MINORS OF A CLASS OF RIORDAN ARRAYS RELATED TO WEIGHTED PARTIAL MOTZKIN PATHS

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Abstract. A partial Motzkin path is a path from $(0,0)$ to (n,k) in the XOY-plane that does not go below the X-axis and consists of up steps $U=(1,1)$, down steps $D=(1,-1)$ and horizontal steps $H=(1,0)$. A weighted partial Motzkin path is a partial Motzkin path with the weight assignment that all up steps and down steps are weighted by 1, the horizontal steps are endowed with a weight x if they are lying on X-axis, and endowed with a weight y if they are not lying on X-axis. Denote by $M_{n,k}(x,y)$ to be the weight function of all weighted partial Motzkin paths from $(0,0)$ to (n,k), and $\mathcal{M}=(M_{n,k}(x,y))_{n\geq k\geq 0}$ to be the infinite lower triangular matrices. In this paper, we consider the sums of minors of second order of the matrix $\mathcal{M}$, and obtain a lot of interesting determinant identities related to $\mathcal{M}$, which are proved by bijections using weighted partial Motzkin paths. When the weight parameters (x,y) are specialized, several new identities are obtained related to some classical sequences involving Catalan numbers. Besides, in the alternating cases we also give some new explicit formulas for Catalan numbers.

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1. Introduction

The starting point for this paper is the observation that the close connections between the Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$ and the Pascal triangle $\mathscr{P}=(\binom{n}{k})_{n\geq k\geq 0}$, that is,

$$C_{n+1}=\sum_{k=0}^{n}N_{n+1,k+1}=\sum_{k=0}^{n}\det\left(\begin{array}{cc}\binom{n}{k} & \binom{n}{k+1}\\ \binom{n+1}{k} & \binom{n+1}{k+1}\end{array}\right),$$

$$C_{n+1}=\sum_{k=0}^{n}N_{n+1,k+1}=\sum_{k=0}^{n}\det\left(\begin{array}{cc}\binom{n}{k} & \binom{n+1}{k+1}\\ \binom{n+1}{k} & \binom{n+2}{k+1}\end{array}\right),$$

where $N_{n,k}=\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ for $n\geq k\geq 1$ are the Narayana numbers.

This makes us to do some numerical verification for other combinatorial triangles. For example, consider Shapiro's Catalan triangle [12], defined by $\mathcal{B}=(B_{n,k})_{n\geq k\geq 0}$ such that $B_{n,k}=\frac{k+1}{n+1}\binom{2n+2}{n-k}$. Table 1 illustrates this triangle for small n and k up to 5.

n/k	0	1	2	3	4	5
0	1					
1	2	1				
2	5	4	1			
3	14	14	6	1		
4	42	48	27	8	1	
5	132	165	110	44	10	1

Table 1. The values of $B_{n,k}$ for n and k up to 5.

Let $\mathcal{X} = (X_{n,k})_{n\geq k\geq 0}$ be the infinite lower triangles defined on the triangle $\mathcal{B}$ by

$$X_{n,k} = \det \begin{pmatrix} B_{n,k} & B_{n,k+1} \\ B_{n+1,k} & B_{n+1,k+1} \end{pmatrix}.$$

Table 1.2 illustrates the triangle $\mathcal{X}$ for small n and k up to 4, together with the row sums. It indicates that the row sums have close relation with the first column of the triangle $\mathcal{B}$.

n/k	0	1	2	3	4	*row sums*
0	1					$1 = 1^2$
1	3	1				$4 = 2^2$
2	14	10	1			$25 = 5^2$
3	84	90	21	1		$196 = 14^2$
4	594	825	308	36	1	$1764 = 42^2$

Table 1.2. The values of $X_{n,k}$ for n and k up to 4, together with the row sums.

The above fact motivates us to consider the following problem.

Question: Let $\mathcal{A} = (A_{n,k})_{n\geq k\geq 0}$ be an infinite lower triangular matrix with nonzero entries on the main diagonal. Given integers m, r, ℓ, p with $m, \ell, p \geq 0$, define a transformation on $\mathcal{A}$ by $\mathcal{A}_p = \left(A_{n,k}^{(p)}(m, r, \ell)\right)_{n\geq k\geq 0}$, where

$$A_{n,k}^{(p)}(m, r, \ell) = \det\left(A_{n+im+jr,k+j\ell}\right)_{0\leq i,j\leq p}.$$

Then how to determine the explicit expression for the n-th row sum of $\mathcal{A}_p$,

$$S_{n,m,r,\ell}^{(p)}(\mathcal{A}_p) = \sum_{k=0}^{n} A_{n,k}^{(p)}(m, r, \ell)?$$

In general, it is not easy to give an exact answer for this question. But, in the case $p = 1$, for some special infinite lower triangular matrices related to weighted partial Motzkin paths, it can produce several surprising results.

The organization of this paper is as follows. The next section gives a brief introduction to weighted partial Motzkin paths, which generates a class of infinite lower triangular matrices. In Section 3, we state our main results and give bijective proofs. When the weight parameters are specialized, several new identities are obtained related to some classical sequences involving Catalan numbers. In Section 4, we consider the alternating sums and give some new explicit formulas for Catalan numbers.

2. Weighted partial Motzkin paths

Recall that a *Motzkin path* is a lattice path from $(0, 0)$ to $(n, 0)$ in the XOY-plane that does not go below the X-axis and consists of up steps $U = (1, 1)$, down steps $D = (1, -1)$ and horizontal steps $H = (1, 0)$. A *partial Motzkin path*, also called a Motzkin path from $(0, 0)$ to (n, k) in [2], is just a Motzkin path but without the requirement of ending on the X-axis. A *weighted partial Motzkin path* [15] is a partial Motzkin path with the weight assignment that the all up steps and down steps are weighted by 1, the horizontal steps are endowed with a weight x if they are lying on X-axis, and endowed with a weight y if they are not lying on X-axis. The *weight* $w(P)$ of a path P is the product of the weight of all its steps. The *weight* of a set of paths is the sum of the total weights of all the paths. If $P = L_1L_2 \ldots L_{n-1}L_n$ is aweighted partial Motzkin path of length n, denoted by $\overline{P} = \overline{L}_n\overline{L}_{n-1} \ldots \overline{L}_2\overline{L}_1$ the reverse of the path P, where $\overline{L}_i = U$ if $L_i = D$, $\overline{L}_i = D$ if $L_i = U$ and $\overline{L}_i = H$ if $L_i = H$. For any step, we say that it is at level i if the Y-coordinate of its end point is i. An up step at level i is *R-visible* [4] if it is the rightmost up step at level i and there are no other up steps at the same level to its right. It is also worth mentioning that another type of weighted partial Motzkin paths is used by Chen, Li, Shapiro and Yan [4] to derive many nice matrix identities related to a class of Riordan arrays.

Let $\mathscr{M}_{n,k}(x,y)$ denote the set of weighted partial Motzkin paths from $(0,0)$ to (n,k), and $M_{n,k}(x,y)$ be its weight. Define $\mathscr{M}_k(x,y)=\bigcup_{n\geq k}\mathscr{M}_{n,k}(x,y)$, it should be pointed out that any $P\in\mathscr{M}_k(x,y)$ has exactly k R-visible up steps. For any $P\in\mathscr{M}_{n,k}(x,y)$, according to the last step U,H or D of P, one can easily deduce the following recurrences for $M_{n,k}(x,y)$,

$$\begin{aligned} M_{n,0}(x,y) &= xM_{n-1,0}(x,y)+M_{n-1,1}(x,y),\ (n\geq 1),\\ M_{n,k}(x,y) &= M_{n-1,k-1}(x,y)+yM_{n-1,k}(x,y)+M_{n-1,k+1}(x,y),\ (n\geq k\geq 1), \end{aligned} \tag{2.1}$$

with $M_{0,0}(x,y)=1$ and $M_{n,k}(x,y)=0$ if $n<k$ or $k<0$.

n/k	0	1	2	3	4
0	1				
1	x	1			
2	x^2+1	$x+y$	1		
3	x^3+2x+y	x^2+xy+y^2+2	$x+2y$	1	
4	$x^4+3x^2+2xy+y^2+2$	$x^3+x^2y+xy^2+3x+y^3+5y$	$x^2+2xy+3y^3+3$	$x+3y$	1

Table 1. The values of $M_{n,k}(x,y)$ for n and k up to 4.

Denote $(M_{n,k}(x,y))_{n\geq k\geq 0}$ by $\mathcal{M}$, then $\mathcal{M}$ is an infinite lower triangular matrix with the main diagonal entries 1. Table 1 illustrates this matrix for small n and k up to 4. In fact, the matrix $\mathcal{M}$ forms a Riordan array. Recall that $\mathcal{R}=(R_{n,k})_{n\geq k\geq 0}$ is a Riordan array [13, 14, 16] if it is an infinite lower triangular matrix with nonzero entries on the main diagonal, such that $R_{n,k}=[t^n]g(t)(f(t))^k$ for $n\geq k$, namely, $R_{n,k}$ equals the coefficient of t^n in the expansion of the series $g(t)(f(t))^k$, where $g(t)=1+g_1t+g_2t^2+\cdots$ and $f(t)=f_1t+f_2t^2+\cdots$ with $f_1\neq 0$ are two formal power series. It is convenient to denote the Riordan array $\mathcal{R}$ by $(g(t),f(t))$. Let $M_k(x,y;t)=\sum_{n\geq k}M_{n,k}(x,y)t^n$ be the generating function of weighted partial Motzkin paths ending at level k. For any $P\in\mathscr{M}_k(x,y)$, according to the k R-visible up steps for $k\geq 1$, P can be uniquely partitioned into $P=P_0UP_1U\ldots UP_k$, where $P_0\in\mathscr{M}_0(x,y)$ and $P_i\in\mathscr{M}_0(y,y)$ for $1\leq i\leq k$. This decomposition produces a relation between $M_k(x,y;t)$ and $M_0(x,y;t)$, namely,

$$M_k(x,y;t)=M_0(x,y;t)(tM_0(y,y;t))^k,\ (k\geq 1),$$

which indicates that $\mathcal{M}=(M_0(x,y;t),tM_0(y,y;t))$ is a Riordan array.

For any $P\in\mathscr{M}_0(x,y)$, P has three cases to be considered, that is (1) $P=\varepsilon$, an empty path; (2) starting with a horizontal step, i.e., $P=HP_1$, where $P_1\in\mathscr{M}_0(x,y)$; (3) $P=UP_2DP_1$, where $P_1\in\mathscr{M}_0(x,y)$ and $P_2\in\mathscr{M}_0(y,y)$. Making use of the so-called symbol method (for details see [11]), we obtain

$$M_0(x,y;t)=1+xtM_0(x,y;t)+t^2M_0(y,y;t)M_0(x,y;t).$$

Solving this equation, we have

$$M_0(y,y;t)=\frac{1-yt-\sqrt{(1-yt)^2-4t^2}}{2t^2}, \tag{2.2}$$

$$M_0(x,y;t)=\frac{1}{1-xt-t^2M_0(y,y;t)}=\frac{1-2xt+yt-\sqrt{(1-yt)^2-4t^2}}{2(y-x)(1-xt)t+2t^2}. \tag{2.3}$$

When the parameters x and y are specialized, $M_0(x,y;t)$ produces generating functions for many classical combinatorial sequences. We give a short list in Table 2, where $C(t)=\frac{1-\sqrt{1-4t}}{2t}$ and $M(t)=\frac{1-t-\sqrt{1-2t-3t^2}}{2t^2}$ are generating functions respectively for Catalan numbers C_n and Motzkin numbers M_n.

(x,y)	$M_0(x,y;t)$	*Sequences*	$\mathcal{M}$
$(0,0)$	$C(t^2)$	$A126120 = C_{\frac{n}{2}}$	$A053121$ [15]
$(0,1)$	$\frac{1+t-\sqrt{1-2t-3t^2}}{2t(1+t)}$	$A005043 = \textit{Riordan numbers } R_n$	$A089942$ [15]
$(0,2)$	$\frac{1-\sqrt{1-4t}}{3-\sqrt{1-4t}}$	$A000957 = \textit{Fine numbers } F_n$	$A126093$ [15]
$(1,0)$	$\frac{1}{1-t-t^2C(t^2)}$	$A001405 = \binom{n}{\lfloor n/2\rfloor}$	$A061554$ [15]
$(1,1)$	$M(t)$	$A001006 = \textit{Motzkin numbers } M_n$	$A064189$ [15]
$(1,2)$	$C(t)$	$A000108 = \textit{Catalan numbers } C_n$	$A039599$ [15]
$(2,2)$	$C^2(t)$	$A000108 = \textit{Catalan numbers } C_{n+1}$	$A039598$ [15]
$(3,2)$	$\frac{C(t)}{\sqrt{1-4t}}$	$A001700 = \binom{2n+1}{n}$	$A111418$ [15]

Table 2. The specializations of (x,y), where $C_{\frac{n}{2}}$ is set to be zero when n is odd.

3. Main results and bijective proofs

Lemma 3.1. *There exists a bijection between the set $\mathscr{M}_{n,0}(y+1,y)$ and the set $\bigcup_{\ell=0}^{n}\mathscr{M}_{n,\ell}(y,y)$.*

Proof. For any $P\in\mathscr{M}_{n,0}(y+1,y)$, each H step of P on X-axis has weight $y+1$, or equivalently, it has weight y or 1. If P has ℓ H steps weighted by 1, replace each of them by a U step, we get a path $P^*\in\mathscr{M}_{n,\ell}(y,y)$.

Conversely, for any $P^*\in\mathscr{M}_{n,\ell}(y,y)$, it has exactly ℓ R-visible up steps, replace each of them by an H step, we get a path P has ℓ H steps which are weighted by 1 and lying on X-axis.

Clearly, the above process indeed forms a bijection between the set $\mathscr{M}_{n,0}(y+1,y)$ and the set $\bigcup_{\ell=0}^{n}\mathscr{M}_{n,\ell}(y,y)$. $\square$

Theorem 3.2. *Let $\mathcal{M}=(M_{n,k}(x,y))_{n\geq k\geq 0}$ be given in Section 2. For any integers $n,r\geq 0$ and $m\geq\ell\geq 0$, set $N_r=\min\{n+r+1,m+r-\ell\}$. Then there hold*

$$\sum_{k=0}^{N_r}\det\begin{pmatrix} M_{n,k}(x,y) & M_{m,k+\ell+1}(x,y)\\ M_{n+r+1,k}(x,y) & M_{m+r+1,k+\ell+1}(x,y)\end{pmatrix}=\sum_{i=0}^{r}M_{n+i,0}(x,y)M_{m+r-i,\ell}(y,y), \tag{3.1}$$

$$\sum_{\ell=0}^{m}\sum_{k=0}^{N_r}\det\begin{pmatrix} M_{n,k}(x,y) & M_{m,k+\ell+1}(x,y)\\ M_{n+r+1,k}(x,y) & M_{m+r+1,k+\ell+1}(x,y)\end{pmatrix}=\sum_{i=0}^{r}M_{n+i,0}(x,y)M_{m+r-i,0}(y+1,y). \tag{3.2}$$

Proof. Define

$$\mathscr{A}^{(r)}_{n,m,k,\ell}(x,y)=\{(P,Q)|P\in\mathscr{M}_{n,k}(x,y),Q\in\mathscr{M}_{m+r+1,k+\ell+1}(x,y)\},$$
$$\mathscr{B}^{(r)}_{n,m,k,\ell}(x,y)=\{(P,Q)|P\in\mathscr{M}_{n+r+1,k}(x,y),Q\in\mathscr{M}_{m,k+\ell+1}(x,y)\},$$

and $\mathscr{C}^{(r,i)}_{n,m,k,\ell}(x,y)$ to be the subset of $\mathscr{A}^{(r)}_{n,m,k,\ell}(x,y)$ such that for any $(P,Q)\in\mathscr{C}^{(r,i)}_{n,m,k,\ell}(x,y)$ $Q=Q_1UQ_2$ with $Q_1\in\mathscr{M}_{i,k}(x,y)$ and $Q_2\in\mathscr{M}_{m+r-i,\ell}(y,y)$ for $k\leq i\leq r$. In other words, for any $(P,Q)\in\mathscr{C}^{(r,i)}_{n,m,k,\ell}(x,y)$, Q satisfies the conditions that (a) the last $(\ell+1)$-th R-visible up step of Q stays at level $k+1$, and (b) there are exactly i steps immediately ahead of the last $(\ell+1)$-th R-visible up step of Q for $k\leq i\leq r$.

It is clear that the weights of the sets $\mathscr{A}^{(r)}_{n,m,k,\ell}(x,y)$ and $\mathscr{B}^{(r)}_{n,m,k,\ell}(x,y)$ are

$$w(\mathscr{A}^{(r)}_{n,m,k,\ell}(x,y)) = M_{n,k}(x,y)M_{m+r+1,k+\ell+1}(x,y),$$
$$w(\mathscr{B}^{(r)}_{n,m,k,\ell}(x,y)) = M_{n+r+1,k}(x,y)M_{m,k+\ell+1}(x,y).$$

Given $0 \le i \le r$, the weight of the set $\bigcup_{k=0}^{i} \mathscr{C}^{(r,i)}_{n,m,k,\ell}(x,y)$ is $M_{n+i,0}(x,y)M_{m+r-i,\ell}(y,y)$. In fact, this claim can be verified by the following argument. For any $(P,Q) \in \mathscr{C}^{(r,i)}_{n,m,k,\ell}(x,y)$, we have $Q = Q_1UQ_2$ as mentioned above with $Q_1 \in \mathscr{M}_{i,k}(x,y)$ and $Q_2 \in \mathscr{M}_{m+r-i,\ell}(y,y)$, then $P\overline{Q}_1 \in \mathscr{M}_{n+i,0}(x,y)$ such that the last $(i+1)$-th step of $P\overline{Q}_1$ is at level k. Summing k for $0 \le k \le i$, all $P\overline{Q}_1 \in \mathscr{M}_{n+i,0}(x,y)$ contribute the total weight $M_{n+i,0}(x,y)$ and all $Q_2 \in \mathscr{M}_{m+r-i,\ell}(y,y)$ contribute the total weight $M_{m+r-i,\ell}(y,y)$. Hence, $w(\bigcup_{k=0}^{i} \mathscr{C}^{(r,i)}_{n,m,k,\ell}(x,y)) = M_{n+i,0}(x,y)M_{m+r-i,\ell}(y,y)$, and then

$$w(\bigcup_{i=0}^{r}\bigcup_{k=0}^{i} \mathscr{C}^{(r,i)}_{n,m,k,\ell}(x,y)) = w(\bigcup_{k=0}^{r}\bigcup_{i=k}^{r} \mathscr{C}^{(r,i)}_{n,m,k,\ell}(x,y)) = \sum_{i=0}^{r} M_{n+i,0}(x,y)M_{m+r-i,\ell}(y,y).$$

Let $\mathscr{A}^{(r)}_{n,m,\ell}(x,y) = \bigcup_{k=0}^{N_r} \mathscr{A}^{(r)}_{n,m,k,\ell}(x,y)$, $\mathscr{B}^{(r)}_{n,m,\ell}(x,y) = \bigcup_{k=0}^{N_r} \mathscr{B}^{(r)}_{n,m,k,\ell}(x,y)$ and $\mathscr{C}^{(r)}_{n,m,\ell}(x,y) = \bigcup_{0\le k\le i\le r} \mathscr{C}^{(r,i)}_{n,m,k,\ell}(x,y)$. In order to prove (3.1), it suffices to construct a simple bijection ϕ between $\mathscr{A}^{(r)}_{n,m,\ell}(x,y) - \mathscr{C}^{(r)}_{n,m,\ell}(x,y)$ and $\mathscr{B}^{(r)}_{n,m,\ell}(x,y)$ such that the ϕ is still preserving the weights.

For any $(P,Q) \in \mathscr{A}^{(r)}_{n,m,k,\ell}(x,y) - \bigcup_{i=k}^{r} \mathscr{C}^{(r,i)}_{n,m,k,\ell}(x,y)$, Q has exactly $k+\ell+1$ R-visible up steps. We claim that there always exists a path Q' of length $r+1$ which is immediately ahead of the last $(\ell+1)$-th (also the $(k+1)$-th along the path) R-visible up step of Q. Otherwise, $Q' \in \mathscr{M}_{i,k}(x,y)$ for some $k \le i \le r$, and then $(P,Q) \in \bigcup_{i=k}^{r} \mathscr{C}^{(r,i)}_{n,m,k,\ell}(x,y)$, a contradiction.

For any $(P,Q) \in \mathscr{A}^{(r)}_{n,m,\ell}(x,y) - \mathscr{C}^{(r)}_{n,m,\ell}(x,y)$, find the path Q' of length $r+1$ which is immediately ahead of the last $(\ell+1)$-th R-visible up step of Q, namely, Q can be uniquely partitioned into $Q = Q_1Q'U^*Q_2$, where $Q_1 \in \mathscr{M}_j(x,y)$ for some $j \ge 0$, $Q_2 \in \mathscr{M}_\ell(y,y)$ and U^* is the last $(\ell+1)$-th R-visible up step of Q. Then we can construct $\phi(P,Q) = (P^*,Q^*) \in \mathscr{B}^{(r)}_{n,m,\ell}(x,y)$ as follows: (1) delete the path Q' in Q to get $Q^* = Q_1U^*Q_2$; (2) annex the reverse path $\overline{Q'}$ of Q' to the end of P to get P^*, that is, $P^* = P\overline{Q'}$. More precisely,

- $(P,Q) \in \mathscr{A}^{(r)}_{n,m,k,\ell}(x,y) - \bigcup_{i=k}^{r} \mathscr{C}^{(r,i)}_{n,m,k,\ell}(x,y)$ leads to $(P^*,Q^*) \in \mathscr{B}^{(r)}_{n,m,j,\ell}(x,y)$, where $Q = Q_1Q'U^*Q_2$ is factored as above.

Note that in this case the last $(\ell+1)$-th R-visible up step of Q are still the one of Q^*.

Conversely, we can recover $(P,Q) \in \mathscr{A}^{(r)}_{n,m,\ell}(x,y) - \mathscr{C}^{(r)}_{n,m,\ell}(x,y)$ from $(P^*,Q^*) \in \mathscr{B}^{(r)}_{n,m,\ell}(x,y)$. For any $(P^*,Q^*) \in \mathscr{B}^{(r)}_{n,m,\ell}(x,y)$, P^* can be uniquely partitioned into $P^* = PP'$ such that $P \in \mathscr{M}_{n,k}(x,y)$ for some $0 \le k \le N_r$ and P' has length $r+1$. Then delete the path P' of P^* to get P, and interpolate the reverse path $\overline{P'}$ of P' immediately ahead of the last $(\ell+1)$-th R-visible up step of Q^* to get Q. In this case, the last $(\ell+1)$-th R-visible up step of Q^* are also the one of Q and there are at least $r+1$ steps immediately ahead of it.

Note that ϕ does not change the weight of any H step, despite ϕ possibly exchange some U steps and D steps, but all U and D steps have the same weight 1. Hence, ϕ is indeed a bijection and also preserves weights. Therefore, (3.1) is proved.

Summing the two sides of (3.1) for $0 \le \ell \le m+r$, by Lemma 3.1, (3.2) follows. □

The special case $r = 0$ in (3.1) produces the following result.

Theorem 3.3. *Let $\mathcal{M} = (M_{n,k}(x, y))_{n\geq k\geq 0}$ be given in Section 2. For any integers $n \geq 0$ and $m \geq \ell \geq 0$, set $N_0 = \min\{n + 1, m - \ell\}$. Then there hold*

$$\sum_{k=0}^{N_0} \det \begin{pmatrix} M_{n,k}(x,y) & M_{m,k+\ell+1}(x,y) \\ M_{n+1,k}(x,y) & M_{m+1,k+\ell+1}(x,y) \end{pmatrix} = M_{n,0}(x,y)M_{m,\ell}(y,y). \tag{3.3}$$

Now we concentrate on the specialization of the parameters (x, y) in Theorem 3.3, which generates many identities involving Catalan numbers.

Example (i). When $(x, y) = (1, 2)$, (2.2) and (2.3) yield that $M(2, 2;t) = C^2(t)$ and $M(1, 2;t) = C(t)$, so $\mathcal{M} = (M_{n,k}(1, 2))_{n\geq k\geq 0}$ is the Riordan array $(C(t), tC^2(t))$. By the series expansion [17],

$$C(t)^{\alpha} = \sum_{n\geq 0} \frac{\alpha}{2n+\alpha}\binom{2n+\alpha}{n} t^n, \tag{3.4}$$

we have

$$M_{n,k}(1,2) = [t^n]C(t)(tC^2(t))^k = [t^{n-k}]C(t)^{2k+1} = \frac{2k+1}{2n+1}\binom{2n+1}{n-k}. \tag{3.5}$$

Then, after some routine simplifications, (3.3) produces the following result.

$$\frac{\ell+1}{m+1}\binom{2m+2}{m-\ell} C_n = \sum_{k=0}^{N_0} \frac{(2k+1)(2k+2\ell+1)\alpha_{n,k}(m,\ell)}{(2n+1)_3(2m+1)_3}\binom{2n+3}{n-k+1}\binom{2m+3}{m-k-\ell}, \tag{3.6}$$

where $\alpha_{n,k}(m, \ell) = 6(m - n)(n + 1)(m + 1) + (\ell + 1)(2k + \ell + 2)(2n + 1)(2n + 2) - 2(m - n)k(k + 1)(2n + 2m + 3)$ and $(x)_k = x(x + 1)\cdots(x + k - 1)$ for $k \geq 1$ and $(x)_0 = 1$.

Taking $\ell = 0$ and $m = n - 1, n$ or $n + 1$ into account, we have

$$\begin{aligned} \alpha_{n,k}(n-1,0) &= (n+k+3)(8nk+2n+2k+2), \\ \alpha_{n,k}(n,0) &= (2k+2)(2n+1)(2n+2), \\ \alpha_{n,k}(n+1,0) &= (n-k+1)(8nk+14n+10k+16). \end{aligned}$$

Then in these three cases, after shifting n to $n + 1$ in the case $m = n - 1$, (3.6) generates

Corollary 3.4. *For any integer $n \geq 0$, there hold*

$$\begin{aligned} C_{n+1}^2 &= \sum_{k=0}^{n} \frac{(2k+1)(2k+3)(8nk+2n+10k+4)}{(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)}\binom{2n+2}{n-k}\binom{2n+5}{n-k+2}, \\ C_nC_{n+1} &= \sum_{k=0}^{n} \frac{(2k+1)(2k+2)(2k+3)}{(2n+1)(2n+2)(2n+3)^2}\binom{2n+3}{n-k}\binom{2n+3}{n-k+1}, \\ C_nC_{n+2} &= \sum_{k=0}^{n} \frac{(2k+1)(2k+3)(8nk+14n+10k+16)}{(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)}\binom{2n+2}{n-k}\binom{2n+5}{n-k+1}. \end{aligned} \tag{3.7}$$

Example (ii). When $(x, y) = (2, 2)$, (2.3) yields that $M(2, 2;t) = C^2(t)$, so $\mathcal{M} = (M_{n,k}(2, 2))_{n\geq k\geq 0}$ is the Riordan array $(C^2(t), tC^2(t))$, it is also Shapiro's Catalan triangle aforementioned. By (3.5), we have

$$M_{n,k}(2,2) = [t^n]C^2(t)(tC^2(t))^k = [t^{n-k}]C(t)^{2k+2} = \frac{2k+2}{2n+2}\binom{2n+2}{n-k}. \tag{3.8}$$

Then, after some routine simplifications, (3.3) produces the following result.

$$\frac{\ell+1}{m+1}\binom{2m+2}{m-\ell}C_{n+1} = \sum_{k=0}^{N_0} \frac{(2k+2)(2k+2\ell+4)\beta_{n,k}(m,\ell)}{(2n+2)_3(2m+2)_3}\binom{2n+4}{n-k+1}\binom{2m+4}{m-k-\ell}, \tag{3.9}$$

where $\beta_{n,k}(m,\ell) = 6(m-n)(n+1)(m+1) + (\ell+1)(2k+\ell+3)(2n+2)(2n+3) - 2(m-n)k(k+2)(2n+2m+5)$.

Taking $\ell = 0$ and $m = n-1, n$ or $n+1$ into account, we have

$$\begin{aligned}
\beta_{n,k}(n-1,0) &= (n+k+3)(8nk+6n+6k+6),\\
\beta_{n,k}(n,0) &= (2k+3)(2n+2)(2n+3),\\
\beta_{n,k}(n+1,0) &= (n-k+1)(8nk+18n+14k+30).
\end{aligned}$$

Then in these three cases, after shifting n to $n+1$ in the case $m = n-1$, (3.9) generates

Corollary 3.5. *For any integer $n \geq 0$, there hold*

$$\begin{aligned}
C_{n+1}C_{n+2} &= \sum_{k=0}^{n} \frac{(2k+2)(2k+4)(8nk+6n+14k+12)}{(2n+2)(2n+3)(2n+4)(2n+5)(2n+6)}\binom{2n+3}{n-k}\binom{2n+6}{n-k+2},\\
C_{n+1}^2 &= \sum_{k=0}^{n} \frac{(2k+2)(2k+3)(2k+4)}{(2n+2)(2n+3)(2n+4)^2}\binom{2n+4}{n-k}\binom{2n+4}{n-k+1},\\
C_{n+1}C_{n+2} &= \sum_{k=0}^{n} \frac{(2k+2)(2k+4)(8nk+18n+14k+30)}{(2n+2)(2n+3)(2n+4)(2n+5)(2n+6)}\binom{2n+3}{n-k}\binom{2n+6}{n-k+1}.
\end{aligned} \tag{3.10}$$

Example (iii). When $(x,y) = (3,2)$, (2.2) and (2.3) yield that $M(2,2;t) = C^2(t)$ and $M(3,2;t) = \frac{C(t)}{\sqrt{1-4t}}$, so $\mathcal{M} = (M_{n,k}(3,2))_{n\geq k\geq 0}$ is the Riordan array $(\frac{C(t)}{\sqrt{1-4t}}, tC^2(t))$. By the series expansion [17],

$$\frac{C(t)^\alpha}{\sqrt{1-4t}} = \sum_{n\geq 0}\binom{2n+\alpha}{n}t^n,$$

we have

$$M_{n,k}(3,2) = [t^n]\frac{C(t)}{\sqrt{1-4t}}(tC^2(t))^k = [t^{n-k}]\frac{C(t)^{2k+1}}{\sqrt{1-4t}} = \binom{2n+1}{n-k}. \tag{3.11}$$

Then, after some routine simplifications, (3.3) produces the following result.

$$\frac{\ell+1}{m+1}\binom{2m+2}{m-\ell}\binom{2n+1}{n} = \sum_{k=0}^{N_0}\frac{\gamma_{n,k}(m,\ell)}{(2n+2)_2(2m+2)_2}\binom{2n+3}{n-k+1}\binom{2m+3}{m-k-\ell}, \tag{3.12}$$

where $\gamma_{n,k}(m) = 2(m-n)(n+1)(m+1) + (\ell+1)(2k+\ell+2)(2n+2)(2n+3) - 2(m-n)k(k+1)(2n+2m+5)$.

Taking $\ell = 0$ and $m = n-1, n$ or $n+1$ into account, we have

$$\begin{aligned}
\gamma_{n,k}(n-1,0) &= (n+k+2)(8nk+6n+6k+6),\\
\gamma_{n,k}(n,0) &= (2k+2)(2n+2)(2n+3),\\
\gamma_{n,k}(n+1,0) &= (n-k+1)(8nk+10n+14k+6).
\end{aligned}$$

Then in these three cases, after shifting n to $n+1$ in the case $m = n-1$, (3.9) generates

Corollary 3.6. *For any integer $n \geq 0$, there hold*

$$\begin{aligned} \binom{2n+3}{n+1} C_{n+1} &= \sum_{k=0}^{n} \frac{(8nk+6n+14k+12)}{(2n+2)(2n+3)(2n+4)} \binom{2n+3}{n-k} \binom{2n+4}{n-k+2}, \\ \binom{2n+1}{n} C_{n+1} &= \sum_{k=0}^{n} \frac{(2k+2)}{(2n+2)(2n+3)} \binom{2n+3}{n-k} \binom{2n+3}{n-k+1}, \\ \binom{2n+1}{n} C_{n+2} &= \sum_{k=0}^{n} \frac{(8nk+10n+14k+6)}{(2n+2)(2n+3)(2n+4)} \binom{2n+4}{n-k} \binom{2n+3}{n-k+1}. \end{aligned} \tag{3.13}$$

Example (iv). When $(x, y) = (0, 0)$, (2.3) yields that $M(0, 0; t) = C(t^2)$, so $\mathcal{M} = (M_{n,k}(0, 0))_{n \geq k \geq 0}$ is the Riordan array $(C(t^2), tC(t^2))$. By (3.5), we have

$$M_{n,k}(0,0) = [t^n]C(t^2)(tC(t^2))^k = [t^{n-k}]C(t^2)^{k+1} = \begin{cases} \frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}}, & \text{if } n-k \ \text{ even }, \\ 0, & \text{otherwise.} \end{cases} \tag{3.14}$$

Replacing n, m, ℓ by $2n, 2m, 2\ell$ respectively, after some routine simplifications, (3.3) produces the following result.

$$\frac{2\ell+1}{2m+1} \binom{2m+1}{m-\ell} C_n = \sum_{k=0}^{N_0} \frac{\lambda_{n,k}(m,\ell)}{(2n+1)_2(2m+1)_2} \binom{2n+2}{n-k} \binom{2m+2}{m-k-\ell}, \tag{3.15}$$

where $\lambda_{n,k}(m, \ell) = 2(2n+1)(2k+\ell+2)(2(k+1)(k+\ell+1) - (m+1)) + 2(m-n)(k+\ell+1)(2k+1)(2k+3)$.

Taking $\ell = 0$ and $n = m$ into account, we have

$$\lambda_{m,k}(m, 0) = (2m+1)(2k+2)((2k+1)(2k+3) - (2m+1)).$$

In this case, (3.15) generates

Corollary 3.7. *For any integer $m \geq 0$, there holds*

$$C_m^2 = \sum_{k=0}^{m} \frac{(2k+2)((2k+1)(2k+3) - (2m+1))}{(2m+1)(2m+2)^2} \binom{2m+2}{m-k}^2. \tag{3.16}$$

Remark 3.8. *It should be pointed out that despite (3.3) is not valid for any integer $\ell \leq -1$, since in these cases $M_{m,\ell}(y, y)$ has not be defined, but (3.6), (3.9), (3.12), (3.15) are all correct for any integer $\ell \leq -1$ if one notices that they hold trivially for any integer $\ell > m$ and both sides of them can be transferred into polynomials on ℓ.*

It is also worth pointing out that the case $m = n$ in (3.6), (3.9) and (3.12) generates the following results respectively.

Corollary 3.9. *For any integers $n \geq \ell \geq 0$, there hold*

$$\frac{1}{n+1} \binom{2n+2}{n-\ell} C_n = \sum_{k=0}^{n-\ell} \frac{(2k+1)(2k+\ell+2)(2k+2\ell+3)}{(2n+1)(2n+2)(2n+3)^2} \binom{2n+3}{n-k-\ell} \binom{2n+3}{n-k+1}, \tag{3.17}$$

$$\frac{1}{n+1} \binom{2n+2}{n-\ell} C_{n+1} = \sum_{k=0}^{n-\ell} \frac{(2k+2)(2k+\ell+3)(2k+2\ell+4)}{(2n+2)(2n+3)(2n+4)^2} \binom{2n+4}{n-k-\ell} \binom{2n+4}{n-k+1}, \tag{3.18}$$

$$\frac{1}{n+1} \binom{2n+2}{n-\ell} \binom{2n+1}{n} = \sum_{k=0}^{n-\ell} \frac{(2k+\ell+2)}{(2n+2)(2n+3)} \binom{2n+3}{n-k-\ell} \binom{2n+3}{n-k+1}. \tag{3.19}$$

Remark 3.10. *Recently, Gutirrez et al.* [7]*, Miana and Romero* [8]*, Chen and Chu* [3]*, and Guo and Zeng* [6] *studied the binomial sums* $\sum_{k=0}^{n}(k+1)^m\binom{2n+2}{n-k}^2$ *related to the classical Catalan triangle* [12]*. Zhang and Pang* [19] *also considered some alternating cases. Later, Miana and Romero* [9] *investigated another binomial sums* $\sum_{k=0}^{n}(2k+1)^m\binom{2n+1}{n-k}^2$*. Clearly our work is closely related to theirs from a different direction. Setting* $p=k+1, \ell=n-i+1$*, and then replacing* n *by* $n-2$*, (3.18) reduces to the main identity obtained by Gutirrez et al.* [7, Theorem 5]*. As mentioned in Remark 3.7, (3.17)-(3.19) also hold for any integer* $\ell<0$*. Specially, in the case* $\ell=-1$*, replacing* $n+1$ *by* n*, after some routine simplifications, (3.17)-(3.19) lead respectively to the following identities,*

$$\sum_{k=0}^{n}\frac{(2k+1)^3}{(2n+1)^2}\binom{2n+1}{n-k}^2=\binom{2n}{n}^2, \qquad [9, \text{Remark } 11] \tag{3.20}$$

$$\sum_{k=0}^{n}\frac{(k+1)^3}{(n+1)^2}\binom{2n+2}{n-k}^2=\binom{2n}{n}\binom{2n+1}{n}, \qquad [7, \text{Corollary } 6] \tag{3.21}$$

$$\sum_{k=0}^{n}\frac{(2k+1)}{(2n+1)}\binom{2n+1}{n-k}^2=\binom{2n}{n}^2. \tag{3.22}$$

Note that (3.20), (3.22) and (4.4) can be regarded as companion ones of an identity obtained by Deng and Yan [5]*,*

$$\sum_{k=0}^{n}\frac{(2k+1)^2}{(2n+1)}\binom{2n+1}{n-k}=4^n.$$

Moreover, (3.17)-(3.19) hold symmetrically on $\ell=-1$*. For example, in the case* $\ell=-2$*, (3.17)-(3.19) can reduce respectively to (3.7) (3.10) and (3.13) which also correspond to the case* $\ell=0$ *in (3.17)-(3.19).*

The special case $r=1$, $\ell=0$ and $m=n$ in (3.1) produces the following result.

Theorem 3.11. *Let* $\mathcal{M}=(M_{n,k}(x,y))_{n\geq k\geq 0}$ *be given in Section 2. Then there holds*

$$\sum_{k=0}^{n}\det\begin{pmatrix} M_{n,k}(y,y) & M_{n,k+1}(y,y) \\ M_{n+2,k}(y,y) & M_{n+2,k+1}(y,y)\end{pmatrix}=2M_{n,0}(y,y)M_{n+1,0}(y,y), \tag{3.23}$$

In the case $y=2$, together with (3.8), after some routine computations, (3.23) generates

Corollary 3.12. *For any integer* $n\geq 0$*, there holds*

$$C_nC_{n+1}=\sum_{k=0}^{n}\frac{(2k+2)(2k+3)(2k+4)}{(2n+2)(2n+3)(2n+6)(2n+7)}\binom{2n+3}{n-k}\binom{2n+7}{n-k+2}.$$

4. Alternating Cases

In this section, we consider some alternating sums related to $\mathcal{M}=(M_{n,k}(x,y))_{n\geq k\geq 0}$. Despite it has no general and unified results as in the previous section, but in several isolated cases, mainly by the creative telescoping algorithm [10, 18], we also obtain some interesting results.

Theorem 4.1. *Let* $\mathcal{M}=(M_{n,k}(x,y))_{n\geq k\geq 0}$ *be given in Section 2. Then there holds*

$$\sum_{k=0}^{n}(-1)^{n-k}\det\begin{pmatrix} M_{n,k}(0,0) & M_{n,k+1}(0,0) \\ M_{n+1,k}(0,0) & M_{n+1,k+1}(0,0)\end{pmatrix}=C_{n+1}, \tag{4.1}$$

or equivalently,

$$C_{2m+1} = \sum_{j=0}^{m} \frac{(2j+2)^2}{(2m+2)^2} \binom{2m+2}{m-j}^2, \tag{4.2}$$

$$C_{2m+2} = \sum_{j=0}^{m} \frac{(2j+2)^2}{(2m+2)(2m+3)} \binom{2m+3}{m-j} \binom{2m+3}{m-j+1}. \tag{4.3}$$

Proof. By (3.14), the cases $n = 2m$ and $n = 2m+1$ in (4.1) are equivalent to (4.2) and (4.3) respectively.

For (4.2), its right side counts the set of pairs (P, Q) such that $P, Q \in \mathscr{M}_{m,j}(2,2)$ for $0 \le j \le m$. Let $\overline{Q}$ be the reverse path of Q. Clearly, $P\overline{Q} \in \mathscr{M}_{2m,0}(2,2)$. The process is obviously reversible. This builds a simple bijection between the sets $\bigcup_{0\le j\le m} \left(\mathscr{M}_{m,j}(2,2) \times \mathscr{M}_{m,j}(2,2)\right)$ and $\mathscr{M}_{2m,0}(2,2)$, which, by (3.8), proves (4.2).

Note that $(j+1)^2 - (m+2)^2 = -(m+j+3)(m-j+1)$, by the trivial identities

$$\sum_{j=0}^{m} \binom{2m+3}{m-j}\binom{2m+3}{m-j+1} = \sum_{j=-m-2}^{-2} \binom{2m+3}{m-j}\binom{2m+3}{m-j+1},$$

$$\sum_{j=0}^{m} \binom{2m+2}{m-j}^2 = \sum_{j=-m-2}^{-2} \binom{2m+2}{m-j}^2,$$

we have

$$\begin{aligned}
&\sum_{j=0}^{m} \frac{(2j+2)^2}{(2m+2)(2m+3)} \binom{2m+3}{m-j}\binom{2m+3}{m-j+1} \\
&= 4\sum_{j=0}^{m} \frac{(m+2)^2}{(2m+2)(2m+3)} \binom{2m+3}{m-j}\binom{2m+3}{m-j+1} \\
&\qquad +4\sum_{j=0}^{m} \frac{(j+1)^2-(m+2)^2}{(2m+2)(2m+3)} \binom{2m+3}{m-j}\binom{2m+3}{m-j+1} \\
&= \frac{4(m+2)^2}{(2m+2)(2m+3)} \sum_{j=0}^{m} \binom{2m+3}{m-j}\binom{2m+3}{m-j+1} - \frac{4(2m+3)}{(2m+2)} \sum_{j=0}^{m} \binom{2m+2}{m-j}^2 \\
&= \frac{2(m+2)^2}{(2m+2)(2m+3)} \left\{ \sum_{j=-m-2}^{m} \binom{2m+3}{m-j}\binom{2m+3}{m-j+1} - \binom{2m+3}{m+1}^2 \right\} \\
&\qquad -\frac{2(2m+3)}{(2m+2)} \left\{ \sum_{j=-m-2}^{m} \binom{2m+2}{m-j}^2 - \binom{2m+2}{m+1}^2 \right\} \\
&= \frac{2(m+2)^2}{(2m+2)(2m+3)} \left\{ \binom{4m+6}{2m+2} - \binom{2m+3}{m+1}^2 \right\} \\
&\qquad -\frac{2(2m+3)}{(2m+2)} \left\{ \binom{4m+4}{2m+2} - \binom{2m+2}{m+1}^2 \right\} \quad \text{(by Vandermonde identity)} \\
&= C_{2m+2},
\end{aligned}$$

which proves (4.3). ∎

Remark 4.2. *Note that $\mathcal{M} = (M_{m,k}(x,y))_{m\geq k\geq 0}$ also forms an admissible matrix defined by Aigner* [1]*, then (4.2) can be obtained easily by the elementary property of the admissible matrix $(M_{m,k}(2,2))_{m\geq k\geq 0}$. What's more, one can also get another one from the admissible matrix $(M_{m,k}(1,2))_{m\geq k\geq 0}$,*

$$\sum_{j=0}^{m} \frac{(2j+1)^2}{(2m+1)^2}\binom{2m+1}{m-j}^2 = C_{2m}, \tag{4.4}$$

which also has a combinatorial proof similar to (4.2). Using the creative telescoping algorithm, one can find another identity related to (3.16) and (4.2), that is,

$$\sum_{j=0}^{m} \frac{(j+1)}{(m+1)}\binom{2m+2}{m-j}^2 = \binom{2m+1}{m}^2, \tag{4.5}$$

which is also obtained implicitly by Chen and Chu [3, Corollary 4]*.*

Note that (3.21), (4.2) and (4.5) can also be regarded as companion ones of an identity obtained by Cameron and Nkwanta [2]*,*

$$\sum_{j=0}^{m} \frac{(j+1)^2}{(m+1)}\binom{2m+2}{m-j} = 4^m.$$

Theorem 4.3. *For any integer $m \geq 0$, there hold*

$$\binom{2m+1}{m} = \sum_{j=0}^{m}(-1)^j \frac{(2j+2)^2}{(2m+2)^2}\binom{2m+2}{m-j}^2, \qquad [19, \text{Theorem 2.2}] \tag{4.6}$$

$$\binom{2m+2}{m+1} = \sum_{j=0}^{m}(-1)^j \frac{(2j+2)^2}{(2m+2)(2m+3)}\binom{2m+3}{m-j}\binom{2m+3}{m-j+1}, \tag{4.7}$$

$$\binom{2m}{m} = \sum_{j=0}^{m}(-1)^j \frac{(2j+1)}{(2m+1)}\binom{2m+1}{m-j}^2. \tag{4.8}$$

Proof. For $0 \leq j \leq m$, define $F_i(m) = \sum_{j=0}^{m} F_i(m,j)$ with $i = 1,2,3$, where

$$\begin{aligned}
F_1(m,j) &= (-1)^j \frac{(2j+2)^2}{(2m+2)^2}\binom{2m+2}{m-j}^2\binom{2m+1}{m}^{-1},\\
F_2(m,j) &= (-1)^j \frac{(2j+2)^2}{(2m+2)(2m+3)}\binom{2m+3}{m-j}\binom{2m+3}{m-j+1}\binom{2m+2}{m+1}^{-1},\\
F_3(m,j) &= (-1)^j \frac{(2j+1)}{(2m+1)}\binom{2m+1}{m-j}^2\binom{2m}{m}^{-1}.
\end{aligned}$$

Then the creative telescoping algorithm quickly finds the recurrence

$$F_i(m+1,j) - F_i(m,j) = G_i(m,j+1) - G_i(m,j),$$

where $G_i(m,j) = R_i(m,j)F_i(m,j)$ for $i = 1,2,3$ and

$$\begin{aligned}
R_1(m,j) &= \frac{j^2(j+1) - j(3m+5)(m+1)}{2(j+1)(m-j+1)^2},\\
R_2(m,j) &= \frac{j^2(j+1) - j(3m+5)(m+2)}{2(j+1)(m-j+1)(m-j+2)},\\
R_3(m,j) &= \frac{j^3 - 3j(m+1)^2}{(2j+1)(m-j+1)^2}.
\end{aligned}$$

If we sum the above recurrence over j, we find that the sums satisfy $F_i(m+1) - F_i(m) = 0$ with $i = 1, 2, 3$, which easily yield $F_i(m) = F_i(0) = 1$. Hence, (4.6), (4.7) and (4.8) are proved. □

Theorem 4.4. *Let $\mathcal{M} = (M_{n,k}(x,y))_{n\geq k\geq 0}$ be given in Section 2. Then for $j = 1$ or 2, there holds*

$$\sum_{k=0}^{n}(-1)^k \det\begin{pmatrix} M_{n,k}(2,2) & M_{n,k+1}(2,2) \\ M_{n+j,k}(2,2) & M_{n+j,k+1}(2,2) \end{pmatrix} = 4^{j-1}C_{n+1},$$

or equivalently, for $j = 1$ there has

$$C_{n+1} = \sum_{k=0}^{n}(-1)^k \frac{(2k+2)(2k+3)(2k+4)}{(2n+2)(2n+3)(2n+4)^2}\binom{2n+4}{n-k}\binom{2n+4}{n-k+1}, \tag{4.9}$$

and for $j = 2$ there has

$$2C_{n+1} = \sum_{k=0}^{n}(-1)^k \frac{(2k+2)(2k+3)(2k+4)}{(2n+2)(2n+3)(2n+6)(2n+7)}\binom{2n+3}{n-k}\binom{2n+7}{n-k+2}. \tag{4.10}$$

Proof. For $0 \leq k \leq n$, define $F_i(n) = \sum_{k=0}^{n} F_i(n,k)$ with $i = 4, 5$, where

$$F_4(n,k) = (-1)^k \frac{(2k+2)(2k+3)(2k+4)}{(2n+2)(2n+3)(2n+4)^2 C_{n+1}}\binom{2n+4}{n-k}\binom{2n+4}{n-k+1},$$
$$F_5(n,k) = (-1)^k \frac{(2k+2)(2k+3)(2k+4)}{2(2n+2)(2n+3)(2n+6)(2n+7)C_{n+1}}\binom{2n+3}{n-k}\binom{2n+7}{n-k+2}.$$

Again the creative telescoping algorithm quickly finds the recurrence

$$F_i(n+1,k) - F_i(n,k) = G_i(n,k+1) - G_i(n,k),$$

where $G_i(n,k) = R_i(n,k)F_i(n,k)$ for $i = 4, 5$ and

$$R_4(n,k) = \frac{k(k+1)^2 - k(3n+7)(n+2)}{(2k+3)(n-k+1)(n-k+2)},$$
$$R_5(n,k) = \frac{2k^2(k+2) - 2k(3n+7)(n+3)}{(2k+3)(n-k+1)(n-k+3)}.$$

Similarly, if summing the above recurrence over k, we find again that the sums satisfy $F_i(n+1) - F_i(n) = 0$ with $i = 4, 5$, which easily generate $F_i(n) = F_i(0) = 1$. Hence, (4.9) and (4.10) are proved. □

Remark 4.5. *Despite (4.3), (4.5)-(4.10) are proved by algebraic methods, but bijective proofs are naturally expected, together with a proof of the following identities involving Motzkin numbers, namely, for $j = 1, 2$, there holds*

$$\sum_{k=0}^{n}(-1)^{n-k} \det\begin{pmatrix} M_{n,k}(1,1) & M_{n,k+1}(1,1) \\ M_{n+j,k}(1,1) & M_{n+j,k+1}(1,1) \end{pmatrix} = 2^{j-1}M_n.$$

Besides, another direction in which this research could be taken is to consider the q-analog of all the identities obtained in this paper.

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