b = 1.$$

The extended Selberg class $\mathcal{S}^\sharp$ is defined as the set of all Dirichlet series (12) which converge for $\mathrm{Re}\, s > 1$ and satisfy axioms (ii) and (iii). This larger class is of minor arithmetical interest. To illustrate this please notice that the Davenport-Heilbronn zeta-function is an example of an element in $\mathcal{S}^\sharp$ (of degree one) which violates the analogue of the Riemann hypothesis; lacking an Euler product representation, this example is not in the Selberg class $\mathcal{S}$. We conclude with an easy

Exercise 6. *Reviewing the proof of Theorem 6, prove that any element $\mathcal{L} \in \mathcal{S}^\sharp$ of degree zero is a Dirichlet polynomial of the following form:*

$$\mathcal{L}(s) = \sum_{n|Q^2} \frac{a(n)}{n^s} \qquad \textit{with} \qquad a(n) = \omega \frac{n}{Q} \overline{a(Q^2/n)}.$$

For another characterization of the small degree elements of the extended Selberg class see [53].

---

3. The Selberg Conjectures and Applications to Arithmetic

The Selberg class is multiplicatively closed. A function $\mathcal{L} \in \mathcal{S}$ is called primitive if it cannot be factored as a product of two elements non-trivially:.

$$\mathcal{L} = \mathcal{L}_1\mathcal{L}_2 \quad \text{with} \quad \mathcal{L}_j \in \mathcal{S} \qquad \Longrightarrow \qquad \mathcal{L}_1 \equiv 1 \quad \text{or} \quad \mathcal{L}_2 \equiv 1.$$

The notion of a primitive function is fruitful for studying the structure of $\mathcal{S}$. The central claim concerning primitive functions is part of

Selberg's Conjectures. *Denote by $a_{\mathcal{L}}(n)$ the coefficients of the Dirichlet series representation of $\mathcal{L} \in \mathcal{S}$.*
A) *For all $1 \neq \mathcal{L} \in \mathcal{S}$ there exists a positive integer $n_{\mathcal{L}}$ such that*

$$\sum_{p \le x} \frac{|a_{\mathcal{L}}(p)|^2}{p} = n_{\mathcal{L}} \log\log x + O(1),$$

where the summation is over prime numbers.
B) *For any primitive functions $\mathcal{L}_1$ and $\mathcal{L}_2 \in \mathcal{S}$,*

$$\sum_{p \le x} \frac{a_{\mathcal{L}_1}(p)\overline{a_{\mathcal{L}_2}(p)}}{p} = \begin{cases} \log\log x + O(1) & \textit{if} \quad \mathcal{L}_1 = \mathcal{L}_2, \\ O(1) & \textit{otherwise}. \end{cases}$$

The stronger Conjecture B reflects a certain kind of orthogonality and is therefore also called Selberg's Orthogonality Conjecture.

In some particular cases it is not too difficult to verify Selberg's Conjecture A. For instance, $\zeta(s)$ satisfies Selberg's Conjecture A which is basically due to Euler who wrote

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots=\log\log\infty.$$

By the Prime Number Theorem 11 for arithmetic progressions (below) it is easy to show that

$$\frac{1}{\varphi(q)}\sum_{p\equiv a \bmod q}\frac{1}{p}=\begin{cases}\log\log x+O(1) & \text{if}\quad \gcd(a,q)=1,\\ O(1) & \text{otherwise,}\end{cases}$$

where $\varphi(q)$ is Euler's totient; hence the same asymptotics hold for Dirichlet L-functions. Taking into account the orthogonality relations for characters, one can also verify Conjecture B for pairs of Dirichlet L-functions. The Rankin-Selberg convolution method shows that L-functions associated with holomorphic modular forms satisfy some kind of orthogonality (in terms of regularity at $s=1$) which is related to Selberg's conjectures. Liu, Wang & Ye [30] proved Conjecture B for automorphic L-functions $L(s,\pi)$ and $L(s,\pi')$, where π and π' are automorphic irreducible cuspidal representations of $\mathsf{GL}_m(\mathbb{Q})$ and $\mathsf{GL}_{m'}(\mathbb{Q})$, respectively; their result holds unconditionally for $m,m'\leq 4$ and in other cases under the assumption of the convergence of

$$\sum_p\frac{|a_\pi(p^k)|^2}{p^k}(\log p)^2$$

for $k\geq 2$, where the $a_\pi(n)$ denote the Dirichlet series coefficients of $L(s,\pi)$. The latter hypothesis is an immediate consequence of the Ramanujan Hypothesis.

An important feature about prime numbers is the unique prime factorization of integers (or rationals). This important concept from arithmetic has a functional analogue: The notion of degree already permits to prove factorization into primitive elements! An important consequence of Selberg's conjectures, due to Conrey & Ghosh [7], is the stronger concept of unique factorization into primitive elements:

Theorem 7 ((Unique) Factorization Theorem). *Every function in the Selberg class has a factorization into primitive functions. If Selberg's conjecture B is true, then this factorization into primitive functions is unique.*

Proof. Suppose that $\mathcal{L}$ is not primitive, then there exist functions $\mathcal{L}_1$ and $\mathcal{L}_2$ in $\mathcal{S}\setminus\{1\}$ such that $\mathcal{L}=\mathcal{L}_1\mathcal{L}_2$. Taking into account the Riemann-von Mangoldt formula, Theorem 5, from

$$N_{\mathcal{L}}(T)=N_{\mathcal{L}_1}(T)+N_{\mathcal{L}_2}(T)$$

we find $\mathrm{d}_{\mathcal{L}}=\mathrm{d}_{\mathcal{L}_1}+\mathrm{d}_{\mathcal{L}_2}$. In view of Structure Theorem 6, both $\mathcal{L}_1$ and $\mathcal{L}_2$ have degree at least 1. Thus, each of $\mathrm{d}_{\mathcal{L}_1}$ and $\mathrm{d}_{\mathcal{L}_2}$ is strictly less than $\mathrm{d}_{\mathcal{L}}$. A continuation of this process terminates since the number of factors is $\leq\mathrm{d}_{\mathcal{L}}$ by Theorem 6, which proves the first claim.

In order to prove the second claim suppose that $\mathcal{L}$ has two factorizations into primitive functions:

$$\mathcal{L}=\prod_{j=1}^{m}\mathcal{L}_j=\prod_{k=1}^{n}\tilde{\mathcal{L}}_k,$$

and assume that no $\tilde{\mathcal{L}}_k$ is equal to $\mathcal{L}_1$. Then it follows from

$$\sum_{j=1}^{m} a_{\mathcal{L}_j}(p) = \sum_{k=1}^{n} a_{\tilde{\mathcal{L}}_k}(p)$$

that

$$\sum_{j=1}^{m} \sum_{p\le x} \frac{a_{\mathcal{L}_j}(p)\overline{a_{\mathcal{L}_1}(p)}}{p} = \sum_{k=1}^{n} \sum_{p\le x} \frac{a_{\tilde{\mathcal{L}}_k}(p)\overline{a_{\mathcal{L}_1}(p)}}{p}.$$

By Selberg's conjecture B, the left-hand side tends to infinity for $x \to \infty$, whereas the right-hand side is bounded, giving the desired contradiction. •

The same argument gives a characterization of primitive functions in terms of the quantity $n_{\mathcal{L}}$ from Selberg's Conjecture A: if the Selberg Conjecture B is true, then $\mathcal{L} \in \mathcal{S}$ is primitive if and only if $n_{\mathcal{L}} = 1$. However, in connection with Theorem 6 it follows much easier that Riemann's zeta-function and Dirichlet L-functions are primitive. A more advanced example of primitive elements are L-functions associated with newforms is due to M.R. Murty [36]. On the contrary, Dedekind zeta-functions to cyclotomic fields $\neq \mathbb{Q}$ are not primitive.

Exercise 7. *Assuming Selberg's Conjectures, show that any function $\mathcal{L} \in \mathcal{S}$ which has a pole of order m at $s = 1$ is divisble by $\zeta(s)^m$ in the sense that $\mathcal{L}(s)/\zeta(s)^m$ is an entire function in $\mathcal{S}$.*

Our next application has purely arithmetical character. Recall the celebrated Prime Number Theorem,

$$\pi(x) \sim \frac{x}{\log x}\ , \qquad \text{resp.} \qquad \psi(x) \sim x,$$

where $\pi(x)$ counts the number of primes $p \le x$ and $\psi(x) := \sum_{n\le x} \Lambda(n)$. The analogue in the Selberg class of this deep result is

Theorem 8 (**Prime Number Theorem for the Selberg Class).** *For $\mathcal{L} \in \mathcal{S}$,*

$$\psi_{\mathcal{L}}(x) := \sum_{n\le x} \Lambda_{\mathcal{L}}(n) \sim k_{\mathcal{L}} x, \tag{27}$$

where $k_{\mathcal{L}} = 0$ if $\mathcal{L}(s)$ is regular at $s = 1$, otherwise $k_{\mathcal{L}}$ is the order of the pole of $\mathcal{L}(s)$ at $s = 1$, and $\Lambda_{\mathcal{L}}(n)$ is the von Mangoldt-function, defined by

$$-\frac{\mathcal{L}'}{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \frac{\Lambda_{\mathcal{L}}(n)}{n^s}.$$

The asymptotic formula (27) is unconditionally true for polynomial Euler products $\mathcal{L}$; otherwise it holds true subject to the truth of Selberg's Conjecture B.

Here an Euler product is said to be polynomial if it is of the form

$$\mathcal{L}(s) = \prod_{p} \prod_{j=1}^{m} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}, \tag{28}$$

where m is a fixed positive integer and for each prime p and $1 \le j \le m$ the $\alpha_j(p)$ are certain complex numbers (it is easily seen that they have absolute value less than or equal to one subject to the Ramanujan hypothesis). Substituting $X = p^{-s}$ each Euler factor is the reciprocal of a polynomial in X of degree m which explains the name polynomial Euler product.

For polynomial Euler products in the Selberg class one can prove this equivalence by standard arguments. We illustrate this by using the following Tauberian theorem.

Theorem 9 (Wiener–Ikehara Theorem). *Let* $F(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ *be a Dirichlet series with non-negative real coefficients and absolutely convergent for* $\mathrm{Re}\, s > 1$*. Assume that* $F(s)$ *can be extended to a meromorphic function in* $\mathrm{Re}\, s \geq 1$ *such that there are no poles except for a possible simple pole at* $s = 1$ *with residue* $r \geq 0$*. Then*

$$A(x) := \sum_{n \leq x} a(n) \sim rx.$$

The proof of this theorem can be found in Appendix C.

As an application of this Tauberian theorem we shall prove Dirichlet's prime number theorem for arithmetic progressions.[¶] Let $\chi \bmod q$ be a character. We consider the logarithmic derivative of a Dirichlet L-functions $L(s, \chi)$, given by

$$\frac{L'}{L}(s, \chi) = -\sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^s},$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with } k \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases}$$

is the von Mangoldt-function. We define

$$\psi(x; \chi) = \sum_{n \leq x} \Lambda(n)\chi(n) \qquad \text{and} \qquad \psi(x; a \bmod q) = \sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \Lambda(n).$$

By the orthogonality relation for characters (Theorem 26 in Appendix A), we find

$$\psi(x; a \bmod q) = \frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a)\psi(x; \chi).$$

Now suppose that a and q are coprime (otherwise the functions in the latter identity are all bounded). We want to apply Theorem 9 with the functions

$$F(s) = -\sum_{\chi \bmod q} \overline{\chi}(a) \frac{L'}{L}(s, \chi) \qquad \text{and} \qquad A(x) = \psi(x; a \bmod q).$$

Notice that the left-hand side has a Dirichlet series representation for $\mathrm{Re}\, s > 1$ with non-negative coefficients. It is well-known that $L(s, \chi)$ is analytic for $\mathrm{Re}\, s \geq 1$ if χ is not a principal character. In the case of the principal character it follows from (6) that

$$L(s, \chi_0) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right). \tag{29}$$

By partial summation (Theorem 28 in Appendix A),

$$\zeta(s) = \sum_{n \leq N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + s\int_N^{\infty} \frac{\lfloor u \rfloor - u}{u^{s+1}} \, \mathrm{d}u.$$

Hence, $\zeta(s)$ has a simple pole at $s = 1$ with residue 1 (which we already noticed in Theorem 1). Moreover,

$$-\frac{L'}{L}(s, \chi_0) = \frac{1}{s-1} + O(1).$$

Finally, we have to assure that any of the appearing $L(s, \chi)$ has no zero on the line $1 + i\mathbb{R}$. The hardest part is the value at $s = 1$:

[¶]The reader who is not familiar with characters and Dirichlet L-functions may follow the reasoning with $q = 1$ and replace everywhere $L(s, \chi)$ by the zeta-function; this leads to the classical prime number theorem $\psi(x) \sim x$.

Theorem 10 (Dirichlet's Theorem). *For any character χ, we have $L(1,\chi)\neq 0$.*

In the sequel we give Mertens' proof from 1897; Dirichlet's original proof from 1837 was a detour through the theory of quadratic forms and led him to his analytic class number formula.

Proof. We may assume that χ is not the principal character. Let $s>1$. In view of the Euler product (6) and the orthogonality relation for characters, Theorem 26, it follows that

$$\frac{1}{\varphi(q)}\sum_{\chi \bmod q}\overline{\chi}(a)\log L(s,\chi)=\sum_p \sum_{\substack{k=1\\ p^k\equiv a \bmod q}}^{\infty}\frac{1}{kp^{ks}}\geq 0.$$

In particular, for $a=1$,

$$\prod_{\chi \bmod q} L(s,\chi)\geq 1. \tag{30}$$

Since $L(s,\chi_0)$ has a simple pole at $s=1$ (inherited from $\zeta(s)$, see (29)) and all other $L(s,\chi)$ are regular, it follows from (30) that there is at most one character χ for which $L(1,\chi)=0$. Since

$$L(1,\overline{\chi})=\overline{L(1,\chi)}$$

such a character has to be real, i.e., $\chi=\overline{\chi}$.

Now suppose χ is real. Then we define $f=\chi*1$, resp. $f(n)=\sum_{d|n}\chi(d)$. Obviously, f is multiplicative. We find $f(p^k)=1$ if p divides q; otherwise, if p does not divide q, then

$$f(p^k)=\begin{cases} k+1 & \text{if} \quad \chi(p)=+1,\\ 1 & \text{if} \quad \chi(p)=-1 \quad \text{and} \quad k\equiv 0 \bmod 2,\\ 0 & \text{if} \quad \chi(p)=-1 \quad \text{and} \quad k\equiv 1 \bmod 2.\end{cases}$$

(Actually, this construction defines a Dirchlet series $L_f(s):=\zeta(s)L(s,\chi)$ which shares many features with, and in some instances is equal to a Dedekind zeta-function of a quadratic number field; see (24).) It follows that $f(n)\geq 0$ and $f(m^2)\geq 1$. Therefore,

$$\sum_{n\leq N^2}\frac{f(n)}{n^{\frac12}}\geq \sum_{m\leq N}\frac{f(m^2)}{m}\geq \sum_{m\leq N}\frac{1}{m},$$

which diverges, as $N\to\infty$. On the contrary, partial summation (Theorem 28) implies

$$\begin{aligned}\sum_{n\leq N^2}\frac{f(n)}{n^{\frac12}} &= \sum_{d\leq N}\frac{\chi(d)}{d^{\frac12}}\sum_{b\leq \frac{N^2}{d}}\frac{1}{b^{\frac12}}+\sum_{b\leq N}\frac{1}{b^{\frac12}}\sum_{N<d\leq\frac{N^2}{b}}\frac{\chi(d)}{d^{\frac12}}\\ &= 2NL(1,\chi)+O(1). \end{aligned}\tag{31}$$

The left-hand side diverges to $+\infty$ which implies $L(1,\chi)\neq 0$ and thus proves the theorem. •

Next we shall show that Dirichlet L-functions do not vanish at any other point of the line $1+i\mathbb{R}$. For an arbitrary point $s=1+it$ with $t\neq 0$ we find via the Euler product (6) that

$$-\frac{L'}{L}(\sigma+it,\chi)=\sum_{n=1}\frac{\Lambda(n)\chi(n)}{n^{\sigma}}\exp(-it\log n).$$

Since $17+24\cos\alpha+8\cos(2\alpha)=(3+4\cos\alpha)^2\geq 0$, it follows that

$$-17\frac{L'}{L}(\sigma,\chi_0)-24\mathrm{Re}\,\frac{L'}{L}(\sigma+it,\chi)-8\mathrm{Re}\,\frac{L'}{L}(\sigma+2it,\chi^2)\geq 0. \tag{32}$$

Assuming that $L(1+it,\chi)$ vanishes for $t=t_0\neq 0$ of order m, it would follow that

$$\mathrm{Re}\,\frac{L'}{L}(\sigma+it_0,\chi)=\frac{m}{\sigma-1}+O(1),$$

which leads to

$$\frac{17}{\sigma-1}-\frac{24m}{\sigma-1}+O(1)\geq -17\frac{L'}{L}(\sigma,\chi_0)-24\mathrm{Re}\,\frac{L'}{L}(\sigma+it,\chi)-8\mathrm{Re}\,\frac{L'}{L}(\sigma+2it,\chi^2),$$

contradicting (32) as $\sigma\to 1+$. Thus, the Dirichlet L-function has no zeros on the line $1+i\mathbb{R}$:

$$L(1+it,\chi)\neq 0 \qquad \text{for} \quad t\in\mathbb{R}.$$

Thus, applying Theorem 9, we obtain $\psi(x;a \bmod q)\sim\varphi(q)^{-1}x$ as in the Prime Number Theorem 8. By partial summation, we may further deduce

Theorem 11 (**Prime Number Theorem for arithmetic progressions**). *Let a and q be coprime integers. Then, as $x\to\infty$,*

$$\pi(x;a \bmod q)\sim\frac{1}{\varphi(q)}\frac{x}{\log x}.$$

It is not difficult to show that the non-vanishing of $L(s,\chi)$ on the line $1+i\mathbb{R}$ is equivalent to the prime number theorem in arithmetic progressions (see [55]). We did not use any information about the behaviour of the involved Dirichlet L-functions from inside the critical strip. Therefore, we do not get an error term. In our situation one can easily prove an asymptotic formula with error term. For this purpose we briefly discuss the case of the Riemann zeta-function. For the beginning we may move the path of integration in Perron's formula: for $x\notin\mathbb{Z}$ and $c>1$,

$$\psi(x)=-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,\mathrm{d}s \tag{33}$$

to the left and apply calculus of residues as Hadamard and and de la Vallée-Poussin (independently) in 1896 when they achieved the first proof of the Prime Number Theorem. Since any zeta zero corresponds to a pole of the logarithmic derivative and hence a contribution to the sum of residues, the error term in the Prime Number Theorem depends heavily on the location of the zeta zeros:

$$\pi(x)-\int_2^x\frac{\mathrm{d}u}{\log u}\ll x^{\theta+\epsilon} \qquad \Longleftrightarrow \qquad \zeta(s)\neq 0 \quad \text{for} \quad \mathrm{Re}\,s>\theta\,,$$

where the logarithmic integral is asymptotically equal to $\frac{x}{\log x}$. Similar formula hold for the prime number counting function for arithmetic progressions. For applications one often wants to have a result which is uniform in the modulus; for instance, for bounds of the least prime in an arithemtic progression. For this purpose one can extend Riemann's approach for the zeta-function to Dirichlet L-functions as sketched above, and the theorem of Page-Siegel-Walfisz provides such an asymptotic formula which is uniform in a small region of values q. As a matter of fact, the character analogue is more delicate than for the zeta-function, since one cannot exclude that certain $L(s,\chi)$ have real zeros on the real axis inside the critical strip. These so-called exceptional zeros (or Siegel zeros) are difficult to deal with. For the explicit formulae and more information on this topic we refer to Davenport [8], resp. [55].

Although the above method via a Tauberian theorem is pretty powerful, it does not imply the Prime Number Theorem 8 for the Selberg class in its full generality. The Selberg conjectures refer to the analytic behaviour at the edge of the critical strip. Conrey

& Ghosh [7] proved the non-vanishing on the line $\mathrm{Re}\, s = 1$ subject to the truth of Selberg's Conjecture B.

Theorem 12. *Let $\mathcal{L} \in \mathcal{S}$. If Selberg's Conjecture B is true, then*

$$\mathcal{L}(s) \neq 0 \qquad \text{for} \quad \mathrm{Re}\, s \geq 1.$$

Proof. With regard to the Euler product representation, in the half-plane $\mathrm{Re}\, s \geq 1$ zeros can only occur on the line $\mathrm{Re}\, s = 1$. In view of Theorem 7 it suffices to consider primitive functions $\mathcal{L} \in \mathcal{S}$. In case of $\zeta(s)$ it is known that there are no zeros on $\mathrm{Re}\, s = 1$. Recall Exercise 7: if Selberg's conjecture B is true and if $\mathcal{L} \in \mathcal{S}$ has a pole at $s = 1$ of order m, then the quotient $\mathcal{L}(s)/\zeta(s)^m$ is an entire function. Hence we may assume that $\mathcal{L}(s)$ is entire. Then $\mathcal{L}(s + i\alpha)$ is for any real α a primitive element of $\mathcal{S}$. Selberg's Conjecture B applied to $\mathcal{L}(s + i\alpha)$ and $\zeta(s)$ yields

$$\sum_{p \leq x} \frac{a_{\mathcal{L}}(p)}{p^{1+i\alpha}} \ll 1. \tag{34}$$

Now suppose that $\mathcal{L}(1 + i\alpha) = 0$. Then

$$\mathcal{L}(s) \sim c(s - (1 + i\alpha))^k \qquad \text{as} \quad s = \sigma + i\alpha \to 1 + i\alpha$$

for some complex $c \neq 0$ and some positive integer k. It follows that

$$\log \mathcal{L}(\sigma + i\alpha) \sim k \log(\sigma - 1) \qquad \text{as} \quad \sigma \to 1+. \tag{35}$$

Since

$$\log \mathcal{L}(s) = \sum_p \frac{a_{\mathcal{L}}(p)}{p^s} + O(1)$$

for $\mathrm{Re}\, s > 1$, we get by partial summation

$$\log \mathcal{L}(\sigma + i\alpha) \sim \sum_p \frac{a_{\mathcal{L}}(p)}{p^{\sigma+i\alpha}} = (\sigma - 1) \int_1^\infty \sum_{p \leq x} \frac{a_{\mathcal{L}}(p)}{p^{1+i\alpha}} \frac{\mathrm{d}x}{x^\sigma}.$$

By (34) the right-hand side is bounded as $\sigma \to 1+$, which contradicts (35). The theorem is proved. •

As we have seen above, the non-vanishing of L-functions on the edge of the critical strip is closely related to Prime Number Theorems. As a matter of fact, the statement of Prime Number Theorem 8 is unconditionally equivalent to the non-vanishing of $\mathcal{L}(s)$ on the 1-line. It is not too difficult to verify this statement (by application of a Tauberian theorem) for polynomial Euler products in the Selberg class. In view of Theorem 12 it follows that

Corollary 13. *Assume Selberg's Conjecture B. Then the Prime Number Theorem 8 holds for elements of the Selberg class of the form (28).*

Exercise 8. *Generalizing the above reasoning for Dirichlet L-functions, prove the Prime Ideal Theorem, that is, the Prime Number Theorem in the case of Dedekind zeta-functions. Moreover, prove Corollary 13. (See* [38] *for an appropriate Tauberian theorem.)*

For a Prime Number Theorem with remainder term one may consult Iwaniec & Kowalski [18] (although their reasoning is slightly more restrictive than the Selberg class).

However, Conjecture B might be a rather strong condition if we are interested in a prime number theorem for a single L-function. Recently, Kaczorowski & Perelli obtained a more satisfying condition. For this aim they introduced a weak form of Selberg's Conjecture A:

Normality conjecture. *For all $1 \neq \mathcal{L} \in \mathcal{S}$ there exists a non-negative integer $k_{\mathcal{L}}$ such that*

$$\sum_{p \leq x} \frac{|a_{\mathcal{L}}(p)|^2}{p} = k_{\mathcal{L}} \log\log x + o(\log\log x).$$

Assuming this hypothesis, they proved the claim of Theorem 12, namely the non-vanishing of any $\mathcal{L}(s)$ on the line $\mathrm{Re}\, s = 1$, and that this statement is equivalent to the Prime Number Theorem for the Selberg class. It should be noted that their proof of $\mathcal{L}(1 + i\mathbb{R}) \neq 0$ for a given $\mathcal{L}$ involves the assumption of their normality conjecture for several elements in $\mathcal{S}$. As already mentioned above, it is conjectured that the Selberg class consists only of automorphic L-functions, and for those Jacquet & Shalika [19] obtained an unconditional non-vanishing theorem.

Exercise 9. *Assuming the Selberg Conjectures, prove Dedekind's conjecture: if $\mathbb{K}$ is a number field and $\zeta_{\mathbb{K}}(s)$ is the associated Dedekind zeta-function (see (23)), then*

$$L(s) := \zeta_{\mathbb{K}}(s)/\zeta(s)$$

is an entire function in $\mathcal{S}$ (compare with Example (24)). For this purpose it might be helpful to recall Exercise 7.

The Dedekind conjecture is known to be true for normal extensions (as a consequence of the Aramata-Brauer theorem).

The observation of the latter exercise is due to Conrey & Ghosh [7]. Another related important application to arithmetic deals with Artin L-functions. Assuming Selberg's conjecture B and applying deep results on Artin L-functions from Algebraic Number Theory, M.R. Murty showed that Artin L-functions are entire elements of the Selberg class which leads to a conditional proof of

Artin's Conjecture. *Let $\mathbb{L}/\mathbb{K}$ be a finite Galois extension with Galois group G. For any irreducible character $\chi \neq 1$ of G the Artin L-function $L(s, \chi, \mathbb{L}/\mathbb{K})$ extends to an analytic function on $\mathbb{C}$ except a possible pole at $s = 1$.*

Appendix D contains an exemplary introduction to Artin L-functions. For Murty's work, which is beyond the scope of this course, we refer to [38].

---

4. Classification of Degree One Elements

We start with a classical theorem for the Riemann zeta-function. In 1921, Hamburger [14] proved that $\zeta(s)$ is characterized by its functional equation (see Theorem 1):

Theorem 14 (Hamburger's Theorem). *Let $G(s)$ be an entire function of finite order, $P(s)$ a polynomial, and suppose that*

$$f(s) := \frac{G(s)}{P(s)} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

the series being absolutely convergent for $\sigma > 1$. Assume that

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) f(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) g(1-s), \tag{36}$$

where

$$g(1-s)=\sum_{n=1}^\infty \frac{b(n)}{n^{1-s}},$$

the series being absolutely convergent for $\operatorname{Re} s<-\alpha$ *for some positive constant* α*. Then* $f(s)=c\zeta(s)$*, where* c *is a constant.*

We shall give here a simplified proof due to Siegel [48] (see also [57]).

Proof. By Perron's formula (see Theorem 31 in Appendix B), we find, for $x>0$,

$$\begin{aligned}\phi(x) &:= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} f(s)\Gamma\left(\frac{s}{2}\right)(\pi x)^{-\frac{s}{2}}\,\mathrm{d}s\\ &= \sum_{n=1}^\infty a(n)\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma\left(\frac{s}{2}\right)(\pi n^2 x)^{-\frac{s}{2}}\,\mathrm{d}s = 2\sum_{n=1}^\infty a(n)\exp(-\pi n^2 x). \qquad (37)\end{aligned}$$

In view of (36) we also have

$$\phi(x)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} g(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{\frac{s-1}{2}}x^{-\frac{s}{2}}\,\mathrm{d}s.$$

Next we move the line of integration to $\operatorname{Re} s=-1-\alpha$. Obviously, $f(s)$ is bounded on the vertical line $\operatorname{Re} s=2$ and $g(1-s)$ is bounded on $\operatorname{Re} s=-1-\alpha$. By Stirling's formula (Theorem 32 from Appendix B),

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \ll |t|^{\operatorname{Re} s-\frac{1}{2}} \qquad \text{for} \quad s=\sigma+it$$

as $|t|\to\infty$. Thus, $g(1-s)\ll |t|^{\frac{3}{2}}$ on $\operatorname{Re} s=2$ as $|t|\to\infty$, and, justified by the Phragmén-Lindelöf principle (see Theorem 33 in Appendix B), we can apply Cauchy's theorem. It follows that

$$\phi(x)=\frac{1}{2\pi i}\int_{-1-\alpha-i\infty}^{-1-\alpha+i\infty} g(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{\frac{s-1}{2}}x^{-\frac{s}{2}}\,\mathrm{d}s+\sum_{j=1}^k R_j, \qquad (38)$$

where $R_1,\ldots,R_k$ are the residues at the poles, say $s_1,\ldots,s_k$. It is easily seen that the sum of residues is of the form

$$\sum_{j=1}^k R_j=\sum_{j=1}^k x^{-\frac{s_j}{2}}P_j(\log x)=:R(x),$$

where the $P_j(\log x)$ are polynomials in $\log x$. We rewrite (38) and find as above

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{x}}\sum_{n=1}^\infty b(n)\frac{1}{2\pi i}\int_{-1-\alpha-i\infty}^{-1-\alpha+i\infty}\Gamma\left(\frac{1-s}{2}\right)\left(\frac{\pi n^2}{x}\right)^{\frac{s-1}{2}}\mathrm{d}s+R(x)\\ &= \frac{2}{\sqrt{x}}\sum_{n=1}^\infty b(n)\exp(-\pi n^2/x)+R(x).\end{aligned}$$

Comparing with (37), we arrive at

$$\sum_{n=1}^\infty a(n)\exp(-\pi n^2 x)-\frac{1}{2}R(x)=\frac{1}{\sqrt{x}}\sum_{n=1}^\infty b(n)\exp(-\pi n^2/x).$$

Multiplying with $\exp(-\pi t^2 x)$ with $t>0$ and integrating over $(0,\infty)$ with respect to x, we get

$$\frac{t}{\pi}\sum_{n=1}^\infty\frac{a(n)}{(t^2+n^2)}-\frac{t}{2}\int_0^\infty P(x)\exp(-\pi t^2 x)\,\mathrm{d}x=\sum_{n=1}^\infty b(n)\exp(-2\pi nt).$$

The integral can be evaluated as a finite sum of terms of the form

$$Q(t;a,b) := \int_0^\infty x^a (\log x)^b \exp(-\pi t^2 x)\,\mathrm{d}x,$$

where the b's are integers and $\mathrm{Re}\, a > -1$; thus, $Q(t;a,b)$ is a sum of terms of the form $t^\alpha (\log t)^\beta$. Hence,

$$\sum_{n=1}^\infty a(n) \left\{ \frac{1}{t-in} - \frac{1}{t+in} \right\} - t\frac{\pi}{2} Q(t;a,b) = \pi \sum_{n=1}^\infty b(n) \exp(-2\pi n t).$$

The left-hand side is a meromorphic function in t with poles at $t = \pm in$ for $n \in \mathbb{N}$. The right-hand side is periodic with period i and, by analytic continuation, the function on the left-hand side is also periodic. Hence, the residues at in and $i(n+1)$ are equal. Thus, $a(n) = a(n+1)$ for all $n \in \mathbb{N}$ and the theorem is proved. •

Exercise 10. *Compare the proof with Riemann's proof for Theorem 1. Try to extend Hamburger's Theorem to the case of Dirichlet L-functions.*

Hamburger's Theorem 14 has been the motivation for Hecke's Converse Theorem 4. One can find quite a few similar converse theorems in the literature (starting from Dirichlet L-functions to higher degree L-functions). We are interested to classify all degree one elements of the Selberg class. Since the degree is defined via the data of the functional equation, we are in a similar position as Hamburger was, however, in our situation we have the additional difficulty that there are infinitely many functional equations associated with degree one (as follows immediately from Theorems 1 and 3).

Theorem 15 (Structure Theorem – Degree One). *If $\mathcal{L}(s) = \sum_{n=1}^\infty a(n) n^{-s}$ has degree one in $\mathcal{S}$, then there exists a positive integer q and a real number θ such that $a(n)n^{-i\theta}$ is q-periodic. Moreover, $\mathcal{L}(s)$ equals either $\zeta(s)$ or a shifted Dirichlet L-function $L(s+i\theta,\chi)$ with a primitve character $\chi \bmod q$. For short:*

$$\mathcal{L} \in \mathcal{S} \quad \textit{with} \quad \mathrm{d}_{\mathcal{L}} = 1 \qquad \Longrightarrow \qquad \mathcal{L}(s) = \zeta(s) \quad \textit{or} \quad = L(s+i\theta,\chi).$$

There are different proofs of this result; we follow Soundararajan [49].

Sketch of proof. For $c > \frac{1}{2}$, we consider the integral

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{L}(\tfrac{1}{2} + it + z) X^z \Gamma(z)\,\mathrm{d}z.$$

If we expand $\mathcal{L}$ into its Dirichlet series and integrate term by term, the integral equals $\sum_{n=1}^\infty a(n) n^{-\frac{1}{2}-it} \exp(-\frac{n}{X})$. Moving the line of integration to $z = -1+\epsilon+i\mathbb{R}$, the pole at $z=0$ yields the residue $\mathcal{L}(\frac{1}{2}+it)$ and the possible pole at $z = \frac{1}{2} - it$ leaves a residue bounded by $X^{\frac{1}{2}+\epsilon}\exp(-|t|)$ by Stirling's formula. Using the latter formula, we may estimate the integral on the vertical $z = -1+\epsilon+i\mathbb{R}$ by $X^{-1+\epsilon}(1+|t|)^{1+\epsilon}$. Hence, we have shown

$$\mathcal{L}(\tfrac{1}{2}+it) = \sum_{n=1}^\infty \frac{a(n)}{n^{\frac{1}{2}+it}} \exp(-\tfrac{n}{X}) + O((1+|t|)^{1+\epsilon} X^{-1+\epsilon} + X^{\frac{1}{2}+\epsilon}\exp(-|t|)), \tag{39}$$

where $X \geq 1$ is arbitrary.

Now define

$$\ell(\alpha,T) = \frac{1}{\sqrt{\alpha}} \int_{\alpha T}^{2\alpha T} \mathcal{L}(\tfrac{1}{2}+it) \exp\left(it \log \frac{t}{2\pi e \alpha} - i\frac{\pi}{4} \right) \mathrm{d}t,$$

where α is positive and $T \geq 1$. Next we use the functional equation for $\mathcal{L}$ in the form

$$\Lambda(s) = Q^s G(s)\mathcal{L}(s) \qquad \text{with} \quad G(s) = \prod_{j=1}^{f} \Gamma(\lambda_j s + \mu_j)$$

such that $\Lambda(s) = \omega\overline{\Lambda(1-\overline{s})}$ (where all involved numbers satisfy axiom (iii)). It follows that

$$\ell(\alpha,T) = \frac{\omega}{\sqrt{\alpha}} \int_{\alpha T}^{2\alpha T} \overline{\mathcal{L}(\tfrac{1}{2}+it)} Q^{-2it} \frac{\overline{G(\frac{1}{2}+it)}}{G(\frac{1}{2}+it)} \exp\left(it \log \frac{t}{2\pi e \alpha} - i\frac{\pi}{4}\right) \mathrm{d}t.$$

Again by Stirling's formula (Theorem 32 in Appendix B) we may rewrite the last but one formula as

$$\frac{\omega \exp(iB)}{\sqrt{\alpha}} \int_{\alpha T}^{2\alpha T} \overline{\mathcal{L}(\tfrac{1}{2}+it)} (\pi C Q^2 \alpha)^{-it} t^{i\theta} (1 + O(\tfrac{1}{T}))\, \mathrm{d}t$$

with certain real numbers θ, B and $C > 0$. Now, using (39) with $X = T^{\frac{4}{3}}$, we find

$$\begin{aligned} \ell(\alpha,T) &= \frac{\omega \exp(iB)}{\sqrt{\alpha}} \int_{\alpha T}^{2\alpha T} \sum_{m=1}^{\infty} \frac{\overline{a(m)}}{m^{\frac{1}{2}}} \exp\left(-\frac{m}{X}\right) \left(\frac{m}{\pi C Q^2 \alpha}\right)^{it} t^{i\theta} (1 + O(\tfrac{1}{T}))\, \mathrm{d}t + \\ &\quad + O(T^{\frac{2}{3}+\epsilon}) \\ &= \frac{\omega \exp(iB)}{\sqrt{\alpha}} \sum_{m=1}^{\infty} \frac{\overline{a(m)}}{m^{\frac{1}{2}}} \exp\left(-\frac{m}{X}\right) \int_{\alpha T}^{2\alpha T} \left(\frac{m}{\pi C Q^2 \alpha}\right)^{it} t^{i\theta}\, \mathrm{d}t + O(T^{\frac{2}{3}+\epsilon}). \qquad (40) \end{aligned}$$

Integration by parts shows, for $x \neq 1$,

$$\int_{\alpha T}^{2\alpha T} x^{it} t^{i\theta}\, \mathrm{d}t \ll \frac{1}{|\log x|},$$

while

$$\int_{\alpha T}^{2\alpha T} t^{i\theta}\, \mathrm{d}t = \frac{(2\alpha T)^{1+i\theta} - (\alpha T)^{1+i\theta}}{1+i\theta}.$$

Now define

$$\ell(\alpha) = \lim_{T\to\infty} T^{-1-i\theta} \ell(\alpha,T)$$

Using our computations of the oscillatory integral in (40), we are led to

$$\begin{aligned} \ell(\alpha) &= \omega \exp(iB) \delta(\pi C Q^2 \alpha) \frac{\overline{a(\pi C Q^2 \alpha)}}{(\pi C)^{\frac{1}{2}} Q} \frac{2^{1+i\theta}-1}{1+i\theta} + \\ &\quad + O\left(\lim_{T\to\infty} T^{-1+\epsilon} \sum_{m=1}^{\infty} |a(m)| m^{\frac{1}{2}} \exp\left(-\frac{m}{X}\right)\right), \end{aligned}$$

where $\delta(x) = 1$ if $x \in \mathbb{N}$ and equals zero otherwise. Since the error term vanishes, we have

$$\ell(\alpha) = \omega \exp(iB) \delta(\pi C Q^2 \alpha) \frac{\overline{a(\pi C Q^2 \alpha)}}{(\pi C)^{\frac{1}{2}} Q} \frac{2^{1+i\theta}-1}{1+i\theta}. \qquad (41)$$

This proves in particular the existence of the limit.

Next we evaluate $\ell(\alpha,T)$ in another way. Using again (40) with $X = T^{\frac{4}{3}}$, we find

$$\ell(\alpha,T) = \frac{1}{\sqrt{\alpha}} \sum_{n=1}^{\infty} \frac{a(n)}{n^{\frac{1}{2}}} \exp\left(-\frac{n}{X}\right) \int_{\alpha T}^{2\alpha T} \exp\left(it \log \frac{t}{2\pi e \alpha} - i\frac{\pi}{4}\right) \mathrm{d}t + O(T^{\frac{2}{3}+\epsilon}). \qquad (42)$$

We may evaluate the oscillatory integral above by standard methods for exponential integrals, namely Theorem 29 in Appendix B. First, the integral is $\ll 1$ for $2\pi n > 3T$ which yields the contribution $O(T^{\frac{2}{3}+\epsilon})$ for such n. In the range $T < 2\pi n \leq 2T$, the integral equals

$$2\pi\sqrt{n\alpha}\exp(-2\pi in\alpha) + O(T^{\frac{2}{5}} + \min\left\{T^{\frac{1}{2}}, \left|\log\frac{T}{2\pi n}\right|^{-1}\right\} + \min\left\{T^{\frac{1}{2}}, \left|\log\frac{T}{\pi n}\right|^{-1}\right\}.$$

For $2\pi n < T$ or between $2T$ and $3T$ the integral is bounded by the error terms above. Hence,

$$\ell(\alpha, T) = 2\pi \sum_{T\leq 2\pi n\leq 2T} a(n)\exp(-2\pi in\alpha) + O(T^{\frac{9}{10}+\epsilon}).$$

We deduce that

$$\ell(\alpha) = \ell(\alpha+1).$$

It follows from (41) that

$$\delta(\pi CQ^2\alpha)\overline{a(\pi CQ^2\alpha)}\alpha^{i\theta} = \delta(\pi CQ^2(\alpha+1))\overline{a(\pi CQ^2(\alpha+1))}(\alpha+1)^{i\theta}.$$

Recall the definition of δ. We immediately deduce that $\pi CQ^2 = q$ has to be a positive integer and that $\overline{a(n)}n^{i\theta}$ is q-periodic which proves the first part of the theorem.

So far we have not used the arithmetic axioms. In view of the Euler product (iv) the coefficients $a(n)$ are multiplicative. Together with the periodicity it follows that

$$a(n)n^{-i\theta} = \chi(n)$$

for a Dirichlet character χ mod q (see Theorem 27 in Appendix A). If $q = 1$, we are done and $\mathcal{L}(s)$ is the zeta-function. Otherwise, if χ' mod q' denotes the primitive character that induces χ, the quotient $\mathcal{L}(s)/L(s+i\theta,\chi')$ is a finite Euler product (over those primes which divide q). It follows from the Euler product axiom that the logarithm of this Euler product converges absolutely for $\mathrm{Re}\, s > \delta$ for some $\delta < \frac{1}{2}$. Setting $a = \frac{1}{2}(1-\chi'(-1))$, we find that

$$f(s) := \frac{Q^s G(s)\mathcal{L}(s)}{(\frac{q'}{\pi})^{\frac{s}{2}}\Gamma(\frac{s+i\theta+a}{2})L(s+i\theta,\chi')}$$

is analytic in the half-plane $\mathrm{Re}\, s > \delta$; obviously, the same is true for $\overline{f(\overline{s})}$ and both functions are zero-free in this region. It follows from the functional equations for $\mathcal{L}(s)$ and $L(s+i\theta,\chi')$ that both, $f(s)$ and $\overline{f(\overline{s})}$ are analytic and zero-free in $\mathrm{Re}\, s < 1-\delta$. Since $\delta < \frac{1}{2}$, it thus follows that both, $f(s)$ and $\overline{f(\overline{s})}$ are entire functions of order one with no zeros. By Hadamard's Product Theorem 34 (see Appendix B), we find

$$f(s) = c_1\exp(c_2 s)$$

for some constants c_1, c_2. The functional equation linking $f(s)$ and $\overline{f(1-\overline{s})}$ implies that $c_2 = 0$ and f is constant. Examination of $f(\frac{1}{2}+it)$ for large t shows that $\mathcal{L}(s) = L(s+i\theta,\chi')$. The theorem is proved. •

Reviewing the proof we observe some ideas from Siegel's proof of Hamburger's theorem. Conrey & Ghosh [7] gave a proof for the subclass of all elements which have necessarily weights $\lambda_j = \frac{1}{2}$ in the Gamma-factor of their functional equation, that is even more close to Siegel's reasoning.

Exercise 11. *Fill in all details left in the sketch of proof of Theorem 15. In particular, prove the estimates for the oscillatory integral (42). Why does this approach not work to classify degree two elements?*

Kaczorowski & Perelli [22] gave a characterization of th degree one elements of the extended selberg class $\mathcal{S}^\sharp$ which is more delicate. They proved that any element in $\mathcal{L} \in \mathcal{S}^\sharp$ of degree one has a unique representation of the form

$$\mathcal{L}(s) = \sum_{\chi \bmod q} P_\chi(s + i\theta) L(s + i\theta, \chi^*),$$

where θ is a real number and the summation is over all characters $\chi \bmod q$ and χ^* denotes the unique primitive or principal character which induces $\chi \bmod q$, and P_χ is a Dirichlet polynomial of degree zero in $\mathcal{S}^\sharp$. Their approach relies on the theory of hypergeometric functions.

---

5. Powerful Tools: Linear and Non-linear Twists

For investigating the structure of the Selberg class beyond Theorem 6 Kaczorowski & Perelli introduced linear and non-linear twists. Given an element $\mathcal{L}(s) = \sum_{n=1}^\infty a(n) n^{-s}$, a linear twist is obtained by twisting with an additive character:

$$\mathcal{L}(s, \alpha) = \sum_{n=1}^\infty \frac{a(n)}{n^s} \exp(-2\pi i \alpha n) \qquad \text{for some } \alpha \in \mathbb{R}.$$

For the Riemann zeta-function this leads to the so-called Lerch zeta-function which has been studied for more than a century sstarting with Lerch, Lipschitz, and Hurwitz; see the monograph [11] of Garunkštis & Laurinčikas for details. Recall that twisting the zeta-function with a multiplicative Dirichlet character leads to Dirichlet L-functions, and in §3 we have seen that their value-distribution contains information about the prime number distribution which with $\zeta(s)$ alone would not have been accessible.

Exercise 12. *Prove the formula*

$$\chi(n)\tau(\overline{\chi}) = \sum_{a \bmod q} \overline{\chi}(a) \exp\left(2\pi i \frac{an}{q}\right),$$

where $\tau(\chi)$ denotes the Gaussian sum associated with χ (see [8, 55] *for help). Use this and related formulae to express Dirichlet L-functions in terms of Lerch zeta-functions and vice versa. Deduce information about Lerch zeta-function from your knowledge of Dirichlet L-functions and compare with* [11].

Kaczorowski & Perelli introduced more advanced twists by replacing the additive character αn with a linear combination of different α's with weights (see (43) below). For the first, the prototype suffices: given $\mathcal{L} \in \mathcal{S}$ of degree d, the standard non-linear twist is given by

$$\mathcal{L}(s, \alpha) = \sum_{n=1}^\infty \frac{a(n)}{n^s} \exp(-2\pi i \alpha n^{1/\mathrm{d}}) \qquad (\mathrm{Re}\, s > 1),$$

where $\alpha > 0$. It is remarkable that those twists carry much information about the Selberg class theory although they are not elements of $\mathcal{S}$. Recently, Kaczorowski & Perelli [25] succeeded in extending Theorem 6 to

Theorem 16 (**Structure Theorem – Larger Degrees**). *If $\mathcal{L} \in \mathcal{S}$ has degree $1 < \mathrm{d}_\mathcal{L} \leq 2$, then $\mathrm{d}_\mathcal{L} = 2$.*

Thus, the Degree Conjecture is true for $[0,2]$. We can give only a sketch of the lengthy and technical proof. Our poor approach cannot replace a detailed study of their deep work. We start with the most simple twist. Given $\mathcal{L} \in \mathcal{S}$ of degree d, the standard non-linear twist is given by

$$\mathcal{L}(s,\alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \exp(-2\pi i \alpha n^{1/\mathrm{d}}) \qquad (\mathrm{Re}\, s > 1),$$

where $\alpha > 0$. We further define the conductor of $\mathcal{L}$ in terms of the data from the functional equation by

$$q_{\mathcal{L}} = (2\pi)^{\mathrm{d}} Q^2 \prod_{j=1}^{f} \lambda_j^{2\lambda_j};$$

in what follows we shall write q in place of $q_{\mathcal{L}}$ for short. The conductor is besides the degree another important invariant. We briefly note $\mathrm{d} = q = 1$ for the zeta-function, and $\mathrm{d} = 1$ and q equals the conductor, resp. the modulus for Dirichlet L-functions $L(s,\chi)$. In the case of Dedekind zeta-functions to a number field $\mathbb{K}$ we have $\mathrm{d} = [\mathbb{K} : \mathbb{Q}]$ and $q = |\Delta_{\mathbb{K}}|$ equals the absolute value of the discriminant of $\mathbb{K}$.

Exercise 13. *What are degree and conductor of L-functions to newforms? What are degree and conductor of Artin L-functions, assuming that they belong to $\mathcal{S}$ (see Appendix D)?*

We further define the quantity

$$n_\alpha := q\,\mathrm{d}^{-\mathrm{d}} \alpha^{\mathrm{d}} \qquad \text{and} \qquad a(n_\alpha) = \begin{cases} a(n) & \text{if } n_\alpha = n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

We call

$$\mathsf{Spec}(\mathcal{L}) := \{\alpha > 0 \,:\, a(n_\alpha) \neq 0\}$$

the spectrum of $\mathcal{L}$. Obviously, the spectrum is an unbounded subset of the positive real numbers if $\mathcal{L}$ has positive degree, i.e., it is not identical 1 what we shall assume for the sequel. Moreover, we shall from now on suppose that $\theta := \sum_{j=1}^{f} \mu_j = 0$, where the μ_j are data from the functional equation; this is no restriction of generality but allows a more convenient presentation. The following theorem due to Kaczorowski & Perelli [24] provides the main analytic properties:

Theorem 17 (Standard Non-Linear Twists). *Let $\mathcal{L} \in \mathcal{S}$ be of degree $\mathrm{d} \geq 1$ (and $\theta = 0$) and $\alpha > 0$. Then $\mathcal{L}(s,\alpha)$ extends to a meromorphic function on $\mathbb{C}$. If $\alpha \notin \mathsf{Spec}(\mathcal{L})$, then $\mathcal{L}(s,\alpha)$ is entire; otherwise, If $\alpha \in \mathsf{Spec}(\mathcal{L})$, then $\mathcal{L}(s,\alpha)$ has at most a simple poles at*

$$s_k := \frac{\mathrm{d}+1}{2\,\mathrm{d}} - \frac{k}{\mathrm{d}} \qquad \textit{for} \quad k = 0,1,2,\ldots$$

with residue equal to $c_{\mathcal{L}}\overline{a(n_\alpha)}$ at $s = s_0$, where $c_{\mathcal{L}}$ is a non-zero constant.

The point s_0 is always a simple pole if $\alpha \notin \mathsf{Spec}(\mathcal{L})$; the other points s_k need not be poles (as follows from the example $\mathcal{L} = \zeta$).

Sketch of Proof. Let X be a positive large parameter. Starting with Mellin's transform (as we already did in the proofs of Theorems 1, 4 and 15) and shifting the line of integration to the left, then applying the functional equation and making use of the Dirichlet series expansion for $\overline{\mathcal{L}(1-\overline{s})}$, one finds

$$\sum_{n=1}^{\infty}\frac{a(n)}{n^s}\exp(-2\pi i\alpha n^{1/d})\exp(-Xn^{1/\mathrm{d}})$$
$$= R_X(s,\alpha)+\omega Q^{1-2s}\sum_{n=1}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}H\left(\frac{n^{1/\mathrm{d}}}{Q^{2/\mathrm{d}}(2\pi i\alpha+X^{-1})},s,\lambda,\mu\right),$$

where $R_X(s,\alpha)$ is a sum of residues and the incomplete hypergeometric function

$$H(z,s,\lambda,\mu):=\frac{1}{2\pi i}\int_{-c-\frac{1}{2}-i\infty}^{-c-\frac{1}{2}+i\infty}\prod_{j=1}^{f}\frac{\Gamma(\lambda_j(1-s)+\overline{\mu_j}-\frac{\lambda_j}{\mathrm{d}}\omega)}{\Gamma(\lambda_j s+\mu_j+\frac{\lambda_j}{\mathrm{d}}\omega)}\Gamma(\omega)z^{\omega}\,\mathrm{d}\omega;$$

here $z=x+iy$ is complex and c is a positive large integer, and λ,μ stand for the data $\lambda_1,\ldots,\lambda_f,\mu_1,\ldots,\mu_f$ from the functional equation. One wishes to let $X\to\infty$ in order to obtain analytic continuation for $\mathcal{L}(s,\alpha)$. However, the integral converges absolutely only for $x>0$. To solve this problem a subtle study of $H(z,s,\lambda,\mu)$ as a function of the two variables z and s is required and finally allows to perform the limit $X\to\infty$. Moreover, it turns out that $H(-iy,s,\lambda,\mu)$ is analytic in s if

$$y\neq\frac{1}{\mathrm{d}}\prod_{j=1}^{f}\lambda_j^{2\lambda_j},$$

while otherwise it has at most simple poles at $s=s_k$. •

Exercise 14. *Fill all gaps in the sketch of proof; good reading for this aim are* [24, 20].

Theorem 17 already allows to show how linear (and later non-linear) twists may be used to show that certain real numbers cannot appear as degree of an L-function in the Sleberg class. For this purpose we give now a short proof of Theorem 6: Assume $\mathcal{L}\in\mathcal{S}$ has degree $\mathrm{d}\in(0,1)$. If $\alpha\in\mathsf{Spec}(\mathcal{L})$, then $\mathcal{L}(s,\alpha)$ has a pole at $s_0=\frac{\mathrm{d}+1}{2\,\mathrm{d}}$ which is >1 for $\mathrm{d}<1$, giving the desired contradiction.

If $\mathrm{d}=1$, then $\mathcal{L}(s,\alpha)$ is a linear twist of $\mathcal{L}$ and hence periodic with respect to α. This allows a very simple proof of Hamburger's Theorem 14 as follows: if additionally $q=1$, then $n_\alpha=\alpha$ and, choosing α as a positive integer m, say, it follows that the residue of $\mathcal{L}(s,m)$ at s_0 equals

$$c_{\mathcal{L}}\overline{a(m)}\qquad\text{for}\quad m=1,2,\ldots\,.$$

In view of the α-periodicity it follows that $a(m)$ is constant, hence $\mathcal{L}(s)=c\zeta(s)$.

This simple reasoning to obtain a rather deep result gives hope. We continue with another application:

Theorem 18 (**Ω-Theorem for Coefficient Sums**)**.** *Let $\mathcal{L}\in\mathcal{S}$ be of degree $\mathrm{d}\geq 1$ (and $\theta=0$) and $\alpha>0$. Then, for any polynomial P,*[‖]

$$\sum_{n\leq x}a(n)=xP(\log x)+\Omega(x^{\frac{\mathrm{d}-1}{2\,\mathrm{d}}}),$$

where $a(n)$ is, as usual, the n-th coefficient of the Dirichlet series for $\mathcal{L}(s)$.

‖We write $f(x)=\Omega(g(x))$ with a positive function $g(x)$ if $f(x)=o(g(x))$ is not true or, alternatively, $\liminf_{x\to\infty}|f(x)|/g(x)>0$.

Sketch of Proof. For the sake of simplicity we may suppose that $\mathcal{L}(s)$ is an entire function. Moreover, we may assume that P vanishes identically (otherwise the statement is trivial). Now suppose that

$$\sum_{n\le x} a(n) = o\left(x^{\frac{\mathrm{d}-1}{2\,\mathrm{d}}}\right).$$

It follows from partial summation that

$$\sum_{n>y} \frac{a(n)}{n^s} = o(y^{\frac{\mathrm{d}-1}{2\,\mathrm{d}}-\mathrm{Re}\, s}) \qquad \text{for} \quad \mathrm{Re}\, s > \frac{\mathrm{d}+1}{2\,\mathrm{d}}.$$

For $\alpha \in \mathsf{Spec}(\mathcal{L})$ we have

$$\begin{aligned} &\mathcal{L}(s,\alpha) - \exp(2\pi i\alpha)\mathcal{L}(s) \\ = \quad & -\frac{2\pi i\alpha}{\mathrm{d}} \int_1^\infty \sum_{n>y} \frac{a(n)}{n^s} y^{1/\,\mathrm{d}-1} \exp(-2\pi i\alpha y^{1/\,\mathrm{d}})\,\mathrm{d}y = o\left(\left(\mathrm{Re}\, s - \frac{\mathrm{d}-1}{2\,\mathrm{d}}\right)^{-1}\right) \end{aligned}$$

as $\mathrm{Re}\, s \to \frac{\mathrm{d}+1}{2\,\mathrm{d}}+$, contradicting the pole at $s = s_0$ by Theorem 17. •

Exercise 15. *Fill the gaps of the latter sketch of proof. How to argue if $\mathcal{L}$ is not entire? Further deduce that the abscissa of convergence of the $\mathcal{L}$ defining Dirichlet series is $\ge \frac{\mathrm{d}-1}{2\,\mathrm{d}}$.*

Here is another type of converse theorem:

Theorem 19. *Assume that $\mathcal{L} \in \mathcal{S}$ is not identically 1. If its Dirichlet series*

$$\sum_{n=1}^\infty \frac{a(n)}{n^s}$$

converges for $\mathrm{Re}\, s > \frac15 - \epsilon$ with some positive ϵ, then $\mathcal{L}(s)$ has degree one and equals a shifted Dirichlet L-function $L(s+i\theta,\chi)$ with real θ and a primitive character χ.

Proof. In Exercise 15 we have seen that the abscissa of convergence of the $\mathcal{L}$ defining Dirichlet series is $\ge \frac{\mathrm{d}_\mathcal{L}-1}{2\,\mathrm{d}_\mathcal{L}}$. In view of the assumption we find $3\,\mathrm{d}_\mathcal{L} \le 5 + 10\,\mathrm{d}_\mathcal{L}\epsilon$, hence $\mathrm{d}_\mathcal{L} = 1$ and the statement follows from Theorem 15. •

The latter theorem is due to Kaczorowski & Perelli [24]. There is also an analogue for the Riemann zeta-function:

Exercise 16. *Assume that $\mathcal{L} \in \mathcal{S}$ is not identically 1. If $a(n)$ denotes the coefficient of the Dirichlet series expansion of $\mathcal{L}$ and the series*

$$\sum_{n=1}^\infty \frac{a(n)-1}{n^s}$$

converges for $\mathrm{Re}\, s > \frac15 - \epsilon$ with some positive ϵ, then $\mathcal{L}(s) = \zeta(s)$.

Now we proceed in direction of Theorem 16. We assume that $\mathcal{L} \in \mathcal{S}$ has degree $\mathrm{d} \in (1,2)$. Our aim is to arrive at a contradiction. The idea is to find a twist that has a pole at the wrong place.

In the sequel we consider the linear twist

$$\mathcal{L}(s,\alpha) = \sum_{n=1}^\infty \frac{a(n)}{n^s} \exp(-2\pi i\alpha n),$$

so we drop the exponent at n in the exponential from the previous notation. Moreover, with the data from above we put

$$\kappa = \frac{1}{\mathrm{d}-1},\ A = (\,\mathrm{d}-1)q^{-\kappa},\ s^* = \kappa(s + \tfrac12\,\mathrm{d} - 1).$$

However, to go further more complicated twists are needed. For real numbers α_j and $\kappa_0 > \kappa_1 > \ldots > \kappa_N > 0$ let

$$\mathcal{L}(s,f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \exp\left(-2\pi i f(n,\alpha,\kappa)\right) \qquad \text{with} \quad f(\xi,\alpha) = \sum_{j=0}^{N} \alpha_j \xi^{\kappa_j}; \tag{43}$$

here the set of κ_j's is fixed whereas α_j are considered as variable. The analytic properties of $\mathcal{L}(s,f)$ depend heavily on the leading exponent κ_0. In fact, $\mathcal{L}(s,f)$ is

- entire if $0 < \mathrm{d}\kappa_0 < 1$,
- meromorphic over $\mathbb{C}$ with control on the poles if $\mathrm{d}\kappa_0 = 1$,
- satisfies a transformation formula if $\mathrm{d}\kappa_0 > 1$.

This can be shown by means of the theory of hypergeometric functions. By a similar reasoning as for Theorem 17, one can show

$$\mathcal{L}(s,\alpha) = \exp(as+b) \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n^{s^*}} \exp\left(2\pi i A \left(\frac{n}{\alpha}\right)^{\kappa}\right) + G(s,\alpha) \tag{44}$$

with certain constants a, b, and where $G(s,\alpha)$ is analytic for $\mathrm{Re}\, s^* > 1-\kappa$. This yields a first glimpse of the general transformation formula of Kaczorowski & Perelli which is to complicated to be given here. Representation (44) transforms the linear twist $\mathcal{L}(s,\alpha)$ into a non-linear twist plus an analytic function. Since the non-linear twist converges in a larger half-plane, this yields an analytic continuation for $\mathcal{L}(s,\alpha)$. It follows from $\mathrm{d} \in (1,2)$ that $\mathrm{Re}\, s^* > 1$ for $\mathrm{Re}\, s > \frac{\mathrm{d}}{2}$, hence the right-hand side of (44) is analytic for $\mathrm{Re}\, s > \frac{\mathrm{d}}{2}$; in particular, $\mathcal{L}(s) = \mathcal{L}(s,1)$ is regular at $s=1$ which shows that any $\mathcal{L}$ with degree $\mathrm{d} \in (1,2)$ must be entire. One can exploit the above transformation formular further. Using Fourier Analysis and Rankin-Selberg convolution besides (44), Kaczorowski & Perelli [23] succeeded to prove that there are no elements of degree $\mathrm{d} \in (1, \frac{5}{3})$.

For the full proof of Theorem 16 Kaczorowski & Perelli [25] introduced a recursive process in which they apply certain operators T and S acting on f in $\mathcal{L}(s,f)$ in order to generate appropriate twists with a pole in a wrong place. The operator S is the shift operator given by

$$f(\xi,\alpha) \mapsto S\, f(\xi,\alpha) = f(\xi,\alpha) + \xi,$$

which acts trivially on the twist: $\mathcal{L}(s, S\, f) = \mathcal{L}(s,f)$. The operator T is self-reciprocal and defined by a complicated algebraic manipulation of the α_j's. Starting with $f_0(\xi,\alpha) := \alpha \xi^{1/\mathrm{d}}$ they construct explicitly expressions as

$$S^{-1} T S^m T S\, f_0(\xi,\alpha) = c_2(m) \xi^{\,\mathrm{d}-1} + c_3(\alpha,m) \xi^{1/\mathrm{d}} + \ldots$$

with certain coefficients c_j. Suitable choices of $\alpha \in \mathsf{Spec}(\mathcal{L})$ lead via the general tarnsformation formula to impossible poles, and hence to the desired contradiction. The construction is too complicated to be reproduced here.

We conclude with a brief look into the future: if we put (formally) $\mathrm{d} = 2$ in (44), some kind of modularity in α appears which matches the expectation that the set of degree elements in the Selberg class consists of analytic and non-analytic modular forms. If we consider the larger extended Selberg class $\mathcal{S}^\sharp$, it is completely open what kind of elements to expect. We may ask which functions lie in $\mathcal{S}^\sharp \setminus \mathcal{S}$. The descriptions of these classes are complete for degree $\mathrm{d} < 2$ thanks to the method of Kaczorowski & Perelli which actually applies to the extended Selberg class. For degree two Kaczorowski et al. [21] gave examples with Dirichlet series associated with cusp forms of certain Hecke groups. Note that for a positive real number λ, the Hecke group $\mathsf{G}(\lambda)$ is defined as the subgroup of $\mathsf{PSL}_2(\mathbb{R})$ generated by the matrices

$$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Kaczorowski et al. showed that the associated Dirichlet series are elements of $\mathcal{S}^\sharp$ or a related class of Dirichlet series where the axiom on the functional equation is appropriately adjusted. Moreover, they showed that the Dirichlet series associated to newforms for $\mathsf{G}(\lambda)$ have an Euler product representation if and only if $\mathsf{G}(\lambda)$ can be arithmetically defined, i.e., if $\lambda \in \{1, \sqrt{2}, \sqrt{3}, 2\}$. Their result is based on Hecke's Converse Theorem 4. For $\lambda \leq 2$, Molteni & Steuding [34] proved that all these Dirichlet series are almost primitive (i.e., primitive up to factors of degree zero) and primitive if $\lambda \notin \{\sqrt{2}, \sqrt{3}, 2\}$; if the latter condition is not fulfilled, there are examples of non-primitive functions.

---

6. A Collection of Open Problems

Selberg (unpublished) proved that the values taken by an appropriate normalization of the Riemann zeta-function on the critical line are normally distributed: let $\mathcal{R}$ be an arbitrary fixed rectangle in the complex plane whose sides are parallel to the real and the imaginary axes, then

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ t \in (0,T] \, : \, \frac{\log \zeta\left(\frac{1}{2}+it\right)}{\sqrt{\frac{1}{2}\log\log T}} \in \mathcal{R} \right\} = \frac{1}{2\pi} \iint_{\mathcal{R}} \exp\left(-\frac{x^2+y^2}{2}\right) \mathrm{d}x \, \mathrm{d}y.$$

In [47], Selberg outlined that if $\mathcal{L} \in \mathcal{S}$ has not too many exceptional zeros of the critical line (more precisely, the Grand density hypothesis) and his Conjecture A, then the values of

$$\frac{\log \mathcal{L}\left(\frac{1}{2}+it\right)}{\sqrt{\pi n_{\mathcal{L}} \log\log t}}$$

are distributed in the complex plane according to the normal distribution, where $n_{\mathcal{L}}$ is the positive integer appearing in Selberg's Conjecture A. Furthermore, Selberg investigated the value-distribution of linear combinations of independent elements of $\mathcal{S}$. His argument was streamlined and extended by Bombieri & Hejhal to independent collections of L-functions having polynomial Euler products with the emphasis just on probabilistic convergence and the goal of applications to the zero-distribution. For this aim they introduced a stronger version of Selberg's conjecture B. Their theorem shows the statistical independence of any collection of independent L-functions in any family of elements of $\mathcal{S}$. Furthermore, Bombieri & Hejhal [3] applied their result to the zero-distribution of linear combinations of independent L-functions. Assuming in addition the Grand Riemann hypothesis and a weak conjecture on the well-spacing of the zeros, they proved that almost all zeros of these linear combinations are simple and lie on the critical line. Bombieri & Perelli [4] considered for the same class of functions the distribution of distinct zeros. They proved, for two different functions $\mathcal{L}_1, \mathcal{L}_2$ of the same degree,

$$\sum_{0\leq\gamma\leq T} \max\{m_{\mathcal{L}_1}(\rho) - m_{\mathcal{L}_2}(\rho), 0\} \gg T \log T,$$

where the sum is taken over the nontrivial zeros $\rho = \beta + i\gamma$ of $\mathcal{L}_1\mathcal{L}_2(s)$ and $m_{\mathcal{L}_j}(\rho)$ denotes the multiplicity of the zero ρ of $\mathcal{L}_j(s)$. An unconditional result with $T/\log\log T$ as lower bound was obtained by Srinivas [50].

We quote from M.R. Murty & V.K. Murty [38] an interesting point which relates unique factorization and zeros: if it could be shown that for any family of primitive L-functions $\mathcal{L}_1, \ldots, \mathcal{L}_k$ in $\mathcal{S}$ there exist complex numbers $s_1, \ldots, s_k$ such that $\mathcal{L}_j(s_\ell) = 0$ if and only if $j = \ell$, then the factorization into primitive elements would be unique, and Theorem 7 would hold unconditionally. Unfortunately, this seems to be out of reach by present day methods.

These lines of investigation are not unrelated to the following rather general

Question: *How to distinguish L-functions?*

A first answer might be: By the coefficients of their Dirichlet series expansion (since Dirichlet series are uniquely determined by their coefficients). It is remarkable, and by no means trivial, that one can identify L-functions already by their Dirichlet coefficients restricted on the set of prime powers. In fact, M.R. Murty & V.K. Murty [37] proved

Theorem 20. *If $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{S}$ have the same Euler factors*

$$\mathcal{L}_p(s) = \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$$

in their Euler product representation for all but finitely many primes p, then they are identical.

The proof follows from the observation that the quotient $\mathcal{L}_1(s)/\mathcal{L}_2(s)$ has to be entire and non-vanishing (thanks to the functional equation), and Hadamard's theory of entire functions (see Appendix B) is applicable. However, we expect more than that:

Strong Multiplicity One – Conjecture : *If $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{S}$ have the same Dirichlet coefficients $a_{\mathcal{L}_1}(p) = a_{\mathcal{L}_2}(p)$ for all but finitely many primes, then they are identical.*

Another way to distinguish L-functions is with respect to their value-distribution.

Recall the famous five value theorem of Rolf Nevanlinna which states that any two non-constant meromorphic functions that share five distinct values are identical. Here two meromorphic functions f and g are said to share a value $c \in \mathbb{C} \cup \{\infty\}$ if the sets of preimages of c under f and under g are equal, for short

$$f^{-1}(c) := \{s \in \mathbb{C} : f(s) = c\} = g^{-1}(c).$$

Furthermore, f and g are said to share the value c counting multiplicities (CM) if the latter identity of sets holds and if the roots of the equations $f(s) = c$ and $g(s) = c$ have the same multiplicities; if there is no restriction on the multiplicities, f and g are said to share the value c ignoring muliplicities (IM). Since the functions $f(s) = \exp(s)$ and $g(s) = \exp(-s)$ share the four values $0, \pm 1, \infty$, the number five in Nevanlinna's statement is best possible. If multiplicities are taken into account, Nevanlinna proved that any two meromorphic functions f and g that share four distinct values $c_1, \ldots, c_4$ CM are identical or can be transformed into one another by a Moebius transformation M in such a way that $g \equiv M \circ f$ and M fixes two of the points c_j while the other two are interchanged. Also the number four of values shared CM is best possible.

In [54], it was shown by means of Nevanlinna Theory that in the special case of L-functions better estimates are possible than those which Nevanlinna's theorems provide (since there is additional information about the functions available):

Theorem 21 (**Uniqueness Theorem – CM**). *If two elements of the extended Selberg class $\mathcal{S}^\sharp$ share a complex value $c \neq \infty$ CM, then they are identical.*

Recent joint work with Garunkštis et al. [10] provides a uniqueness theorem for L-functions from the Selberg class for small degrees where the rather restrictive condition on sharing a value CM is dropped:

Theorem 22 (**Uniqueness Theorem – IM – Degree One**). *If two elements of the Selberg class $\mathcal{S}$, both of degree one, share a complex value $c \neq \infty$ IM, then they are identical.*

In view of Theorem 15 this shows that if $L(s, \chi_1)$ and $L(s, \chi_2)$ are L-functions associated with primitive Dirchlet characters $\chi_j \bmod q_j$ and

$$L(s + i\theta_1, \chi_1) = c \qquad \Longleftrightarrow \qquad L(s + i\theta_2, \chi_2) = c$$

for some fixed $c \in \mathbb{C}$, then $\theta_1 = \theta_2$ and $\chi_1 = \chi_2$; hence, the Dirichlet L-functions are identical. Moreover, the same statement holds if one of the shifted Dirichlet L-functions is replaced by the zeta-function. The idea of proof relies on Voronin's universality theorem which states, roughly speaking, that any non-vanishing analytic function can be uniformly approximated by certain shifts of the Riemann zeta-function: for suitable functions f defined on compact subsets $\mathcal{K}$ of the right half of the critical strip and any $\epsilon > 0$, there exists a real number τ such that

$$\max_{s \in \mathcal{K}} |\zeta(s + i\tau) - f(s)| < \epsilon.$$

Actually, we need Voronin's extension to a simultaneous approximation theorem for a family of Dirichlet L-functions associated with non-equivalent characters which we formulate here in a slightly more general form (according to [54]):

Theorem 23 (**Joint Universality for Dirichlet L-functions**). *Let $\chi_1, \ldots, \chi_\ell$ be pairwise non-equivalent Dirichlet characters, $\mathcal{K}_1, \ldots, \mathcal{K}_\ell$ be compact subsets of the strip $\frac{1}{2} < \mathrm{Re}\, s < 1$ with connected complements. Further, for each $1 \leq j \leq \ell$, let $f_j(s)$ be a continuous non-vanishing function on $\mathcal{K}_j$ which is analytic in the interior of $\mathcal{K}_j$. Then, for any $\epsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] \, : \, \max_{1 \leq j \leq \ell} \max_{s \in \mathcal{K}_j} |L(s + i\tau, \chi_j) - f_j(s)| < \epsilon \right\} > 0.$$

For the sophisticated and lengthy proof see Voronin [58], resp. [54].

Proof of Theorem 22. Let us consider two shifted Dirichlet L-functions $L(s + i\theta_j, \chi_j)$ associated with either primitive characters or the principal character $\bmod 1$, where the θ_j are real numbers and $\theta_j = 0$ if $\chi_j \equiv \chi_0 \bmod 1$. Now assume that $L(s + i\theta_1, \chi_1)$ and $L(s + i\theta_2, \chi_2)$ share a complex value c. If $\chi_1 = \chi_2$, then $L(s + i\theta_1, \chi_1) = c$ whenever $L(s + i\theta_2, \chi_1) = c$, and it follows that either $\theta_1 = \theta_2$ or the c-points of $L(s, \chi_1)$ are periodically distributed with period $i(\theta_1 - \theta_2)$ which is absurd. Therefore, we may assume that $\chi_1 \neq \chi_2$; hence, being primitive, they are non-equivalent.

Now suppose $c \neq 0$ and that c' is another non-zero complex number different from c. We shall show the existence of some complex number s' such that

$$L(s' + i\theta_1, \chi_1) = c \neq L(s' + i\theta_2, \chi_2).$$

For this purpose let $\mathcal{K}$ be the closed disk centered at $\frac{3}{4}$ of radius $r \in (0, \frac{1}{4})$. Moreover, define target functions by setting $f_1(s) = c + \lambda(s - i\theta_1 - \frac{3}{4})$ and $f_2(s) = c'$ on sets $\mathcal{K}_j$, where

$$\mathcal{K}_j = \mathcal{K} + i\theta_j := \{s + i\theta_j \, : \, s \in \mathcal{K}\} \tag{45}$$

and λ is a positive real number for which $\lambda r < |c|$. By the latter condition $f_1(s)$ does not vanish on $\mathcal{K}_1$. Thus, an application of Theorem 15 with $0 < \epsilon < \min\{\lambda r, |c - c'|\}$ yields a real number τ such that

$$\max_{1\le j\le 2}\max_{s\in\mathcal{K}_j}|L(s+i\tau,\chi_j) - f_j(s)| < \epsilon. \tag{46}$$

We first deduce that

$$\max_{s\in\mathcal{K}}|L(s+i\theta_1+i\tau,\chi_1) - f_1(s+i\theta_1)| < \epsilon.$$

Since the absolute value of $f_1(s+i\theta_1) - c = \lambda(s-\frac{3}{4})$ on the boundary of $\mathcal{K}$ equals λr which is strictly larger than ϵ, it follows that

$$\max_{s\in\partial\mathcal{K}}|L(s+i\theta_1+i\tau,\chi_1) - c - \{f_1(s+i\theta_1)-c\}| < \epsilon < \min_{s\in\partial\mathcal{K}}|f_1(s+i\theta_1) - c|,$$

and an application of Rouché's theorem gives the existence of a c-point of $L(s+i\theta_1,\chi_1)$ inside $\mathcal{K}+i\tau := \{s+i\tau\,:\,s\in\mathcal{K}\}$. Secondly, we deduce from (46) that

$$\max_{s\in\mathcal{K}}|L(s+i\theta_2+i\tau,\chi_2) - c'| < \epsilon.$$

Consequently, $L(s+i\theta_2,\chi_2)$ does not assume the value c in $\mathcal{K}+i\tau$ since $\epsilon < |c-c'|$. This already shows that $L(s+i\theta_1,\chi_1)$ and $L(s+i\theta_2,\chi_2)$ do not share any complex value $c\neq 0$.

Since Dirichlet L-functions are expected to have no zeros to the right of the critical line $\frac{1}{2}+i\mathbb{R}$, universality is not an appropriate tool to discuss the remaining case of a shared value $c=0$.

To conclude let us assume that $L(s+i\theta_1,\chi_1)$ and $L(s+i\theta_2,\chi_2)$ share the value $c=0$. In view of the trivial zeros of Dirichlet L-functions on the negative real axis it follows that $\theta_1=\theta_2$. Next we may use a formula due to Fujii [9], resp. its unconditional version [52],

$$\lim_{T\to\infty}\frac{\pi}{T\log T}\sum_{|\gamma_{\chi_1}|\le T} L(\rho_{\chi_1},\chi_2) = 1 - \frac{1}{\varphi([q_1,q_2])}\sum_{\substack{a \bmod [q_1,q_2]\\ (a,[q_1,q_2])=1}}(\overline{\chi_1}\chi_2)(a),$$

where q_j is the modulus of χ_j. It follows from the orthogonality relation for characters (Theorem 26) that the right-hand side does not vanish for $\chi_1\neq\chi_2$. Hence $L(s+i\theta_1,\chi_1)$ and $L(s+i\theta_2,\chi_2)$ do not share the value 0. This proves Theorem 22. •

Alternatively, we could have used a joint universality theorem due to Sander & Steuding [45] which applies to a family of Dirichlet series with periodic coefficients and analytic continuation beyond the abscissa of absolute convergence. This theorem covers indeed the degree one case of the extended Selberg class $\mathcal{S}^\sharp$; however, since elements in $\mathcal{S}^\sharp$ may be linearly dependent, they cannot be jointly universal in general without any restriction. In fact, the joint universality theorem of Sander & Steuding is conditional subject to a linear independence condition on the target functions. Moreover, since $\zeta(s)$ and $\lambda\zeta(s)$ for some complex $\lambda\neq 0$ are both elements of $\mathcal{S}^\sharp$ of degree one and therefore the statement of Theorem 22 does not hold for the extended Selberg class.

Problem: *Extend the above Uniqueness Theorem 22 to elements of the Selberg class of higher degree.*

It is reasonable to expect that *independent* L-functions cannot share any complex value. However, it is not clear what the correct meaning of *independence* should be. Joint universality seems to be an interesting approach to this question. Hence, it might be an interesting line of investigation to consider universality phenomena generalizing Voronin's Universality Theorem.

Theorem 24 (Universality Theorem for the Selberg Class). *Assume $\mathcal{L} \in \mathcal{S}$ satisfies*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} |a(p)|^2 = \kappa$$

for some positive constant κ, where the $a(p)$ are the Dirichlet series coefficients of $\mathcal{L}$ on the primes. Furthermore, let $\mathcal{K}$ be a compact subset of the strip $\max\{\frac{1}{2}, 1-\frac{1}{\mathrm{d}_{\mathcal{L}}}\} < \mathrm{Re}\, s < 1$ with connected complement, and suppose that $g(s)$ is a non-vanishing continuous function on $\mathcal{K}$ which is analytic in the interior of $\mathcal{K}$. Then

$$\limsup_{T\to\infty} \frac{1}{T} \mathrm{meas}\, \{\tau \in [0,T] \,:\, \max_{s\in\mathcal{K}} |\mathcal{L}(s+i\tau) - g(s)| < \epsilon\} > 0.$$

This universality theorem is due to Nagoshi & Steuding [39]; it improves upon a previous one for polynomial Euler products in $\mathcal{S}$ (see [54]). It should be noticed that the range for the approximation is restricted to the mean-square half-plane. If $\mathcal{L} \in \mathcal{S}$, then existence of the mean-square is, by Carlson's theorem, equivalent to

$$\limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\mathcal{L}(\sigma+it)|^2 \,\mathrm{d}t < \infty,$$

which is known to be true only for $\sigma > \max\{\frac{1}{2}, 1-\frac{1}{\mathrm{d}_{\mathcal{L}}}\}$. This follows from [54], resp. classical work of Chandrasekharan & Narasimhan [6] on Dedekind zeta-functions. In this context we shall mention recent work of Mazhouda & Omar [33] who proved asymptotics for the mean-square of $\mathcal{L}(s)$ on and to the right of the the critical line. For instance, they obtained the unconditional bound

$$\int_0^T |\mathcal{L}(\tfrac{1}{2}+it)|^2 \,\mathrm{d}t \ll T^{\frac{1}{2}\mathrm{d}_{\mathcal{L}}+\epsilon}$$

for any positive ϵ, where the implicit constant depends on ϵ and $\mathcal{L}$, and a conditional improvement subject to an estimate for $\sum_{n\leq X} |a_{\mathcal{L}}(n)|^2$.

Furthermore, we may ask for generalizations of Voronin's Universality Theorem. For example, if $\mathcal{L}(s)$ is universal, is the standard linear twist also universal? In the case of the Riemann zeta-function, this twist $\zeta(s,\alpha)$ is a Lerch zeta-function and is indeed universal if α is rational or transcendental, however, the case of algebraic irrational α is unsettled; see the monograph [11] of Garunkštis & Laurinčikas.

What about generalizations of the Joint Universality Theorem for Dirichlet L-functions from above? The known methods fail to prove, for example, joint universality for $\zeta(s)$ and an arbitrary L-function from the Selberg class. More generally, we ask for a necessary and sufficient condition that a given finite family of L-functions is jointly universal? In the context of the Selberg class we expect that Selberg's Conjecture B (or a suitable quantitative extension) could be used to answer this question. By Selberg's Conjecture B, primitive functions are expected to form an orthonormal system. Recall that Bombieri & Hejhal [3] proved, assuming a stronger version of Selberg's conjecture B, the statistical independence of any collection of independent L-functions in any family of independent elements of $\mathcal{S}$. With regard to this statistical independence, predicted by Selberg's Conjecture B, we recall from [54] the following

Joint Universality Conjecture: *Any finite collection of distinct primitive functions in the Selberg class is jointly universal. Moreover, any two functions $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{S}$ are jointly universal, if and only if*

$$\sum_{p\leq x} \frac{a_1(p)\overline{a_2(p)}}{p} = O(1),$$

as $x \to \infty$, where the Dirichlet series coefficients of $\mathcal{L}_j(s)$ are denoted by $a_j(n)$.

In the example of $\zeta(s)$ and $\zeta(s)^2$ the latter asymptotics does not hold; needles to say that this pair is not jointly universal. For more details to this question and how it is related to other results in this direction we refer to [54].

We conclude with an interesting number field-analogue of the Selberg class, proposed by M.R. Murty [36]. Let $\mathbb{K}$ be a number field. The Dedekind zeta-function (23) satisfies the following axioms:

(i') Dirichlet Series & Ramanujan Hypothesis: for $\operatorname{Re} s > 1$,

$$\mathcal{L}(s) = \sum_{\mathfrak{a}} \frac{c(\mathfrak{a})}{\mathrm{N}(\mathfrak{a})^s},$$

where the summation is over all integral ideals $\mathfrak{a}$ and the coefficients satisfy $c(1) = 1$ and $c(\mathfrak{a}) \ll \mathbb{N}(\mathfrak{a})^\epsilon$.

(ii) Analytic Continuation: there exists a non-negative integer k such that $(s-1)^k\mathcal{L}(s)$ is an entire function of finite order.

(iii) Functional Equation: $\mathcal{L}(s)$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s) = \omega\overline{\Lambda_{\mathcal{L}}(1-\overline{s})}, \qquad \text{where} \quad \Lambda_{\mathcal{L}}(s) := \mathcal{L}(s)Q^s \prod_{j=1}^{f} \Gamma(\lambda_j s + \mu_j)$$

with positive real numbers Q, λ_j, and complex numbers μ_j, ω with $\operatorname{Re}\mu_j \geq 0$ and $|\omega| = 1$.

(iv') Euler Product: $\mathcal{L}(s)$ has a product representation

$$\mathcal{L}(s) = \prod_{\mathfrak{p}} \mathcal{L}_{\mathfrak{p}}(s), \qquad \text{where} \quad \mathcal{L}_{\mathfrak{p}}(s) = \exp\left(\sum_{k=1}^{\infty} \frac{b(\mathfrak{p}^k)}{\mathrm{N}(\mathfrak{p})^{ks}}\right)$$

with suitable coefficients $b(\mathfrak{p}^k)$ satisfying $b(\mathfrak{p}^k) \ll \mathrm{N}(\mathfrak{p})^{k\theta}$ for some $\theta < \frac{1}{2}$, where $\mathfrak{p}$ is a prime ideal.

Observe that the arithmetic axioms (i') and (iv') are more restictive than the corresponding axioms (i) and (iv) in the definition of the Selberg class $\mathcal{S}$. M.R. Murty [36] showed that one can find Artin L-functions $\mathcal{L}$ (see Appendix D for an exemplary introduction) associated with the splitting field of the polynomial $X^8 + 9X^6 + 23X^4 + 14X^2 + 1$ which satisfy the above properties but

$$\sum_{\mathrm{N}(\mathfrak{p}) \leq x} \frac{|c(\mathfrak{p})|^2}{\mathrm{N}(\mathfrak{p})} = \frac{1}{4}\log\log x + O(1).$$

Hence, the analogue of Selberg's Conjecture A fails. For this purpose M.R. Murty [36] imposed as further axiom

(v') Rankin–Selberg Property:* for some positive integer n,

$$\sum_{\mathrm{N}(\mathfrak{p}) \leq x} \frac{|c(\mathfrak{p})|^2}{\mathrm{N}(\mathfrak{p})} = n\log\log x + O(1).$$

*These names are in honour of the inventors of the so-called Rankin-Selberg convolution which is closely related to this type of asymptotics and the Selberg Conjectures in particular.

The set of all functions $\mathcal{L}$ satisfying the above axioms is said to be the Selberg class over $\mathbb{K}$ and will be denoted as $\mathcal{S}_{\mathbb{K}}$. Generalizing from the Selberg class, we call an element $\mathcal{L}$ a $\mathbb{K}$-primitive function if all factorizations of $\mathcal{L}$ in $\mathcal{S}_{\mathbb{K}}$ are trivial. Following M.R. Murty, we introduce the

The Number Field-Analogue of Selberg's Conjecture B: Let $\mathbb{K}$ be a number field. *For any $\mathbb{K}$-primitive functions $\mathcal{L}_1$ and $\mathcal{L}_2 \in \mathcal{S}_{\mathbb{K}}$,*

$$\sum_{\mathrm{N}(\mathfrak{p})\leq x} \frac{c_{\mathcal{L}_1}(\mathfrak{p})\overline{c_{\mathcal{L}_2}(\mathfrak{p})}}{\mathrm{N}(\mathfrak{p})} = \begin{cases} \log\log x + O(1) & \textit{if} \quad \mathcal{L}_1 = \mathcal{L}_2, \\ O(1) & \textit{otherwise}, \end{cases}$$

where $c_{\mathcal{L}_j}(\mathfrak{p})$ denotes the Dirichlet coefficient of $\mathcal{L}_j$ at the prime ideal $\mathfrak{p}$,

Then it is not difficult to see that also the anaologue of Theorem 7 holds true:

Theorem 25. *Every function in $\mathcal{S}_{\mathbb{K}}$ has a factorization into primitive functions. If the analogue of Selberg's conjecture B is true, then this factorization into $\mathbb{K}$-primitive functions is unique.*

Problem: *Develop a theory for the Selberg class over $\mathbb{K}$. What is the analogue of the Degree Conjecture?*

Further problems and questions can be found in the excellent surveys of Kaczorowski [20] and Perelli [42, 43]. We conclude with a last task which may take a while:

Exercise 17. *Prove any of these open problems or any of the many other conjectures.* **:-)**

In the following appendices we have tried to provide apart from standard results all mathematics necessary to make these course notes self-contained. For some results we have decided to present their proofs; for others we only refer to literature which contains a proof.

Appendix A: Characters and Other Tools from Number Theory

A character χ is a non-trivial group homomorphism from a finite (for the sake of simplicity) abelian group G onto $\mathbb{C}^*$. By the structure theorem for finite abelian groups any such group G is the direct product of cyclic groups. Often we will be concerned with the the multiplicative group of the ring of residue classes mod q, i.e., the group of prime residue classes modulo q,

$$(\mathbb{Z}/q\mathbb{Z})^* := \{a \bmod q \,:\, \mathsf{gcd}(a,q) = 1\}.$$

By the chinese remainder theorem,

$$(\mathbb{Z}/q\mathbb{Z})^* = \prod_{p|q} (\mathbb{Z}/p^{\nu(q;p)}\mathbb{Z})^*,$$

where $\nu(q;p)$ denotes the exponent of the prime p in the prime factorization of the integer q. In this case the decomposition into a product of cyclic groups is much easier to obtain. Gauss proved that the group of residue classes modulo q is cyclic if and only if $q = 2, 4, p^\nu$ or $2p^\nu$, where $p \neq 2$; a generator of such a cyclic group $(\mathbb{Z}/q\mathbb{Z})^*$ is called a primitive root mod q. In the case $q = 2^\nu$ one has $(\mathbb{Z}/2^\nu\mathbb{Z})^* = \langle -1 \rangle \times \langle 5 \rangle$ (which leads to a cyclic group if $\nu = 1, 2$, since then $-1 \equiv 5 \bmod 2^2$). In any case, the group of prime residue classes mod q is a product of finitely many cyclic groups.

For the first, however, we shall argue more generally. Assume that

$$\mathsf{G} = \prod_{j=1}^{r} \mathsf{G}_j \qquad \text{with} \quad \mathsf{G}_j = \langle g_j \rangle.$$

In particular, any $g \in \mathsf{G}$ has a unique representation of the form

$$g = \prod_{j=1}^{r} g_j^{t_j} \qquad \text{with} \qquad 0 < t_j \leq \ell_j,$$

where $\ell_j = \sharp \mathsf{G}_j$ is the group order of G_j. Since a character on G is a group homomorphism, i.e.,

$$\chi(a \cdot b) = \chi(a) \cdot \chi(b) \qquad \text{for all} \quad a, b \in \mathsf{G},$$

it follows that

$$\chi(g) = \prod_{j=1}^{r} \chi(g_j)^{t_j} \qquad \text{for} \qquad g = \prod_{j=1}^{r} g_j^{t_j}.$$

Therefore, a character is uniquely determined by its values on the generators. Since the order of any element of a finite abelian group is a divisor of the group order, we find

$$1 = \chi(1) = \chi(g_j^{\ell_j}) = \chi(g_j)^{\ell_j},$$

and thus $\chi(g_j)$ is an ℓ_j-th root of unity, i.e.,

$$\chi(g_j) = \exp\left(2\pi i \frac{k_j}{\ell_j}\right) \qquad \text{for some} \quad k_j \in \mathbb{Z} \quad \text{with} \quad 0 < k_j \leq \ell_j.$$

Consequently, there are at most $\ell_1 \cdot \ldots \cdot \ell_r$ many characters χ on G. On the contrary, any choice of $k_1, \ldots, k_r$ with $0 < k_j \leq \ell_j$ defines via $\chi(g_j) = \exp(2\pi i \frac{k_j}{\ell_j})$ such a character. Hence, the number of characters χ on G is equal to the group order $\sharp \mathsf{G} = \ell_1 \cdot \ldots \cdot \ell_r$. We may define the product of two characters $\bmod\, q$ by setting $(\chi \cdot \psi)(g) = \chi(g) \cdot \psi(g)$; this gives the set of characters $\chi \bmod q$ the structure of a group, the character group (resp. dual group) of G, for short $\hat{\mathsf{G}}$. Its unit element, the principal character, is the character constant 1 and is denoted by χ_0. Since $|\chi(g)| = 1$, the inverse of a character $\chi \in \hat{\mathsf{G}}$ is given by conjugation:

$$\overline{\chi}(g) = \overline{\chi(g)} = \chi(g)^{-1}.$$

Given

$$\chi_k(g_j) = \begin{cases} \exp\left(2\pi i \frac{1}{\ell_j}\right) & \text{if} \quad j = k, \\ 1 & \text{otherwise,} \end{cases}$$

the mapping $g_j \mapsto \chi_j$ is an isomorphism between G and its character group $\hat{\mathsf{G}}$. We illustrate these observations with the example $\mathsf{G} = (\mathbb{Z}/5\mathbb{Z})^*$:

	χ_0	χ_1	χ_2	χ_3
$1 \equiv 2^0$	+1	+1	+1	+1
$2 \equiv 2^1$	+1	-1	+i	-i
$4 \equiv 2^2$	+1	+1	-1	-1
$3 \equiv 2^3$	+1	-1	-i	+i

We find $\langle 2 \rangle \cong \langle \chi_2 \rangle$ (of course, here we can also replace 2 by 3 or χ_2 by χ_3).

Now we switch to the group of prime residue classes $(\mathbb{Z}/q\mathbb{Z})^*$ and state the important orthogonality relations for characters. Via the natural embedding of $(\mathbb{Z}/q\mathbb{Z})^*$ in $\mathbb{Z}$ we can define characters $\chi \bmod q$ on the whole of $\mathbb{Z}$ by setting

$$\chi(n) = \begin{cases} \chi(n + q\mathbb{Z}) & \text{if} \quad \gcd(n, q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The new objects are called Dirichlet characters $\chi \bmod q$. The function $n \mapsto \chi(n)$ is completely multiplicative; moreover, it is a q-periodic function on $\mathbb{Z}$, i.e., $\chi(n+q) = \chi(n)$ for any $n \in \mathbb{Z}$. Notice that $\sharp(\mathbb{Z}/q\mathbb{Z})^* = \varphi(q)$.

Theorem 26 (Orthogonality Relation For Characters). *If a and q are coprime, then*

$$\frac{1}{\varphi(q)} \sum_{\chi \bmod q} \overline{\chi}(a)\chi(n) = \begin{cases} 1 & \text{if} \quad n \equiv a \bmod q, \\ 0 & \text{otherwise.} \end{cases} \tag{47}$$

A proof can be found in almost any book on Number Theory, e.g., Davenport [8], resp. in [55]. With this tool one can sieve prime residue classes from the set of positive integers as we did in the proof of Theorem 11. Hence, characters are at the heart of Dirichlet's approach to prove the infinitude of primes in arithmetic progressions.

An arithmetical function $f : \mathbb{N} \to \mathbb{C}$ is said to be multiplicative, if $f(mn) = f(m)f(n)$ for all coprime integers m, n; if the latter condition on the coprimality can be dropped, then f is said to be completely multiplicative. Dirichlet Characters are examples of completely multiplicative functions.

Theorem 27. *A multiplicative function f is periodic if, and only if, there exists an integer q and a Dirichlet character $\chi \bmod q$ satisfying the following properties:*

- *$f(p^k) = 0$ if p is a prime divisor of q;*
- *if p is prime and $p \not| \, q$, then the function $k \mapsto \overline{\chi}(p^k)f(p^k)$ is constant and $\neq 0$;*
- *there are at most finitely many primes p for which $\overline{\chi}(p^k)f(p^k) \neq 1$ for some exponent k.*

This result is due to de Bruijn and (independently) Leitmann & Wolke [28] (cf. Schwarz & Spilker [46]). It follows immediately, that a completely multiplicative function which is q-periodic, is a Dirichlet character $\bmod\, q$.

Another very important tool in Number Theory is

Theorem 28 (Abel's Partial Summation). *Let $\lambda_1 < \lambda_2 < \ldots$ be a divergent sequence of real numbers, define for $\alpha_n \in \mathbb{C}$ the function $A(u) := \sum_{\lambda_n \leq u} \alpha_n$, and let $F : [\lambda_1, \infty) \to \mathbb{C}$ be a continuous differentiable function. Then*

$$\sum_{\lambda_n \leq x} \alpha_n F(\lambda_n) = A(x)F(x) - \int_{\lambda_1}^{x} A(u)F'(u)\,\mathrm{d}u.$$

The proof is by rewriting the difference between the integral and the sum by using the fundamental theorem of analysis or, simply, by Stieltjes integration.

As an easy application we find

$$-\frac{\zeta'}{\zeta}(s) = s\int_1^{\infty} \frac{\psi(x)}{x^{s+1}}\,\mathrm{d}x \tag{48}$$

with the summatory function $\psi(x) = \sum_{n \leq x} \lambda(n)$ of the Dirichlet coefficients.

We conclude this appendix with estimates for exponential integrals:

Theorem 29. *i) Let $f(x)$ be a real differentiable function with monotonic derivative which satisfies either $f'(x) \geq m > 0$ or $f'(x) \leq -m < 0$ throughout the interval $[a, b]$ for some constant m. Then*

$$\left| \int_a^b \exp(if(x))\,\mathrm{d}x \right| \leq \frac{4}{m}.$$

ii) Let $f(x)$ be real differentiable with derivatives up to the third order such that $|f'''(x)| \leq c\lambda_2$ and either $0 < \lambda_1 \leq f''(x) < c\lambda_1$ or $0 < \lambda_1 \leq -f''(x) \leq c\lambda_1$ throughout the intervall $[a,b]$ for some constants λ, c. Moreover, let $f'(\xi) = 0$ for some $\xi \in [a,b]$. Then

$$\begin{aligned}\int_a^b \exp(if(x))\,\mathrm{d}x &= (2\pi)1\frac{1}{2}\frac{\exp(\pm\frac{\pi i}{4} + if(\xi))}{|f''(\xi)|^{\frac{1}{2}}} + O(\lambda_1^{-\frac{4}{5}}\lambda_2^{\frac{1}{5}}) + \\ &\quad + O\left(\min\{\lambda_2^{-\frac{1}{2}}, |f'(a)|^{-1}\} + \min\{\lambda_2^{-\frac{1}{2}}, |f'(b)|^{-1}\}\right),\end{aligned}$$

where the sign $\pm$ is according to the sign of f''; if f' does not vanish in $[a,b]$, then the latter formula holds without leading term.

For the elementary proof see Lemmas 4.2 and 4.6 in Titchmarsh's monograph [57].

Appendix B: Poisson Summation Formula and Further Results From Analysis

Suppose $f : \mathbb{R} \to \mathbb{C}$ is an integrable function satisfying $f(z) \ll |z|^{-2}$ as $|z| \to \infty$ (actually, this is a strong restriction but it allows to do the next step). Then we may define its Fourier transform by

$$\hat{f}(y) = \int_{-\infty}^{+\infty} f(z)\exp(-2\pi i yz)\,\mathrm{d}z.$$

The Poisson summation formula is a useful tool in Fourier theory with many applications in real and complex analysis.

Theorem 30 (**Poisson Summation Formula).** *Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function with $f(z) \ll |z|^{-2}$ as $|z| \to \infty$. Further, assume that the integral*

$$\int_{-\infty}^{+\infty} |f''(z)|\,\mathrm{d}z$$

exists. Then, for any $\alpha \in \mathbb{R}$,

$$\sum_{n\in\mathbb{Z}} f(n+\alpha) = \sum_{m\in\mathbb{Z}} \hat{f}(m)\exp(2\pi i\alpha m).$$

Proof. It suffices to prove the formula in question only for $\alpha = 0$. In fact, writing $g(z) = f(z+\alpha)$ for fixed $\alpha \in \mathbb{R}$, we have $\hat{g}(y) = \hat{f}(y)\exp(2\pi i\alpha y)$. Therefore, we may assume $\alpha = 0$.

First of all, for $r > 0$, define

$$P(y,r) = \sum_{m=-\infty}^{\infty} r^{|m|}\exp(2\pi i m y).$$

This series is the sum of the term for $m = 0$ plus two infinite geometric series, one for $m < 0$ and one for $m > 0$, both being absolutely convergent for $r \in [0,1)$. Hence, we can compute the value of the infinite series $P(y,r)$ by

$$P(y,r) = 1 + \frac{r\exp(2\pi i y)}{1 - r\exp(2\pi i y)} + \frac{r\exp(-2\pi i y)}{1 - r\exp(-2\pi i y)} = \frac{1-r^2}{1 - 2r\cos(2\pi y) + r^2}.$$

This implies $P(y,r) \geq 0$ for any y (since the denominator is equal to $(r - \cos 2\pi y)^2 + (\sin 2\pi y)^2$). Using

$$\int_0^1 \exp(2\pi i m y)\,\mathrm{d}y = \begin{cases} 1 & \text{if} \quad m = 0, \\ 0 & \text{otherwise,}\end{cases}$$

we find

$$\int_0^1 P(y,r)\,\mathrm{d}y = 1$$

for all $r \in [0,1)$. Further note that $P(y,r)$ is 1-periodic with respect to y. Hence,

$$P(y,r) \leq \frac{1-r^2}{(\sin 2\pi\delta)^2}$$

for $0 < \delta \leq |y| \leq \frac{1}{2}$.

Since $f(z) \ll z^{-2}$, we have

$$\sum_{m=-\infty}^{+\infty} r^{|m|}\hat{f}(m) = \int_{-\infty}^{+\infty} P(y,r)f(y)\,\mathrm{d}y = \sum_{m=-\infty}^{+\infty} \int_{[m-\frac{1}{2},m+\frac{1}{2}]} P(y,r)f(y)\,\mathrm{d}y;$$

interchanging summation and integration is justified with respect to the absolute convergence. We want to show that the right-hand side converges to $\sum_m f(m)$ as $r \to 1-$. For this purpose we note that

$$\begin{aligned} \int_{[m-\frac{1}{2},m+\frac{1}{2}]} P(y,r)f(y)\,\mathrm{d}y &\leq \max_{m-\frac{1}{2}\leq y\leq m+\frac{1}{2}} |f(y)| \int_0^1 P(y,r)\,\mathrm{d}y \\ &\leq \max_{m-\frac{1}{2}\leq y\leq m+\frac{1}{2}} |f(y)| \ll m^{-2}, \end{aligned}$$

as $|m| \to \infty$. Hence, given $\epsilon > 0$, there exists $M > 0$ such that

$$\sum_{|m|>M} \int_{[m-\frac{1}{2},m+\frac{1}{2}]} P(y,r)f(y)\,\mathrm{d}y < \epsilon \qquad \text{and} \qquad \sum_{|m|>M} |f(m)| < \epsilon.$$

Now assume $|m| \leq M$. Of course,

$$\int_{[m-\frac{1}{2},m+\frac{1}{2}]} P(y,r)f(y)\,\mathrm{d}y - f(m) = \int_{[m-\frac{1}{2},m+\frac{1}{2}]} P(y,r)(f(y)-f(m))\,\mathrm{d}y.$$

Take some $\delta > 0$ for which $|f(y) - f(z)| < \frac{\epsilon}{3M}$ for all m with $|m| \leq M$ and all y, z with $|y - z| \leq \delta$. Then

$$\sum_{|m|\leq M} \left| \int_{[m-\frac{1}{2},m+\frac{1}{2}]} P(y,r)f(y)\,\mathrm{d}y - f(m) \right| \leq \sum_{|m|\leq M} (J_1(m) + J_2(m)), \tag{49}$$

where

$$\begin{aligned} J_1(m) &:= \int_{m-\delta}^{m+\delta} P(y,r)|f(y)-f(m)|\,\mathrm{d}y, \\ J_1(m) &:= \int_{W(m)} P(y,r)|f(y)-f(m)|\,\mathrm{d}y \end{aligned}$$

with $W(m) := \{y \in \mathbb{R} : \delta < |y - m| \leq \frac{1}{2}\}$. By construction,

$$J_1(m) \leq \frac{\epsilon}{3M} \int_{m-\delta}^{m+\delta} P(y,r)\,\mathrm{d}y \leq \frac{\epsilon}{3M}.$$

Moreover,

$$J_2(m) \leq \frac{1-r^2}{(\sin 2\pi\delta)^2} \int_{W(m)} |f(y)-f(m)|\,\mathrm{d}y \ll \frac{1-r^2}{m^2},$$

where the implicit constant depends only on δ, ϵ and f. Thus, the right-hand side of (49) can be made less than 2ϵ for some r sufficiently close to 1. Hence, letting $\epsilon \to 0$, we obtain

$$\lim_{r\to 1-} \sum_{m=-\infty}^{+\infty} r^{|m|}\hat{f}(y) = \sum_{m=-\infty}^{+\infty} f(m). \tag{50}$$

Partial integration shows $\hat{f}(m) \ll m^{-2}$. Consequently, the series on the left-hand side of (50) converges absolutely and uniformly for $r \in [0,1)$ and we may interchange summation and take the limit. This proves the theorem. •

The (most simple) theta-function is given by the infinite series

$$\theta(x) = \sum_{n\in\mathbb{Z}} \exp(-\pi x n^2).$$

We apply Poisson's summation formula, Theorem 30, to the function $f(z) := \exp(-\pi z^2/x)$ with $x > 0$. We compute the Fourier transform by quadratic substitution:

$$\begin{aligned} \hat{f}(y) &= \int_{-\infty}^{+\infty} \exp(-\pi(z^2/x + 2iyz))\,\mathrm{d}z \\ &= x\exp(-\pi xy^2)\int_{-\infty}^{+\infty} \exp(-\pi x(w+iy)^2)\,\mathrm{d}w. \end{aligned} \tag{51}$$

Next we consider the integral

$$I(\lambda) := \int_{-\infty}^{+\infty} \exp(-\pi x(w+\lambda)^2)\,\mathrm{d}w,$$

where λ is any complex number. Consider the integral

$$\int_{\mathcal{R}} \exp(-x\omega^2)\,\mathrm{d}\omega,$$

where $\mathcal{R}$ is the rectangular contour with vertices $\pm r, \pm r + i\mathrm{Im}\,\lambda$, where r is a positive real number. By Cauchy's theorem, the integral is equal to zero. On the line $\mathrm{Re}\,\omega = r$, the integrand tends uniformly to zero as $r \to \infty$. Hence, $I(\lambda) = I(0)$, and thus the integral $I(\lambda)$ does not depend on λ. This gives in (51)

$$\hat{f}(y) = x\exp(-\pi xy^2)\int_{-\infty}^{+\infty} \exp(-\pi x w^2)\,\mathrm{d}w = C\sqrt{x}\exp(-\pi xy^2),$$

where

$$C := \int_{-\infty}^{+\infty} \exp(-\pi z^2)\,\mathrm{d}z.$$

Applying Poisson's summation formula leads to

$$\sum_{n\in\mathbb{Z}} \exp(-\pi(n+\alpha)^2/x) = C\sqrt{x}\sum_{n\in\mathbb{Z}} \exp(-\pi x n^2 + 2\pi i n\alpha);$$

here we have introduced the parameter α by the trick from the proof of Theorem 30. Choosing $\alpha = 0$ and $x = 1$, both sums are equal; thus, $C = 1$ and we have just proved the functional equation (3) for the theta-function: *for any $x > 0$,*

$$\theta(x) = \frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right).$$

Another important analytic tool is

Theorem 31 (Perron's formula). *For positive real numbers c, y, T, define*

$$I(y,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\,\mathrm{d}s \qquad \textit{and} \qquad \delta(y) = \begin{cases} 0 & \textit{if} \quad 0 < y < 1, \\ \frac{1}{2} & \textit{if} \quad y = 1, \\ 1 & \textit{if} \quad y > 1. \end{cases}$$

Then

$$|I(y,T) - \delta(y)| < \begin{cases} y^c \min\{1, (T|\log y|)^{-1}\} & \textit{if} \quad y \neq 1, \\ c/T & \textit{otherwise.} \end{cases}$$

The proof follows from Cauchy's theorem and can be found in Tichmarsh's monograph [56], resp. [55].

We shall apply this with the logarithmic derivative of the zeta-function in order to obtain, for $x \notin \mathbb{Z}$ and $c > 1$,

$$\int_{c-i\infty}^{c+i\infty} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \frac{x^s}{s} \, \mathrm{d}s = \sum_{n=1}^{\infty} \Lambda(n) \int_{c-i\infty}^{c+i\infty} \left(\frac{x}{n}\right)^s \frac{\mathrm{d}s}{s};$$

here interchanging integration and summation is allowed by the absolute convergence of the series. In view of Theorem 31 with $T \to \infty$ it follows that

$$\sum_{n \leq x} \Lambda(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \frac{x^s}{s} \, \mathrm{d}s,$$

resp. Formula (33) which is also known as Perron's formula. Please notice that this gives in some sense the inversion of Formula (48).

The Gamma-function plays a central role in the analytic theory of L-functions. We shall not give a full account of all properties needed, since they can be found in any textbook on Complex Analysis, but state a convenient asymptotic formula for its growth:

Theorem 32 (**Stirling's formula**). *We have*

$$\begin{aligned} \log \Gamma(z) &= \left(z - \frac{1}{2}\right) - z + \frac{1}{2} \log 2\pi + \int_0^{\infty} \frac{[u] - u + \frac{1}{2}}{u + z} \, \mathrm{d}u \\ &= \left(z - \frac{1}{2}\right) - z + \frac{1}{2} \log 2\pi + O\left(|z|^{-1}\right), \end{aligned}$$

uniformly in z with $-\pi + \epsilon \leq \arg z \leq \pi - \epsilon$.

A proof can be found in Titchmarsh [56].

The next statement is a maximum principle for unbounded domains:

Theorem 33. *Let $f(s)$ be analytic in the strip $\sigma_1 \leq \operatorname{Re} s \leq \sigma_2$ with $f(\sigma + it) \ll \exp(\epsilon|t|)$. If*

$$f(\sigma_1 + it) \ll |t|^{c_1} \qquad \text{and} \qquad f(\sigma_2 + it) \ll |t|^{c_2},$$

then $f(s) \ll |t|^{c(\sigma)}$ uniformly in $\sigma_1 \leq \sigma \leq \sigma_2$, where $c(\sigma)$ is linear with $c(\sigma_1) = c_1$ and $c(\sigma_2) = c_2$.

A proof can be found in Titchmarsh [56]. Note that there are counterexamples if the growth condition $f(s) \ll \exp(\epsilon|t|)$ is not fulfilled.

Weierstrass proved that any non-zero entire function can be factored into a product over its zeros (times an exponential function). In the case of polynomials this is just another formulation of the fundamental theorem of algebra (that any polynomial over $\mathbb{C}$ has a root in $\mathbb{C}$) and is known since Gauss' first proof in his doctorate. However, a generic entire function has infinitely many zeros and hence its so-called Weierstrass product is infinite and the analysis much more difficult. As part of his theory of entire functions, Hadamard's Product Theorem 34 obtained for entire functions of finite order a more explicit form for Weierstrass' products. For our purpose it suffices to consider only functions of order one.

Theorem 34 (**Hadamard's Product Theorem**). *Let $f(s)$ be an entire function of order one with zeros $\rho_0 = 0$ with multiplicity m_0 and $\rho_1, \rho_2, \ldots$ arranged so that $0 < |\rho_1| \leq$*

$|\rho_2| \leq \ldots$ *and repeated according their multiplicities. Then there are constants* A, B *such that*

$$f(s) = s^{m_0} \exp(A + Bs) \prod_{j=1}^{\infty} \left(1 - \frac{s}{\rho_j}\right) \exp\left(\frac{s}{\rho_j}\right).$$

A proof of this theorem can be found in many textbooks, e.g., Titchmarsh [56], as well as in [55].

It should be noticed that all the above results were introduced with respect to investigations in Number Theory. It is amazing how Complex Analysis was pushed forwards in the second half of the nineteenth century thanks to the Riemann zeta-function and attempts to prove the Prime Number Theorem!

Appendix C: The Wiener-Ikehara Tauberian Theorem

Around 1825, Abel proved $\sum_{n=0}^{\infty} a(n)x^n$ tends to 1 as $x \to 1-$ provided that $\sum_{n=0}^{\infty} a(n) = 1$. In 1897, Tauber proved that the converse implication holds if $na(n) = o(1)$ as $n \to \infty$. After Tauber plenty of similar results were proven, many of them with direct applications to number theory (created with number theoretical motivation in mind). Here we shall prove

Theorem 35 (Wiener–Ikehara Theorem – original version). *Let* $A(x)$ *be a non-negative, non-decreasing function of* $x \in [0, \infty)$. *Suppose that the integral*

$$\int_0^{\infty} A(x) \exp(-sx)\, \mathrm{d}x$$

converges to the function $f(s)$ *and that* $f(s)$ *is analytic in the half-plane* $\sigma \geq 1$, *except for a simple pole at* $s = 1$ *with residue* 1. *Then*

$$\lim_{x \to \infty} A(x) \exp(-x) = 1.$$

Proof. Define $B(x) = A(x) \exp(-x)$. First we shall prove that, for any positive λ,

$$\lim_{y \to \infty} \int_{-\infty}^{\lambda y} B\left(y - \frac{v}{\lambda}\right) \left(\frac{\sin v}{v}\right)^2 \mathrm{d}v = \pi. \tag{52}$$

For $\sigma > 1$, we have

$$f(s) = \int_0^{\infty} A(x) \exp(-sx)\, \mathrm{d}x \qquad \text{and} \qquad \frac{1}{s-1} = \int_0^{\infty} \exp((1-s)x)\, \mathrm{d}x.$$

Thus, after a short computation,

$$F(s) := f(s) - \frac{1}{s-1} = \int_0^{\infty} (B(x) - 1) \exp((1-s)x)\, \mathrm{d}x.$$

By assumption $F(s)$ is analytic for $\sigma \geq 1$. Now define $F_\epsilon(t) = F(1 + \epsilon + it)$ for $\epsilon > 0$. For $\lambda > 0$, we obtain

$$\begin{aligned} &\int_{-2\lambda}^{2\lambda} F_\epsilon(t) \left(1 - \frac{|t|}{2\lambda}\right) \exp(iyt)\, \mathrm{d}t \\ &\quad = \int_{-2\lambda}^{2\lambda} \left(1 - \frac{|t|}{2\lambda}\right) \exp(iyt) \left(\int_0^{\infty} (B(x) - 1) \exp(-(\epsilon + it)x)\, \mathrm{d}x\right) \mathrm{d}t. \end{aligned} \tag{53}$$

Next we want to interchange the order of integration on the right-hand side. Since $A(x)$ is non-negative and non-decreasing, for real s and $x > 0$,

$$f(s) \geq A(x) \int_x^\infty \exp(-su)\,\mathrm{d}u = \frac{A(x)\exp(-sx)}{s},$$

resp. $A(x) \leq sf(s)\exp(sx)$. Since $f(s)$ is analytic for $\sigma > 1$, this implies $A(x) = O(\exp(sx))$ for any $s > 1$ and

$$B(x)\exp(-\delta x) = A(x)\exp(-(1+\delta)x) = o(1)$$

for every $\delta > 0$. It follows that the integral

$$\int_0^\infty (B(x)-1)\exp(-(\epsilon+it)x)\,\mathrm{d}x$$

converges uniformly for $-2\lambda \leq t \leq 2\lambda$. Thus, we can interchange the order of integration in (53) and obtain

$$\int_0^\infty (B(x)-1)\exp(-\epsilon x)\left(\int_{-2\lambda}^{2\lambda} \exp(i(y-x)t)\left(1-\frac{|t|}{2\lambda}\right)\mathrm{d}t\right)\mathrm{d}x.$$

This leads with (53) to

$$\begin{aligned}&\int_{-2\lambda}^{2\lambda} F_\epsilon(t)\left(1-\frac{|t|}{2\lambda}\right)\exp(iyt)\,\mathrm{d}t\\ &\quad = 2\int_0^\infty (B(x)-1)\exp(-\epsilon x)\frac{(\sin(\lambda(y-x)))^2}{\lambda(y-x)^2}\,\mathrm{d}x. \qquad (54)\end{aligned}$$

Since $F(s)$ is analytic in $\sigma \geq 1$, it follows that $F_\epsilon(t)$ tends to $F(1+it)$ as $\epsilon \to 0$, uniformly for $-2\lambda \leq t \leq 2\lambda$. Moreover,

$$\lim_{\epsilon\to 0}\int_0^\infty \exp(-\epsilon x)\frac{(\sin(\lambda(y-x)))^2}{\lambda(y-x)^2}\,\mathrm{d}x = \int_0^\infty \frac{(\sin(\lambda(y-x)))^2}{\lambda(y-x)^2}\,\mathrm{d}x.$$

Applying the theorem on monotone convergence, we deduce

$$\lim_{\epsilon\to 0}\int_0^\infty B(x)\exp(-\epsilon x)\frac{(\sin(\lambda(y-x)))^2}{\lambda(y-x)^2}\,\mathrm{d}x = \int_0^\infty B(x)\frac{(\sin(\lambda(y-x)))^2}{\lambda(y-x)^2}\,\mathrm{d}x.$$

By (54),

$$\frac{1}{2}\int_{-2\lambda}^{2\lambda} F(1+it)\left(1-\frac{|t|}{2\lambda}\right)\exp(iyt)\,\mathrm{d}t = \int_0^\infty (B(x)-1)\frac{(\sin(\lambda(y-x)))^2}{\lambda(y-x)^2}\,\mathrm{d}x. \qquad (55)$$

The Riemann-Lebesgue lemma states that

$$\lim_{y\to\infty}\int_{-\infty}^\infty f(x)\exp(ixy)\,\mathrm{d}x = 0$$

for any absolutely integrable function f. Thus, letting $y \to \infty$, the left-hand side of (55) tends to zero while

$$\lim_{y\to\infty}\int_0^\infty \frac{(\sin(\lambda(y-x)))^2}{\lambda(y-x)^2}\,\mathrm{d}x = \lim_{y\to\infty}\int_{-\infty}^{\lambda y}\left(\frac{\sin v}{v}\right)^2\mathrm{d}v = \pi. \qquad (56)$$

Hence,

$$\lim_{y\to\infty}\int_0^\infty B(x)\frac{(\sin(\lambda(y-x)))^2}{\lambda(y-x)^2}\,\mathrm{d}x = \pi;$$

this proves (52).

In order to prove the theorem we have to show

$$1 \leq \liminf_{x\to\infty} B(x) \leq \limsup_{x\to\infty} B(x) \leq 1. \qquad (57)$$

Clearly, this implies the existence of the limit $\lim_{x\to\infty} B(x)$ and that this limit is equal to 1. For given positive numbers a and λ let $y > \frac{a}{\lambda}$. By (52),

$$\limsup_{y\to\infty} \int_{-a}^{a} B\left(y-\frac{v}{\lambda}\right)\left(\frac{\sin v}{v}\right)^2 \,\mathrm{d}v \le \pi$$

(the integrand is non-negative). Since $A(u) = B(u)\exp(u)$ is non-decreasing, we have, for $-a \le v \le a$,

$$B\left(y-\frac{a}{\lambda}\right)\exp\left(y-\frac{a}{\lambda}\right) \le B\left(y-\frac{v}{\lambda}\right)\exp\left(y-\frac{v}{\lambda}\right).$$

This implies

$$B\left(y-\frac{v}{\lambda}\right) \ge B\left(y-\frac{a}{\lambda}\right)\exp\left(\frac{v-a}{\lambda}\right) \ge B\left(y-\frac{a}{\lambda}\right)\exp\left(-\frac{2a}{\lambda}\right).$$

Hence,

$$\begin{aligned}&\limsup_{y\to\infty} B\left(y-\frac{a}{\lambda}\right)\exp\left(-\frac{2a}{\lambda}\right)\int_{-a}^{a}\left(\frac{\sin v}{v}\right)^2 \,\mathrm{d}v\\ &= \limsup_{y\to\infty}\int_{-a}^{a} B\left(y-\frac{v}{\lambda}\right)\left(\frac{\sin v}{v}\right)^2 \,\mathrm{d}v \le \pi.\end{aligned}$$

For fixed a and λ we have $\limsup_{y\to\infty} B(y-\frac{a}{\lambda}) = \limsup_{y\to\infty} B(y)$. Thus,

$$\exp\left(-\frac{2a}{\lambda}\right)\limsup_{y\to\infty} B(y)\int_{-a}^{a}\left(\frac{\sin v}{v}\right)^2 \,\mathrm{d}v \le \pi,$$

being valid for all $a > 0$ and $\lambda > 0$. Since the right-hand side is independent of a and λ, letting $a, \lambda \to \infty$ such that $\frac{a}{\lambda} \to 0$, we deduce

$$\limsup_{y\to\infty} B(y)\int_{-\infty}^{\infty}\left(\frac{\sin v}{v}\right)^2 \,\mathrm{d}v \le \pi.$$

Now (56) implies the desired upper bound for $\limsup_{y\to\infty} B(y)$. The just proved inequality yields the existence of a constant c such that $|B(x)| \le c$. Hence, for fixed positive a and λ and a sufficiently large y,

$$\begin{aligned}&\int_{-\infty}^{\lambda y} B\left(y-\frac{v}{\lambda}\right)\left(\frac{\sin v}{v}\right)^2 \,\mathrm{d}v\\ &\le c\left\{\int_{-\infty}^{-a}+\int_{a}^{\infty}\right\}\left(\frac{\sin v}{v}\right)^2 \,\mathrm{d}v + \int_{-a}^{a} B\left(y-\frac{v}{\lambda}\right)\left(\frac{\sin v}{v}\right)^2 \,\mathrm{d}v.\end{aligned} \tag{58}$$

As above, we have $B(y-\frac{v}{\lambda}) \le B(y+\frac{a}{\lambda})\exp(\frac{2a}{\lambda})$ for $-a \le v \le a$. Therefore,

$$\int_{-a}^{a} B\left(y-\frac{v}{\lambda}\right)\left(\frac{\sin v}{v}\right)^2 \,\mathrm{d}v \le B\left(y+\frac{a}{\lambda}\right)\exp\left(\frac{2a}{\lambda}\right)\int_{-a}^{a}\left(\frac{\sin v}{v}\right)^2 \,\mathrm{d}v.$$

From (52), (58) and the latter inequality it follows that

$$\begin{aligned}\pi &\le c\left\{\int_{-\infty}^{-a}+\int_{a}^{\infty}\right\}\left(\frac{\sin v}{v}\right)^2 \,\mathrm{d}v +\\ &\quad + \liminf_{y\to\infty} B\left(y+\frac{a}{\lambda}\right)\exp\left(\frac{2a}{\lambda}\right)\int_{-a}^{a}\left(\frac{\sin v}{v}\right)^2 \,\mathrm{d}v.\end{aligned}$$

Here we may replace $\liminf_{y\to\infty} B(y+\frac{a}{\lambda})$ by $\liminf_{y\to\infty} B(y)$. Then, after sending $a, \lambda \to \infty$ such that $\frac{a}{\lambda} \to 0$, we get the desired lower bound for $\liminf_{y\to\infty} B(y)$. The theorem is proved. •

Now we shall derive Theorem 9 to which we also refer to as the Theorem of Wiener & Ikehara. For the sake of convenience, we recall its formulation: *Let* $F(s)=\sum_{n=1}^{\infty}a(n)n^{-s}$ *be a Dirichlet series with non-negative real coefficients and absolutely convergent for* $\mathrm{Re}\, s>1$. *Assume that* $F(s)$ *can be extended to a meromorphic function in* $\mathrm{Re}\, s\geq 1$ *such that there are no poles except for a possible simple pole at* $s=1$ *with residue* $r\geq 0$. *Then*

$$A(x):=\sum_{n\leq x}a(n)\sim rx.$$

Proof. Without loss of generality we may suppose that the residue is positive, $r>0$, since otherwise we consider the function $F(s)+\zeta(s)$ (which then has residue $r+1=1$). Furthermore, we may assume that $r=1$ simply by replacing $a(n)$ by $a(n)/r$. By partial summation,

$$F(s)=s\int_1^{\infty}\frac{A(x)}{x^{s+1}}\,\mathrm{d}x,$$

resp.

$$\frac{F(s)}{s}=\int_0^{\infty}A(\exp(y))\exp(-ys)\,\mathrm{d}y$$

with $x=\exp(y)$. Taking all assumptions on $F(s)$ into account it follows from Theorem 35 that

$$\lim_{y\to\infty}A(\exp(y))\exp(-y)=1.$$

Re-substituting $x=\exp(y)$ we get the assertion. •

Our presentation followed Chandrasekharan [5]. The standard reference for Tauberian theorems is Korevaar's book [27]. Another interesting approach is offered by M.R. Murty & V.K. Murty [38].

Appendix D: Artin L-Functions

We have included Artin L-functions for several reasons. Firstly, they are important objects in Algebraic Number Theory, resp. the analytic theory of algebraic numbers. Secondly, they are related to certain elements of the Selberg class in a rather sophisticated way. Exploiting these deep relations, one may hope to prove that they themselves are elements of the Selberg class too. Our approach, however, is exemplarily, and follows the articles of Heilbronn [15] and Stark [51].

Already the definition of Artin L-functions is non-trivial. Let $\mathbb{L}/\mathbb{K}$ be a Galois extension of number fields with Galois group G. Further, let $\rho:\mathsf{G}\to\mathsf{GL}_m(V)$ be a representation (group homomorphism) of G on a finite dimensional complex vector space V. Then the Artin L-function is defined by

$$L(s,\rho,\mathbb{L}/\mathbb{K})=\prod_{\mathfrak{p}}\det\left(1-\frac{\rho(\sigma_{\mathfrak{p}})}{\mathrm{N}(\mathfrak{p})^s}\right)^{-1}, \tag{59}$$

where $\mathfrak{p}$ runs through the prime ideals of the ring of integers in $\mathbb{K}$, and $\sigma_{\mathfrak{P}}$ is the Frobenius automorphism; for a precise definition of Artin L-functions we refer to [38]. To give an idea about these objects, let us briefly consider an example due to Stark. Assume that $\mathbb{L}/\mathbb{Q}$ is normal with Galois group equal to the symmetric group S_3 on three letters:

$$\mathsf{G}:=\{1,(\alpha\beta\gamma),(\alpha\gamma\beta),(\alpha\beta),(\alpha\gamma),(\beta\gamma)\}$$

say. For instance, one may consider the cubic field $\mathbb{K} = \mathbb{Q}(2^{\frac{1}{3}})$ and its normal closure $\mathbb{L} = \mathbb{Q}(\alpha, e^{\frac{2\pi i}{3}}) = \mathbb{Q}(\alpha, \beta, \gamma)$, where

$$\alpha = 2^{\frac{1}{3}}, \quad \beta = e^{\frac{2\pi i}{3}} 2^{\frac{1}{3}}, \quad \gamma = e^{\frac{4\pi i}{3}} 2^{\frac{1}{3}}.$$

Since automorphisms of $\mathbb{L}$ are determined by their action on α, β and γ, we find $\mathsf{G} = \mathsf{Gal}(\mathbb{L}/\mathbb{Q})$ for the Galois group of $\mathbb{L}$. The splitting of primes from $\mathbb{Q}$ to $\mathbb{K}$, and likewise from $\mathbb{K}$ to L, is ruled by the Frobenius automorphisms. The conjugacy classes of the symmetric group on α, β, γ are precisely the conjugacy classes of Frobenius automorphisms arising from prime numbers; that are

$$\mathsf{C}_1 : \{1\}, \qquad \mathsf{C}_2 : \{(\alpha\beta\gamma), (\alpha\gamma\beta)\}, \qquad \mathsf{C}_3 : \{(\alpha\beta), (\alpha\gamma), (\beta\gamma)\}.$$

For each of them we associated Euler factors corresponding to the splitting of the prime numbers. Since the field extension $\mathbb{K}/\mathbb{Q}$ has degree 3, there are the following possibilities to consider.

- The prime p splits completely into three different prime divisors; e.g., $(31) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$ with
$$\mathfrak{p}_1 = (31, \alpha - 4), \quad \mathfrak{p}_2 = (31, \alpha - 7), \quad \mathfrak{p}_3 = (31, \alpha - 20).$$
In this case the local Euler factor at p is of the form
$$\left(1 - \frac{1}{p^s}\right)^{-3} = \det \left(1 - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{p^s}\right)^{-1}.$$
Obviously, the appearing matrix has the eigenvalue $+1$ with multiplicity three. This Euler factor corresponds to the class C_1.
- The prime p can be factored into a product of two factors, one of first degree and one of second degree; for example, $(5) = \mathfrak{p}_1\mathfrak{p}_2$ with
$$\mathfrak{p}_1 = (5, \alpha - 3), \quad \mathfrak{p}_2 = (5, \alpha^2 + 3\alpha + 9).$$
Here we have
$$\left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{1}{p^{2s}}\right)^{-1} = \det \left(1 - \mathsf{M}\frac{1}{p^s}\right)^{-1},$$
for any of the matrices
$$\mathsf{M} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
corresponding to C_3. The eigenvalues of the (similar) matrices are -1 and $+1$ with multiplicities one and two, respectively.
- The prime p is a prime ideal of third degree; e.g., $(7) = \mathfrak{p}$. In this case we have
$$\left(1 - \frac{1}{p^{3s}}\right)^{-1} = \det \left(1 - \mathsf{M}\frac{1}{p^s}\right)^{-1}$$
for the matrices associated with C_2:
$$\mathsf{M} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$
Here the eigenvalues of the (similar) matrices are the third roots of unity.

To find a more convenient notation of Artin L-functions, for any representation ρ of G, we associate a character χ of G by setting

$$\chi(g) = \mathrm{trace}(\rho(g))$$

for $g \in \mathsf{G}$. The degree of a character is defined by $\deg \chi = \chi(1)$. These characters χ of G are constant on the conjugacy classes. Two representations are said to be equivalent if they have the same character. If ρ_1 and ρ_2 are representations of G with characters χ_1 and χ_2, then

$$\rho(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$$

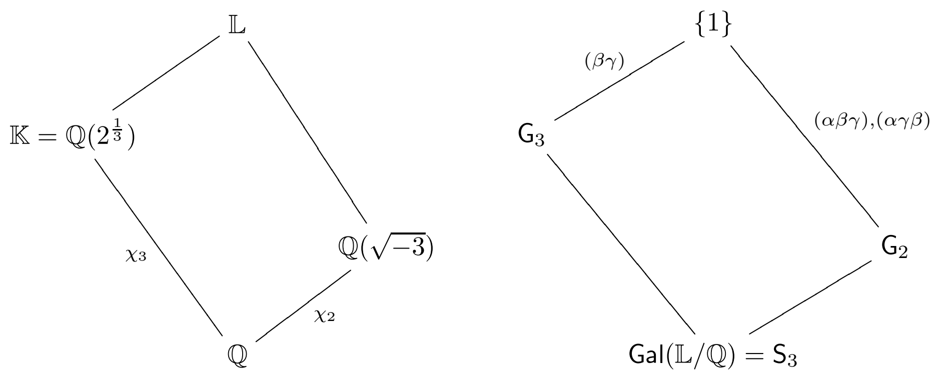
also defines a representation of G with character $\chi_1 + \chi_2$, and in this case ρ is said to be reducible; any representation which is not reducible is called irreducible. We shall use the same attributes for the associated character. It turns out that any conjugacy class of G corresponds to an irreducible representation and there are not more; of course, distinct irreducible representations are non-equivalent (these observations are analogous to the case of Dirichlet characters and the group of residue classes of $\mathbb{Z}$). In our example we find for the the three conjugacy classes of G:

	C_1	C_2	C_3
χ_1	$+1$	$+1$	$+1$
χ_2	$+1$	$+1$	-1
χ_3	$+2$	-1	0

Hence, for $\mathsf{G} = \mathsf{S}_3$, there are three irreducible characters (in some literature 'simple characters'). For simplicity, we write $L(s, \chi)$ for the Artin L-function (59). One can construct more characters from the irreducible characters listed above, for example, a third degree character χ related to the permutation representation $(\alpha\beta)$. Taking the character relations into account we find $\chi = \chi_1 + \chi_3$. For the related Artin L-functions we note that

$$L(s, \chi) = L(s, \chi_1 + \chi_3) = L(s, \chi_1)L(s, \chi_3).$$

For the field $\mathbb{L} = \mathbb{Q}(\alpha, \beta, \gamma)$ there are four subfields up to conjugacy: firstly, the field $\mathbb{Q}$ itself, fixed by all of G, secondly, $\mathbb{Q}(\sqrt{-3})$ fixed by $\mathsf{G}_2 := \{1, (\alpha\beta\gamma), (\alpha\gamma\beta)\}$ (corresponding to the conjugacy class C_2), thirdly, $\mathbb{K} = \mathbb{Q}(2^{\frac{1}{3}})$ fixed by $\mathsf{G}_3 := \{1, (\beta\gamma)\}$ (corresponding to the conjugacy class C_3), and finally $\mathbb{L}$ fixed just by $\{1\}$.



We obtain the following factorizations of the associated Dedekind zeta-functions into products of Artin L-functions to $\mathbb{L}/\mathbb{Q}$:

$$\begin{aligned}\zeta(s) = \zeta_{\mathbb{Q}}(s) &= L(s,\chi_1),\\ \zeta_{\mathbb{Q}(\sqrt{-3})}(s) &= L(s,\chi_1)\,L(s,\chi_2),\\ \zeta_{\mathbb{Q}(2^{\frac{1}{3}})}(s) &= L(s,\chi_1)\,L(s,\chi_3),\\ \zeta_{\mathbb{L}}(s) &= L(s,\chi_1)\,L(s,\chi_2)\,L(s,\chi_3).\end{aligned}$$

We observe that any of the Dedekind zeta-functions on the left-hand side is divisible by the Riemann zeta-function. It follows from these factorizations and the analytic behaviour of Dedekind zeta-functions that each of the involved Artin L-functions with $\chi \neq \chi_0$ possesses a meromorphic continuation to the whole complex plane; the only possible poles can occur at zeros of other Artin L-functions. Furthermore, one can deduce functional equations of the Riemann-type. For instance, in our example we may deduce directly from the functional equations for the Dedekind zeta-function

$$a^s\Gamma(s)L(s,\chi_3) = a^{1-s}\Gamma(1-s)L)1-s,\chi)$$

with some constant a which can be computed by means of Algebraic Number Theory; in a similar manner further identities hold for the other Artin L-functions. This is a remarkable way to deduce analytic properties for L-functions! If $\mathbb{L}/\mathbb{K}$ is abelian, then it follows from Artin reciprocity law that $L(s,\mathbb{L}/\mathbb{K},\rho)$ coincides with a suitable Hecke L-function. With (24) we have seen a toy example of this phenomenon in §2 where the Dedekind zeta-function to a quadratic number field split into the product of the Riemann zeta-function and a Dirichlet L-function.

The Langlands program has emerged in the late 1960s of the last century in a series of far-reaching conjectures tying together seemingly unrelated objects in number theory, algebraic geometry, and the theory of automorphic forms. These disciplines are linked by Langlands' L-functions associated with automorphic representations, and by the relations between the analytic properties and the underlying algebraic structures. There are two kinds of L-functions: *motivic* L-functions which generalize Artin L-functions and are defined purely arithmetically, and *automorphic* L-functions, defined by transcendental data. In its comprehensive form, an identity between a motivic L-function and an automorphic L-function is called a reciprocity law. Langlands' reciprocity conjecture claims, roughly, that every L-function, motivic or automorphic, is equal to a product of L-functions attached to automorphic representations. One instance is Wiles' et al. celebrated proof of the Shimura-Taniyama conjecture on the modularity of elliptic curves with Fermat's last theorem on the integer solutions of the diophantine equation $X^n + Y^n = Z^n$ as corollary; see . For an introduction to the Langlands program we refer to the excellent surveys of Gelbart [12] and M.R. Murty [35]. This is the universe of the Selberg class $\mathcal{S}$ and the general converse problem whether $\mathcal{S}$ is built exactly from the automorphic L-functions may be regarded as an analytic version of the Langlands program.

References

[1] E.P. Balanzario, Remark on Dirichlet series satisfying functional equations, *Divulg. Mat.* **8** (2000), 169-175

[2] E. Bombieri, The Classical Theory of Zeta and L-Functions, *Milan J. Math.* **78** (2010), 11-59

[3] E. Bombieri, D.A. Hejhal, On the distribution of zeros of linear combinations of Euler products, *Duke Math. J.* **80** (1995), 821-862

[4] E. Bombieri, A. Perelli, Distinct zeros of L-functions, *Acta Arith.* **83** (1998), 271-281

[5] K. Chandrasekharan, *Arithmetical Functions*, Springer 1970

[6] K. Chandrasekharan, R. Narasimhan, The approximate functional equation for a class of zeta-functions, *Math. Ann.* **152** (1963), 30-64

[7] J.B. Conrey, A. Ghosh, On the Selberg class of Dirichlet series: small degrees, *Duke Math. J.* **72** (1993), 673-693

[8] H. Davenport, *Multiplicative Number Theory*, Springer 2000, 3rd ed.

[9] A. Fujii, Zeta zeros and Dirichlet L-functions, *Proc. Japan Acad. Sci.* **64** (1988), 215-218

[10] R. Garunkštis, J. Grahl, J. Steuding, Uniqueness Theorems For L-Functions, submitted

[11] R. Garunkštis, A. Laurinčikas, *The Lerch zeta-function*, Kluwer Academic Publishers, Dordrecht 2002

[12] S.S. Gelbart, An elementary introduction to the Langlands program, *Bull. Amer. Math. Soc.* **10** (1984), 177-219

[13] S.S. Gelbart, S.D. Miller, Riemann's zeta function and beyond, *Bull. Amer. Math. Soc.* **41** (2004), 59-112

[14] H. Hamburger, Über die Riemannsche Funktionalgleichung der ζ-Funktion. I, II., *Math. Z.* **10** (1921), 240-254; **11** (1921), 224-245

[15] H. Heilbronn, Zeta-functions and L-functions, in: '*Algebraic Number Theory*', J.W.S. Cassels and A. Fröhlich (eds.), Academic Press 1967, 204-230

[16] M.N. Huxley, Exponential sums and the Riemann zeta-function, V, *Proc. London Math. Soc.* **90** (2005), 1-41

[17] H. Iwaniec, *Topics in classical automorphic forms*, Amer. Math. Soc., Providence 1997

[18] H. Iwaniec, E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc., Providence 2004

[19] H. Jacquet, J.A. Shalika, A non-vanishing theorem for zeta-functions on GL_n, *Invent. math.* **38** (1976), 1-16

[20] J. Kaczorowski, Axiomatic theory of L-functions: the Selberg class, in: *Ananlytic Number Theory*, Lecture Notes in Math. **1891**, Springer 2006, 133-209

[21] J. Kaczorowski, G. Molteni, A. Perelli, J. Steuding, J. Wolfart, Hecke's theory and the Selberg class, *Funct. Approx. Comment. Math.* **XXXV** (2006), 183-194

[22] J. Kaczorowski, A. Perelli, On the structure of the Selberg class, I: $0 \leq d \leq 1$, *Acta Math.* **182** (1999), 207-241

[23] J. Kaczorowski, A. Perelli, On the structure of the Selberg class, VII: $1 \leq d < 5/3$, *Invent. Math.* **150** (2002), 485-516

[24] J. Kaczorowski, A. Perelli, On the structure of the Selberg class, VI: non-linear twists, *Acta Arith.* **116** (2005), 315-341

[25] J. Kaczorowski, A. Perelli, On the structure of the Selberg class, VII: $1 \leq d < 2$, *Ann. of Math.* (to appear)

[26] N. Koblitz, *Introduction to elliptic curves and modular forms*, Springer 1993, 2nd ed.

[27] J. Korevaar, *Tauberian theory. A century of developments*, Springer 2004

[28] D. Leitmann, D. Wolke, Periodische und multiplikative zahlentheoretische Funktionen, *Monatsh. Math.* **81** (1976), 279-289

[29] X.-J. Li, The positivity of a sequence of numbers and the Riemann hypothesis, *J. Number Theory* **65** (1997), 325-333

[30] J. Liu, Y. Wang, Y. Ye, A proof of Selberg's orthogonality for automorphic L-functions, *Manuscripta math.* **118** (2005), 135-149

[31] K. Mazhouda, S. Omar, Le critére de Li et l'hypothése de Riemann pour la classe de Selberg, *J. Number Theory* **125** (2007), 50-58; corrigendum et addendum, *J. Number Theory* **130** (2010), 1109-1114

[32] K. Mazhouda, S. Omar, The Li criterion and the Riemann hypothesis for the Selberg class, II, *J. Number Theory* **130** (2010), 1098-1108

[33] K. Mazhouda, S. Omar, Mean-square of L-functions in the Selberg class, in: *New directions in value-distribution theory of zeta and L-functions*, Proceedings of Würzburg Conference 2008, R. & J. Steuding (eds.), Shaker Verlag 2009, 249-263

[34] G. Molteni, J. Steuding, (Almost) primitivity of Hecke L-functions, *Monatshefte Math.* **152** (2007), 63-71

[35] M.R. Murty, A motivated introduction to the Langlands program, in: '*Advances in Number theory*', F. Gouvea and N. Yui, eds., Clarendon Press Oxford 1993, 37-66

[36] M.R. Murty, Selberg's conjectures and Artin L-functions, II, in *Current trends in mathematics and physics. A tribute to Harish-Chandra*, Adhikari, S.D. (ed.), New Delhi: Narosa Publishing House. 154-168 (1995)

[37] M.R. Murty, V.K. Murty, Strong multiplicity one for Selberg's class, *C.R. Acad. Sci.* **319** (1994), 315-320

[38] M.R. Murty, V.K. Murty, *Non-vanishing of L-functions and applications*, Birkhäuser 1997

[39] H. Nagoshi, J. Steuding, Universality for L-functions in the Selberg class, *Lith. Math. J.* **50** (2010), 293-311

[40] W. Narkiewicz, *Elementary and analytic theory of algebraic numbers*, Springer 2004, 3rd ed.

[41] A. Ogg, *Modular forms and Dirichlet series*, W.A. Benjamin, Inc. XVI, 1969

[42] A. Perelli, A survey of the Selberg class of L-functions, Part I, *Milan J. Math.* **73** (2005), 19-52

[43] A. Perelli, A survey of the Selberg class of L-functions, Part II, *Riv. Mat. Univ. Parma* **3** (2004), 83-118

[44] A. Perelli, Non-Linear Twists of L-Functions, *Milan J. Math.* **78** (2010), 117-134

[45] J. Sander, J. Steuding, Joint universality for sums and products of Dirichlet L-functions, *Analysis* **26** (2006), 295-312

[46] W. Schwarz, J. Spilker, *Arithmetical Functions*, Cambridge University Press 1994

[47] A. Selberg, Old and new conjectures and results about a class of Dirichlet series, in: '*Proceedings of the Amalfi Conference on Analytic Number Theory*', Maiori 1989, E. Bombieri et al. (eds.), Università di Salerno 1992, 367-385

[48] C.L. Siegel, Bemerkungen zu einem Satz von Hamburger über die Funktionalgleichung der Riemannschen Zetafunktion, *Math. Ann.* **86** (1922), 276-279

[49] K. Soundararajan, Degree 1 elements of the Selberg class, *Expo. Math.* **23** (2005), 65-70

[50] K. Srinivas, Distinct zeros of functions in the Selberg class, *Acta Arith.* **103** (2002), 201-207

[51] H.M. Stark, Galois theory, algebraic numbers and zeta functions, in: '*From number teory to physics*', Waldschmitd et al. (eds.), Springer 1989, 313-393

[52] J. Steuding, Dirichlet series associated to periodic arithmetic functions and the zeros of Dirichlet L-functions, in *Analytic and probabilistic methods in number theory. Proceedings of the third international conference in honour of J. Kubilius, Palanga, Lithuania, September 24–28, 2001*, Dubickas, A. (ed.) et al., TEV Vilnius 2002, 282-296

[53] J. Steuding, A note on the extended Selberg class (degrees 0 and 1), *Arch. Math.* **81** (2003), 22-25

[54] J. Steuding, *Value-distribution of L-functions*, Lecture Notes in Mathematics **1877**, Springer 2007.

[55] J. Steuding, *L-functions*, www.mathematik.uni-wuerzburg.de/~steuding/seminario0.pdf.

[56] E.C. Titchmarsh, *The theory of functions*, Oxford University Press, 2nd ed. 1939

[57] E.C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford University Press 1986, 2nd ed., revised by D.R. Heath-Brown

[58] S.M. Voronin. On the functional independence of Dirichlet L-functions. *Acta Arith.* **27** (1975), 493–503 (in Russian).

[59] A. Weil, Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, *Math. Ann.* **168** (1967), 149-156

[60] A. Weil, *Elliptic functions according to Eisenstein and Kronecker*, Springer 1976

Last, but not least, I would like to express my gratitude for the kind invitation to Tunisia and for the opportunity to lecture at Monastir University. In particular, I would like to thank Prof. Kamel Mazhouda for the perfect organization. As more or less any longer text there are misprints or even errors in these course notes for which I apologize. Corrections and remarks are most welcome!

Jörn Steuding[†]

[†]Department of Mathematics, Würzburg University, Am Hubland, 97 218 Würzburg, Germany, steuding@mathematik.uni-wuerzburg.de