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Some Fibonacci-like Sequences

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A random look at Fibonacci sequences

Each term of the familiar Fibonacci sequence

$$1, 1, 2, 3, 5, 8, \ldots$$

is obtained as the sum of the two previous terms. This sequence has many beautiful properties and even has a journal, *The Fibonacci Quarterly*, devoted to its study. Many of its fundamental properties are considered in reference 1. In this article, we consider some sequences related to the Fibonacci sequence and pose some interesting questions for further study.

The Fibonacci sequence can be written in the form of a recurrence relation. Specifically, since the nth term $x(n)$ relies on only the two previous terms, we can write

$$x(n) = x(n-1) + x(n-2)\,. \tag{1}$$

To get the recurrence relation going, we need to specify the first two terms. Here $x(0) = 1$ and $x(1) = 1$.

Recurrence relations such as (1) are useful for describing a sequence, but are often difficult to use when we wish to compute a term for a large value of n. For instance, to find the 100th term of the Fibonacci sequence, we would have to compute the previous 99 terms. It is thus desirable to express $x(n)$ in a *closed form*, that is, a form that depends only upon n.

Finding a closed form for the recurrence relation (1) is similar to finding the solution of a homogeneous linear differential equation with constant coefficients. The details of the method can be found in reference 2.

We apply the method to the Fibonacci recurrence by rewriting it as a difference equation

$$x(n) - x(n-1) - x(n-2) = 0\,,$$

and then substituting $x(n) = C\lambda^n$, where C is a constant, to obtain

$$C\lambda^n - C\lambda^{n-1} - C\lambda^{n-2} = 0\,.$$

Factoring out $C\lambda^{n-2}$ gives the *characteristic equation*

$$\lambda^2 - \lambda - 1 = 0\,,$$

a quadratic which we can solve easily. The solution gives the *eigenvalues* for the recurrence relation. The eigenvalues can be viewed as 'growth rates' of the recurrence relation. For the Fibonacci sequence, they are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}\,.$$

Notice that $\lambda_1 \approx 1.618$ and $\lambda_2 \approx -0.618$, so $\lambda_1 > \lambda_2$. We call λ_1 the *dominant* eigenvalue. In general, the root of a characteristic equation with largest modulus will be termed the dominant eigenvalue and denoted λ^*.

The value

$$\lambda_1 = \frac{1+\sqrt{5}}{2}$$

is also known in geometry as the *golden ratio*.

The closed form of $x(n)$ is

$$x(n) = C_1\left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

where we know that the initial values are $x(0) = 1$ and $x(1) = 1$. Using these we have the following simultaneous equations for the constants C_1 and C_2:

$$C_1 + C_2 = 1 \quad \text{and} \quad C_1\frac{1+\sqrt{5}}{2} + C_2\frac{1-\sqrt{5}}{2} = 1\,.$$

Solving gives $C_1 = (5+\sqrt{5})/10$ and $C_2 = (5-\sqrt{5})/10$. Thus, the closed-form solution to the Fibonacci recurrence relation (1) is

$$x(n) = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{2}$$

One surprising feature of this closed-form solution (2) is that, despite the fact that all of its factors are irrational, it produces integer values for every n.

The long-term behaviour of the Fibonacci sequence can be determined by (2). Considering each of its terms, we see that, since

$$\frac{1+\sqrt{5}}{2} > 1 \quad \text{and} \quad \frac{1-\sqrt{5}}{2} < 1\,,$$

the behaviour of the sequence as n increases resembles that of exponential growth with a base of $(1+\sqrt{5})/2$, which is the value of the dominant eigenvalue. Moreover,

$$\lim_{n\to\infty} \sqrt[n]{x(n)} = \frac{1+\sqrt{5}}{2}\,. \tag{3}$$

In general, the growth of a recurrence relation will be determined by its dominant eigenvalue λ^* with an expression similar to (3); indeed,

$$\lim_{n\to\infty} \sqrt[n]{|x(n)|} = \lambda^*\,. \tag{4}$$

Problem 1. Suppose that we generate a sequence using the Fibonacci recurrence relation

$$x(n) = x(n-1) + x(n-2)$$