DEMONSTRATIO MATHEMATICA
Vol. XLVI No 4 2013

Paweł J. Szabłowski

ON THE q-HERMITE POLYNOMIALS AND THEIR RELATIONSHIP WITH SOME OTHER FAMILIES OF ORTHOGONAL POLYNOMIALS

Abstract. We review properties of the q-Hermite polynomials and indicate their links with the Chebyshev, Rogers–Szegö, Al-Salam–Chihara, continuous q-utraspherical polynomials. In particular, we recall the connection coefficients between these families of polynomials. We also present some useful and important finite and infinite expansions involving polynomials of these families including symmetric and non-symmetric kernels. In the paper, we collect scattered throughout literature useful but not widely known facts concerning these polynomials. It is based on 43 positions of predominantly recent literature.

1. Introduction

The aim of this paper is to review basic properties of the q-Hermite polynomials and collect their not always widely known properties scattered throughout the recent literature. The q-Hermite polynomials constitute a 1-parameter family of orthogonal polynomials that for $q = 1$ are equal to the well known Hermite polynomials, more precisely the probabilistic Hermite polynomials i.e. orthogonal with respect to the density of $N(0,1)$ distributions $(\exp(-x^2/2)/\sqrt{2\pi})$. For $q = 0$, they are equal to the re-scaled Chebyshev polynomials of the second kind, again more precisely, polynomials orthogonal with respect to the Wigner measure i.e. the one with the density $2\sqrt{4-x^2}/\pi$. On the other hand, these polynomials are related to the so called Rogers–Szegö polynomials or Rogers (continuous q-ultraspherical) polynomials and other important families of polynomials such as the Al-Salam–Chihara polynomials.

Why these polynomials are important? For one thing, they are very simple and as it will be shown in the sequel, many more complicated (i.e.

2010 *Mathematics Subject Classification*: Primary 33D45, 05A30; Secondary 42C05.
Key words and phrases: q-Hermite, Al-Salam–Chihara, Rogers–Szegö, Chebyshev polynomials, q-ultraspherical polynomials, summing kernels, generalization of Poisson–Mehler kernel, generalization of Kibble–Slepian formula.

having more parameters) families of orthogonal polynomials can be expressed as linear combinations of the q-Hermite polynomials. Secondly, since they are simple, many facts concerning them are known. Thirdly, during last 15 years they appeared within several interesting applications that came from the theories quite distant from the classical q-series theory or combinatorics.

They appeared long time ago by the end of XIX-th century as a version of the Rogers polynomials (see [28], [27], [29]), their important properties were examined by Szegö [40] and Carlitz [9], [10], [8] through XX-th century, but only recently it appeared that they are important in non-commutative probability (see e.g. [3], [43], [42]), quantum physics (see e.g. [13], [12]), combinatorics (see e.g. [18], [31], [17]) and last but not least ordinary, classical probability theory (see e.g. [4], [5], [7], [6], [23], [24]) extending the spectrum of known, finite support measures.

To define these polynomials and briefly describe their properties one has to adopt notation used in the so called q-series theory. Moreover, the terminology concerning these polynomials is not fixed and under the same name appear sometimes different, but related to one another families of polynomials. Thus, one has to be aware of these differences.

That is why the next section of the paper is devoted to notation, definitions and discussion of different families of polynomials that function under the same name. The following section is dedicated to various 'finite expansions' formulae, establishing relationships between these families of polynomials, including so called 'connection coefficients' and 'linearization' formulae. The last section is dedicated to some infinite expansions involving discussed polynomials. It consists of three subsections, the first of which is devoted to different generalizations of the Mehler expansion formula, the second one to some useful infinite expansions including reciprocals of some kernels that have auxiliary meaning. Finally, the third subsection is dedicated to an attempt of generalization of the 3-dimensional Kibble–Slepian formula with the Hermite polynomials replaced by the q-Hermite ones.

2. Notation and definitions

q is a parameter. We will assume that $-1 < q \leq 1$ unless otherwise stated. The case $q = 1$ may not always be considered directly but sometimes as left hand side limit (i.e. $q \longrightarrow 1^-$). We will point out these cases.

We will use traditional notation of the q-series theory i.e. $[0]_q = 0$; $[n]_q = 1 + q + \ldots + q^{n-1}$, $[n]_q! = \prod_{j=1}^{n} [j]_q$, with $[0]_q! = 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{[n]_q!}{[n-k]_q![k]_q!}, & n \geq k \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

$\binom{n}{k}$ will denote ordinary, well known binomial coefficient.

It is useful to use the so called q-Pochhammer symbol for $n \geq 1$:

$$(a;q)_n = \prod_{j=0}^{n-1}\left(1 - aq^j\right), \quad (a_1, a_2, \ldots, a_k; q)_n = \prod_{j=1}^{k}(a_j;q)_n,$$

with $(a;q)_0 = 1$.

Often $(a;q)_n$ as well as $(a_1, a_2, \ldots, a_k; q)_n$ will be abbreviated to $(a)_n$ and $(a_1, a_2, \ldots, a_k)_n$, if it will not cause misunderstanding.

It is easy to notice that $(q)_n = (1-q)^n [n]_q!$ and that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_{n-k}(q)_k}, & n \geq k \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The above mentioned formula is just an example where direct setting $q = 1$ is senseless however passage to the limit $q \longrightarrow 1^-$ makes sense.

Notice that in particular $[n]_1 = n, [n]_1! = n!$, $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$, $(a)_1 = 1 - a$, $(a;1)_n = (1-a)^n$ and $[n]_0 = \begin{cases} 1 & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$, $[n]_0! = 1$, $\begin{bmatrix} n \\ k \end{bmatrix}_0 = 1$, $(a;0)_n = \begin{cases} 1 & \text{if } n = 0 \\ 1-a & \text{if } n \geq 1 \end{cases}$.

i will denote imaginary unit, unless otherwise clearly stated.

In the sequel, we shall also use the following useful notation:

$$S(q) = \begin{cases} [-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}] & \text{if } |q| < 1, \\ \mathbb{R} & \text{if } q = 1. \end{cases}$$

Sometimes we will define sets of polynomials only on bounded intervals. Of course they can be naturally extended to whole real line.

Basically, each considered family of polynomials will be of one of two kinds. The first 'kind' will be orthogonal on $S(q)$ and their names generally start with the capitals. The polynomials of the second 'kind' will be orthogonal on $[-1,1]$ and their names will generally start with the lower case letters. There will be in fact 4 exceptions for the two kinds of Chebyshev polynomials (traditionally denoted by T and U), the so called Rogers or continuous q-ultraspherical polynomials traditionally denoted by C and the Al-Salam–Chihara polynomials in its 'lower case version' traditionally denoted by Q. These polynomials are orthogonal on $[-1,1]$ but their names as mentioned before traditionally start with the capital letter. The difference between those two kinds of polynomials are minor. Besides the 'lower case letters' polynomials either do not allow the case $q = 1$ or this case leads to some trivialities. On the contrary, for the 'upper case letters' polynomials, the case $q=1$ either considered directly or understood as a limit $q\longrightarrow 1^{-}$ leads to important simplifications and supports intuition.

As of now, it seems that the 'upper case' polynomials are more important in the applications that appeared recently such as probability theory both commutative and non-commutative or quantum physics, while the 'lower case' polynomials are more typical in the special functions theory and the combinatorics.

In brief description of certain functions, given by infinite products, important for the discussed families of polynomials, we will use the following families of auxiliary polynomials of degree at most 2.

In fact, they are again of two different forms (as are the families of polynomials) that are connected with the fact if considered polynomials are orthogonal on $[-1,1]$ regardless of q or on $S\left(q\right)$. As the families of polynomials, these auxiliary polynomials will be denoted by the name starting with the capital if the case concerns orthogonality on $S\left(q\right)$.

Hence, we will consider for $k\geq 0$:

$$\begin{aligned}
v\left(x|a\right) &= 1-2ax+a^{2},\ V_{q}(x|a)=1-(1-q)ax+(1-q)a^{2},\\
l\left(x|a\right) &= (1+a)^{2}-4xa^{2},\ L_{q}(x|a)=(1+a)^{2}-(1-q)x^{2}a,\\
w\left(x,y|t\right) &= (1-t^{2})^{2}-4xyt(1+t^{2})+4t^{2}(x^{2}+y^{2}),\\
W_{q}(x,y|t) &= (1-t^{2})^{2}-(1-q)xyt(1+t^{2})+(1-q)t^{2}(x^{2}+y^{2}).
\end{aligned}$$

Let us notice that

$$\begin{aligned}
w\left(x,x|t\right) &= (1-t)^{2}l(x|t),\\
W_{q}\left(x,x|t\right) &= (1-t)^{2}L_{q}(x|t),
\end{aligned}$$

and that also

$$\begin{aligned}
(ae^{i\theta},ae^{-i\theta})_{1} &= v(x|a),\\
(ae^{i\theta+i\eta)},ae^{-i\theta+i\eta},ae^{i\theta-i\eta},ae^{-i\theta-i\eta})_{1} &= w(x,y|a),\\
(ae^{i2\theta},ae^{-i2\theta})_{1} &= l(x|a),
\end{aligned}$$

$$\left(ae^{i\theta},ae^{-i\theta}\right)_{\infty}=\prod_{k=0}^{\infty}v\left(x|aq^{k}\right), \tag{2.1}$$

$$\left(te^{i(\theta+\phi)},te^{i(\theta-\phi)},te^{-i(\theta-\phi)},te^{-i(\theta+\phi)}\right)_{\infty}=\prod_{k=0}^{\infty}w\left(x,y|tq^{k}\right), \tag{2.2}$$

$$\left(ae^{2i\theta},ae^{-2i\theta}\right)_{\infty}=\prod_{k=0}^{\infty}l\left(x|aq^{k}\right), \tag{2.3}$$

where, as usually in the q-series theory, $x=\cos\theta$ and $y=\cos\phi$.

The following convention will help in ordered listing of the properties of the discussed families of polynomials. Namely, the family of polynomials whose names start with, say a letter A, will be referred to as A (similarly for the lower case a). There will be one exception, namely members of the so called family of big q-Hermite polynomials are traditionally denoted by letter H (or h) as members of the family of q-Hermite polynomials. So, family of big q-Hermite polynomials will be referred to by bH.

Below we define sets of polynomials, and present their generating functions and measures with respect to which these polynomials are orthogonal, provided these measures are positive.

2.1. Hermite. The Hermite polynomials are defined by the following 3-term recurrence (2.4), below:

$$xH_n(x) = H_{n+1}(x) + nH_{n-1}, \tag{2.4}$$

$n \geq -1$, with $H_{-1}(x) = 0$ and $H_0(x) = 1$. They slightly differ from the Hermite polynomials h_n considered in most of the books on special functions. Namely

$$2xh_n(x) = h_{n+1}(x) + 2nh_{n-1}(x),$$

with $h_{-1}(x) = 0$, $h_0(x) = 1$.

It is known that polynomials $\{h_n\}$ are orthogonal with respect to $\exp\left(-x^2\right)$ while $\{H_n\}$ are orthogonal with respect to $\exp\left(-x^2/2\right)$. Moreover $H_n(x) = h_n\left(x/\sqrt{2}\right) / \left(\sqrt{2}\right)^n$. Besides we have

$$\exp\left(xt - t^2/2\right) = \sum_{k\geq 0} \frac{t^k}{k!} H_k(x), \tag{2.5}$$

$$\exp\left(2xt - t^2\right) = \sum_{k\geq 0} \frac{t^k}{k!} h_k(x). \tag{2.6}$$

2.2. Chebyshev. They are of two kinds. The Chebyshev polynomials of the first kind $\{T_n\}_{n\geq -1}$ are defined by the following 3-term recursion

$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x), \tag{2.7}$$

for $n \geq 1$, with $T_0(x) = 1$, $T_1(x) = x$. One can define them also in the following way:

$$T_n(\cos\theta) = \cos(n\theta). \tag{2.8}$$

The Chebyshev polynomials $\{U_n(x)\}_{n\geq 0}$ of the second kind are defined by the same 3-term recurrence i.e. (2.7) with the different initial conditions, namely $U_0(x) = 1$ and $U_1(x) = 2x$. One shows that they can be defined also by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{2.9}$$

We have

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{dx}{\pi\sqrt{1-x^2}} = \begin{cases} 1 & \text{if } m=n=0, \\ 1/2 & \text{if } m=n\neq 0, \\ 0 & \text{if } m\neq n, \end{cases}$$

$$\int_{-1}^{1} U_n(x)U_m(x)\frac{2\sqrt{1-x^2}}{\pi}dx = \begin{cases} 1 & \text{if } m=n, \\ 0 & \text{if } m\neq n, \end{cases}$$

and for $|t|\leq 1$

$$\sum_{k=0}^{\infty} t^k T_k(x) = \frac{1-tx}{1-2tx+t^2}, \tag{2.10}$$

$$\sum_{k=0}^{\infty} t^k U_k(x) = \frac{1}{1-2tx+t^2}. \tag{2.11}$$

2.3. q-Hermite. The q-Hermite polynomials are defined by:

$$2xh_n(x|q) = h_{n+1}(x|q) + (1-q^n)h_{n-1}(x|q), \tag{2.12}$$

for $n\geq 1$ with $h_{-1}(x|q)=0$, $h_0(x|q)=1$.

The polynomials h_n are often called the continuous q-Hermite polynomials. Since the terminology is not fixed and also since we will consider only two types of them (defined by (2.12) and (2.14)) we will use the name q-Hermite polynomials for the brevity.

In fact we will also use the following transformed form of the polynomials h_n, namely the polynomials:

$$H_n(x|q) = (1-q)^{-n/2}h_n\left(\frac{x\sqrt{1-q}}{2}|q\right). \tag{2.13}$$

It is easy to notice that the polynomials $\{H_n(x|q)\}$ satisfy the following 3-term recurrence

$$xH_n(x|q) = H_{n+1}(x|q) + [n]_q H_{n-1}(x), \tag{2.14}$$

for $n\geq 1$ with $H_{-1}(x|q)=0$, $H_1(x|q)=1$. The name is justified since one can easily show that $n\geq -1$

$$H_n(x|1) = H_n(x).$$

Notice that since $[n]_0 = 1$ for $n\geq -1$, we have

$$H_n(x|0) = U_n(x/2). \tag{2.15}$$

It is known that (see e.g. [22](14.26.2)):

$$\int_{-1}^{1} h_n(x|q)\, h_m(x|q)\, f_h(x|q)\, dx = \begin{cases} (q)_n & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

$$\int_{S(q)} H_n(x|q) H_m(x|q) f_N(x|q) dx = \begin{cases} [n]_q! & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

where we denote

$$f_h(x|q) = \frac{2(q)_\infty \sqrt{1-x^2}}{\pi} \prod_{k=1}^{\infty} l\left(x|q^k\right), \tag{2.16}$$

$$f_N(x|q) = \begin{cases} \sqrt{1-q} f_h(x\sqrt{1-q}/2|q)/2 & \text{if } |q| < 1, \\ \exp\left(-x^2/2\right)/\sqrt{2\pi} & \text{if } q = 1, \end{cases} \tag{2.17}$$

and (see [22](14.26.11))

$$\sum_{j=0}^{\infty} \frac{t^j}{(q)_j} h_j(x|q) = \frac{1}{\prod_{k=0}^{\infty} v(x|tq^k)}, \tag{2.18}$$

$$\sum_{j=0}^{\infty} \frac{t^j}{[j]_q!} H_j(x|q) = \frac{1}{\prod_{k=0}^{\infty} V_q(x|tq^k)}. \tag{2.19}$$

Convergence is in the above formulae for $|x|, |t| \leq 1$ in (2.18) and $x \in S(q)$ and $\left|t\sqrt{1-q}\right| \leq 1$ in (2.19).

One proves also that

$$\lim_{q->1^-} f_N(x|q) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \tag{2.20}$$

$$\lim_{q->1^-} \frac{1}{\prod_{k=0}^{\infty} V_q(x|tq^k)} = \exp\left(xt - \frac{t^2}{2}\right). \tag{2.21}$$

Rigorous and easy proofs of these facts can be found in [18]. The convergence in distribution is obvious since we have $\forall n \geq 0$: $\lim_{q->1^-} H_n(x|q) = H_n(x|1)$ consequently we have the convergence of moments.

Let us remark that the density f_N is a real probabilistic density i.e. integrates to 1. Since we have (2.20), it is sometimes called q-Gaussian or q-Normal. It appeared, in non-commutative probability context, in an important paper [3]. Later in classical probability context appeared in [4] and [5]. Its further properties, including algorithm how to simulate sequences if independent random observations having f_N as its density, were presented in [39].

One considers also the small generalization of the q-Hermite polynomials namely the so called big continuous q-Hermite polynomials. i.e. the polynomials defined by the following 3-term recurrence:

$$(2x - aq^n)h_n(x|a,q) = h_{n+1}(x|a,q) + (1-q^n)h_{n-1}(x|a,q),$$
$$(x - aq^n)H_n(x|a,q) = H_{n+1}(x|a,q) + [n]_q H_{n-1}(x|a,q),$$

$n \geq 0$, with initial conditions: $h_{-1}(x|a,q) = H_{-1}(x|a,q) = 0$ and $h_0(x|a,q) = H_0(x|a,q) = 1$. For the sake of brevity, we will call them simply big q-Hermite polynomials. They are obviously inter-related by

$$H_n(x|a,q) = h_n\left(\frac{\sqrt{1-q}x}{2}|a\sqrt{1-q},q\right)/(1-q)^{n/2}.$$

Notice that, using well known properties of the ordinary Hermite polynomials, we have:

$$H_n(x|a,1) = H_n(x-a).$$

One can easily show (by calculating generating function and comparing it with (2.22), below and then applying (2.1)) that

$$h_n(x|a,q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(ae^{i\theta}\right)_k e^{i(n-2k)\theta},$$

$n \geq -1$, where as usually $x = \cos\theta$.

We have (see e.g. [22](14.18.13))

$$\sum_{j=0}^{\infty} \frac{t^j}{(q)_j} h_j(x|a,q) = \frac{(at)_\infty}{\prod_{k=0}^{\infty} v(x|tq^k)}, \tag{2.22}$$

$$\sum_{j=0}^{\infty} \frac{t^j}{[j]_q!} H_j(x|a,q) = \frac{((1-q)at)_\infty}{\prod_{k=0}^{\infty} V_q(x|tq^k)}. \tag{2.23}$$

We have also the following orthogonality relationships for $|a| < 1$ (again e.g. [22](14.18.2))

$$\int_{S(q)} H_n(x|a,q) H_m(x|a,q) f_{bN}(x|a,q) = \begin{cases} 0 & \text{if } n \neq m, \\ [n]_q! & \text{if } n = m, \end{cases}$$

where

$$f_{bN}(x|a,q) = f_N(x|q) \frac{1}{\prod_{k=0}^{\infty} V_q(x|aq^j)},$$

with similar formula for the polynomials $h_n(x|a,q)$.

It should be mentioned also that if $a > 1$ then the measure that makes polynomials $h_n(x|a,q)$ orthogonal has, apart from absolutely continuous part with the apropriety modified density f_{bN}, also $\#\{k : 1 < aq^k \leq a\}$ atoms at points

$$x_k = (aq^k + a^{-1}q^{-k})/2, \tag{2.24}$$

with weights

$$\hat{w}_k = \frac{(1-a^2q^{2k})(a^{-2})_\infty (a^2)_k}{(1-a^2)(q)_k} q^{-(3k^2+k)/2} \left(\frac{-1}{a^4}\right)^k .$$

The family of the big q-Hermite polynomials will be referred to by symbol bH. For details see [22](14.18.3).

2.4. Al-Salam–Chihara. Following literature on special functions, Al-Salam–Chihara polynomials are defined by the following recursion:

$$(2.25)\qquad (2x-(a+b)q^n)Q_n(x|a,b,q) \\ = Q_{n+1}(x|a,b,q) + (1-abq^{n-1})(1-q^n)Q_{n-1}(x|a,b,q),$$

$n \geq 0$, with $Q_{-1}(x|a,b,q) = 0$, $Q_0(x|a,b,q) = 1$. From Favard's theorem ([16]), it follows that if $|ab| \leq 1$, then there exists positive measure with respect to which polynomials Q_n are orthogonal. Further, when $|a|,|b| < 1$ then this measure has density.

As in the case of big q-Hermite polynomials, if one of the parameters a and b is greater than 1 then the measure that makes ASC polynomials orthogonal has $\#\{k : 1 < aq^k \leq a\}$ atoms located at points x_k defined by (2.24) with weights given by:

$$\hat{w}_k = \frac{(a^{-2})_\infty (1-a^2q^{2k})(a^2,ab)_k}{(q,ab,b/a)_\infty (1-a^2)(q,aq/b)_k} q^{-k^2} \left(\frac{1}{a^3b}\right)^k .$$

For details see [22](14.8.3).

Since we are interested in the ASC polynomials in connection with the q-Hermite polynomials, we will consider only the case $|a|,|b| < 1$.

We will more often use these polynomials with new parameters ρ and y defined by $a = \frac{\sqrt{1-q}}{2}\rho\left(y - i\sqrt{\frac{4}{1-q}-y^2}\right)$, $b = \frac{\sqrt{1-q}}{2}\rho\left(y + i\sqrt{\frac{4}{1-q}-y^2}\right)$, such that $y^2 \leq 4/(1-q)$, $|\rho| < 1$. To support intuition let us remark:

$$a+b = \sqrt{1-q}\rho y, \quad ab = \rho^2.$$

More precisely, we will also consider the polynomials

$$(2.26)\qquad P_n(x|y,\rho,q) = Q_n\Bigg(x\frac{\sqrt{1-q}}{2} | \frac{\rho\sqrt{1-q}}{2}\left(y - i\sqrt{\frac{4}{1-q}-y^2}\right), \\ \frac{\rho\sqrt{1-q}}{2}\left(y + i\sqrt{\frac{4}{1-q}-y^2}\right), q\Bigg)/(1-q)^{n/2}.$$

It is also of use to consider another version of the ASC polynomials, namely for $|x|,|y|,|\rho|,|q| < 1$:

$$(2.27)\qquad p_n(x|y,\rho,q) = P_n\left(\frac{2x}{\sqrt{1-q}} | \frac{2y}{\sqrt{1-q}}, \rho, q\right).$$

One can easily show that the polynomials P_n and p_n satisfy the following 3-term recurrence:

$$(x-\rho yq^{n})P_{n}(x|y,\rho,q) = P_{n+1}(x|y,\rho,q)+(1-\rho^{2}q^{n-1})[n]_{q}P_{n-1}(x|y,\rho,q), \tag{2.28}$$

$$2(x-\rho yq^{n})p_{n}(x|y,\rho,q) = p_{n+1}(x|y,\rho,q)+\left(1-\rho^{2}q^{n-1}\right)(1-q^{n})p_{n-1}(x|y,\rho,q), \tag{2.29}$$

with $P_{-1}(x|y,\rho,q) = p_{-1}(x|y,\rho,q) = 0$, $P_0(x|y,\rho,q) = p_0(x|y,\rho,q) = 1$ since as stated above $a+b = \rho y\sqrt{1-q}$ and $ab = \rho^2$ in the case of the polynomials P and $a+b = 2\rho y$ and $ab = \rho^2$ in the case of the polynomials p.

The polynomials $\{P_n\}$ have a nice probabilistic interpretation see e.g. [5]. To support intuition let us notice that for $n \geq 1$:

$$P_{n}(x|y,\rho,1) = (1-\rho^{2})^{n/2}H_{n}\left(\frac{x-\rho y}{\sqrt{1-\rho^{2}}}\right),$$

$$P_{n}(x|y,\rho,0) = U_{n}(x/2)-\rho yU_{n-1}(x/2)+\rho^{2}U_{n-2}(x/2),$$

if we define $U_{-r}(x) = 0$, $r \geq 1$.

We have the following orthogonality relationships (see [22](14.8.2)) satisfied for $|a|, |b| < 1$:

$$\int_{-1}^{1} Q_{n}(x|a,b,q)\,Q_{m}(x|a,b,q)\,\omega(x|a,b,q)\,dx = \begin{cases} 0 & \text{if } n\neq m, \\ (q)_{n}(ab)_{n} & \text{if } m=m, \end{cases}$$

where

$$\omega(x|a,b,q) = \frac{(q)_{\infty}(ab)_{\infty}}{2\pi\sqrt{1-x^{2}}} \times \prod_{k=0}^{\infty}\frac{l\left(x|q^{k}\right)}{((1-abq^{2k})^{2}-2x(a+b)q^{k}(1+abq^{2k})+q^{2k}ab(4x^{2}+(a+b)^{2}/(ab))}.$$

Also after passing to the parameters ρ and y, we get (see [5]):

$$\int_{S(q)} P_{n}(x|y,\rho,q)P_{m}(x|y,\rho,q)\,f_{CN}(x|y,\rho,q)\,dx = \begin{cases} 0 & \text{if } m\neq n, \\ [n]_{q}!\left(\rho^{2}\right)_{n} & \text{if } m=n, \end{cases} \tag{2.30}$$

where we denoted for $|q| < 1$:

$$f_{CN}(x|y,\rho,q) = \frac{\sqrt{1-q}\,(q)_{\infty}\left(\rho^{2}\right)_{\infty}}{2\pi\sqrt{L_{q}(x|1)}}\prod_{k=0}^{\infty}\frac{L_{q}\left(x|q^{k}\right)}{W_{q}\left(x,y|\rho q^{k}\right)}. \tag{2.31}$$

Let us notice that :

$$f_{CN}(x|y,\rho,q) = f_N(x|q)\frac{(\rho^2)_\infty}{\prod_{k=0}^{\infty} W_q(x,y|\rho q^k)}.$$

We also set

$$f_{CN}(x|y,\rho,1) = \frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left(-\frac{(x-\rho y)^2}{2(1-\rho^2)}\right). \tag{2.32}$$

Notice that we have also

$$f_{CN}(x|y,\rho,0) = \frac{(1-\rho^2)\sqrt{4-x^2}}{2\pi W_q(x,y|\rho)},$$

which is called Kesten–McKay density.

Again one shows (see e.g. [18]) that

$$f_{CN}(x|y,\rho,q) \underset{q\to 1^-}{\longrightarrow} f_{CN}(x|y,\rho,1).$$

Following [36], we have: $\forall\, |q| < 1,\ x, y \in S(q)$:

$$0 < \frac{(\rho^2)_\infty}{(-|\rho|)_\infty^4} \leq \frac{f_{CN}(x|y,\rho,q)}{f_N(x|q)} \leq \frac{(\rho^2)_\infty}{(|\rho|)_\infty^4}.$$

One shows (see e.g. [5]) that for $|x|,\ |z| \in S(q)$:

$$\int_{S(q)} f_{CN}(x|y,\rho_1,q)\, f_{CN}(y|z,\rho_2,q)\, dy = f_{CN}(x|z,\rho_1\rho_2,q).$$

This property is nothing else but Chapman–Kolmogorov property satisfied by the density f_{CN} interpreted as the density of the transition distribution of some Markov chain.

Distribution with the density f_{CN} is sometimes called conditional q-Gaussian or conditional q-Normal since we have (2.32). It appeared in [3] and later was analyzed in [4] and [5].

We also have

$$\sum_{k=0}^{\infty} \frac{t^k}{(q)_k} Q_k(x|a,b,q) = \frac{(at,bt)_\infty}{\prod_{j=0}^{\infty} v(x|tq^j)},$$

and for the parameters ρ and y:

$$\sum_{k=0}^{\infty} \frac{t^k}{[k]_q!} P_k(x|y,\rho,q) = \prod_{j=0}^{\infty} \frac{V_q(y|\rho t q^j)}{V_q(x|tq^j)}.$$

2.5. Continuous q-utraspherical polynomials. It turns out that the polynomials $\{H_n\}_{n\geq -1}$ are also related to another family of orthogonal polynomials $\{C_n(x|\beta,q)\}_{n\geq -1}$ which was considered by Rogers in 1894 (see [27]).

Now they are called the continuous q-utraspherical polynomials. The polynomials C_n can be defined through their 3-recurrence (see [22](14.10.19))

$$2(1-\beta q^n)xC_n(x|\beta,q) = (1-q^{n+1})C_{n+1}(x|\beta,q)+(1-\beta^2q^{n-1})C_{n-1}(x|\beta,q),$$

for $n\geq 0$, with $C_{-1}(x|\beta,q)=0$, $C_0(x|\beta,q)=1$, where β is a real parameter such that $|\beta|<1$. One shows (see e.g. [16](13.2.1)) that for $|q|,|\beta|<1$, $n\geq 1$:

$$C_n(x|\beta,q)=\sum_{k=0}^{n}\frac{(\beta)_k(\beta)_{n-k}}{(q)_k(q)_{n-k}}e^{i(n-2k)\theta},$$

where $x=\cos\theta$. Hence we have (following formula (2.40)):

$$C_n(x|0,q)=\frac{h_n(x|q)}{(q)_n}.$$

In fact, we will consider slightly modified polynomials C_n. Namely, we will consider polynomials $R_n(x|\beta,q)$ related to polynomials C_n through the relationship:

$$C_n(x|\beta,q)=(1-q)^{n/2}R_n\left(\frac{2x}{\sqrt{1-q}}|\beta,q\right)/(q)_n,n\geq 1. \tag{2.33}$$

One can easily check that the polynomials $\{R_n\}$ satisfy the following 3-term recurrence:

(2.34)
$$(1-\beta q^n)xR_n(x|\beta,q)=R_{n+1}(x|\beta,q)+\left(1-\beta^2q^{n-1}\right)[n]_qR_{n-1}(x|\beta,q).$$

We have an easy proposition

Proposition 1. *For $n\geq 1$:*

i) $R_n(x|0,q)=H_n(x|q)$,
ii) $R_n(x|q,q)=(q)_nU_n(x\sqrt{1-q}/2)$,
iii) $\lim_{\beta->1^-}\frac{R_n(x|\beta,q)}{(\beta)_n}=2\frac{T_n\left(x\sqrt{1-q}/2\right)}{(1-q)^{n/2}}$.

Proof. i) direct calculation.

ii) We have for $\beta=q$: $\tilde{R}_{n+1}(x|q,q)=x\tilde{R}_n(x|q,q)-\tilde{R}_{n-1}(x|q,q)$, where we denoted $\tilde{R}_n(x|q,q)=(1-q)^{n/2}R_n(x|q,q)/(q)_n$. Now recall (2.7). Since we have $R_0(x|q,q)=\tilde{R}_0(x|q,q)=1$ and $R_1(x|q,q)=\tilde{R}_1(x|q,q)=x$, we deduce that $\tilde{R}_n(x|q,q)=U_n(x/2)$.

iii) Let us first denote $F_n(x|\beta,q)=\frac{R_n(x|\beta,q)}{(\beta)_n}$, write the 3-term recurrence for it obtaining

$$F_{n+1}(x|\beta,q)=xF_n(x|\beta,q)-\frac{(1-q^n)(1-\beta^2q^{n-1})}{(1-q)(1-\beta q^n)(1-\beta q^{n-1})}F_{n-1}(x|\beta,q),$$

with $F_{-1}(x|\beta,q)=0,\ F_0(x|\beta,q)=1$ and let $\beta->1^{-}$. We immediately see that the limit, denote it by $F_n(x|1,q)$, satisfies the following the 3-term recurrence:

$$F_{n+1}(x|1,q)=xF_n(x|1,q)-\frac{F_{n-1}(x|1,q)}{(1-q)},$$

which, confronted with the 3-term recurrence satisfied by the polynomials T_n, proves our assertion. ■

It is known that (see e.g. [16] (13.2.4)):

$$\int_{-1}^{1}C_n(x|\beta,q)C_m(x|\beta,q)f_C(x|\beta,q)\,dx=\begin{cases}0 & \text{if } m\neq n,\\ \frac{(\beta^2)_n}{(1-\beta q^n)(q)_n} & \text{if } m=n,\end{cases}$$

$$\int_{S(q)}R_n(x|\beta,q)R_m(x|\beta,q)f_R(x|\beta,q)=\begin{cases}0 & \text{when } n\neq m,\\ \frac{(1-\beta)(\beta^2)_n[n]_q!}{(1-\beta q^n)} & \text{when } n=m,\end{cases}$$

where we denote

$$f_C(x|\beta,q)=\frac{(\beta^2)_\infty}{(1-\beta)(\beta,\beta q)_\infty}f_h(x|q)\,/\prod_{j=1}^{\infty}l(x|\beta q^j), \tag{2.35}$$

$$f_R(x|\beta,q)=\sqrt{1-q}f_C(x\sqrt{1-q}/2|q)/2 \tag{2.36}$$

$$=\frac{\left(q,\beta^2\right)_\infty\sqrt{1-q}}{(\beta,\beta q)_\infty2\pi\sqrt{L_q(x|1)}}\prod_{k=0}^{\infty}\frac{L_q\left(x|q^k\right)}{L_q\left(x|\beta q^k\right)}. \tag{2.37}$$

Let us remark that

$$f_R(x|\beta,q)=f_N(x|q)\frac{\left(\beta^2\right)_\infty}{(\beta,\beta q)_\infty\prod_{k=0}^{\infty}L_q\left(x|\beta q^k\right)}.$$

Notice also that examining the 3-term recurrence satisfied by P_n and R_n, we see $\forall n\geq-1$:

$$P_n(x|x,\rho,q)=R_n(x|\rho,q),$$

and that for $|x|,|y|\in S(q)$

$$f_{CN}(x|x,\rho,q)/(1-\rho)=f_R(x|\rho,q),$$

since we have $(1-\rho^2q^{2k})^2-(1-q)\rho q^k(1+\rho^2q^{2k})x^2+2(1-q)\rho^2x^2q^{2k}=\left(1-\rho q^k\right)^2\left(\left(1+\rho q^k\right)^2-(1-q)\rho x^2q^k\right)$ and the fact that $\frac{(\rho)_\infty(\rho q)_\infty}{(\rho)_\infty^2}=\frac{1}{1-\rho}$.

We also have

$$\sum_{k=0}^{\infty}t^kC_k(x|\beta,q)=\prod_{k=0}^{\infty}\frac{v\left(x|\beta tq^k\right)}{v\left(x|tq^k\right)}, \tag{2.38}$$

$$\sum_{k=0}^{\infty}\frac{t^k}{[k]_q!}R_k\left(x|\beta,q\right)=\prod_{j=0}^{\infty}\frac{V_q\left(x|\beta tq^k\right)}{V_q\left(x|q^k t\right)}. \tag{2.39}$$

Remark 1. Assertion ii) of Proposition 1 could have been deduced also from (2.39), namely putting $\beta = q$ we get $\sum_{k=0}^{\infty}\frac{t^k}{[k]_q!}R_k\left(x|q,q\right) = \frac{1}{(1-(1-q)tx+(1-q)t^2)}$ which confronted with (2.11) and formula $(q)_n = (1-q)^n\left[n\right]_q!$ leads to the conclusion that $R_k\left(x|q,q\right)/\left(q\right)_k = U_k\left(x\sqrt{1-q}/2\right)$. Following this idea we see that

$$\sum_{k=0}^{\infty}\frac{t^k}{[k]_q!}R_k\left(x|q^2,q\right) = \frac{1}{(1-(1-q)tx+(1-q)t^2)(1-(1-q)txq+(1-q)t^2q^2)}.$$

Hence $R_k\left(x|q^2,q\right)/\left(q\right)_k=\sum_{k=0}^{n}q^kU_k\left(x\sqrt{1-q}/2\right)U_{n-k}\left(x\sqrt{1-q}/2\right)$ using common knowledge on the properties of the generating functions. Simple 'generating functions' argument shows that

$$\sum_{k=0}^{n}q^kU_k\left(x\sqrt{1-q}/2\right)U_{n-k}\left(x\sqrt{1-q}/2\right)$$

simplifies to $\sum_{j=0}^{\lfloor n/2\rfloor}q^j\left[n+1-2j\right]_qU_{n-2j}\left(x\sqrt{1-q}/2\right)$. On the other hand since these polynomials are proportional to $R_k\left(x|q^2,q\right)$, we know their 3-term recurrence and the density that makes them orthogonal. Similarly for other cases $R_k\left(x|q^m,q\right)$, $m\geq 3$. Besides notice that $\forall n\geq -1, x\in\mathbb{R}$ $\lim_{m\to\infty}R_k\left(x|q^m,q\right)=H_n(x|q)$.

We will need also two families of auxiliary polynomials.

2.6. Rogers–Szegö. These polynomials are defined by the equality:

$$s_n\left(x|q\right)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q x^k,$$

for $n\geq 0$ and $s_{-1}\left(x|q\right)=0$. They will be playing here an auxiliary rôle. In particular, one shows (see e.g. [16](13.1.7)) that:

$$h_n\left(x|q\right)=e^{in\theta}s_n\left(e^{-2i\theta}|q\right), \tag{2.40}$$

where $x=\cos\theta$, and that:

$$\sup_{|x|\leq 1}\left|h_n\left(x|q\right)\right|\leq s_n(1|q).$$

In the sequel, the following identities discovered by Carlitz (see Exercise 12.2(b) and 12.2(c) of [16]), are true for $|q|,|t|<1$:

$$\sum_{k=0}^{\infty}\frac{s_k\left(1|q\right)t^k}{(q)_k}=\frac{1}{(t)_{\infty}^{2}},\quad \sum_{k=0}^{\infty}\frac{s_k^2\left(1|q\right)t^k}{(q)_k}=\frac{\left(t^2\right)_{\infty}}{(t)_{\infty}^{4}},\tag{2.41}$$

will allow to show convergence of many considered in the sequel series.

2.7. q^{-1}-Hermite. First of all notice that polynomials $\{H_n(x|q)\}_{n\geq -1}$ are orthogonal for $q>1$ since then $[n]_q>0$. With polynomials h_n it is not so simple. For $q>1$ one has to consider polynomials

$$\left\{(-i)^n(q-1)^{n/2}H_n\Big(-\frac{2ix}{\sqrt{q-1}}|q\Big)\right\}_{n\geq -1}.$$

Discussion of this case is thoroughly done in [20]. Note that in this case the measure that makes these polynomials orthogonal is not unique.

We will need, for auxiliary purposes, polynomials closely related to $H_n(x|q^{-1})$ for $q\in(-1,1]\backslash\{0\}$ and their 'lower case version'. Namely let polynomials $\{B_n\left(x|q\right)\}_{n\geq -1}$ be defined by (compare [5]) $B_n(x|q)=i^nq^{n(n-2)/2}H_n(i\sqrt{q}\,x|q^{-1})$ for $q\neq 0$ and satisfying the following 3-term recurrence:

$$B_{n+1}\left(y|q\right)=-q^nyB_n\left(y|q\right)+q^{n-1}\left[n\right]_qB_{n-1}(y|q),\tag{2.42}$$

for all $n\geq 0$ and with $B_{-1}\left(y|q\right)=0$, $B_0\left(y|q\right)=1$. One easily shows that $B_n\left(x|1\right)=i^nH_n\left(ix\right)$. We will also need the 'continuous' or 'lower case version' of these polynomials namely $b_n\left(y|q\right)=(1-q)^{n/2}B_n\left(2y/\sqrt{1-q}|q\right)$. Polynomials b_n satisfy the following 3-term recurrence:

$$b_{n+1}\left(y|q\right)=-2q^nyb_n\left(y|q\right)+q^{n-1}(1-q^n)b_{n-1}(y|q),\tag{2.43}$$

for all $n\geq 0$ and with $b_{-1}\left(y|q\right)=0$, $b_0\left(y|q\right)=1$. Following [35](2.24), we have $b_n\left(x|q\right)=(-1)^n\,q^{\binom{n}{2}}h_n\left(x|q^{-1}\right)$. (2.44) and (2.45) allow extension of the definitions of b_n and B_n to the case $q=0$. Namely we set $b_0(x|0)=1$, $b_1(x|0)=-2x$, $b_2(x|0)=1$ and $b_n(x|0)=0$, for $n=-1,3,4,\ldots$ and $B_0(x|q)=1$, $B_1(x|q)=-x$ and $B_n(x|q)=0$, $n\geq 2$.

By [5](1.7), we have:

$$\sum_{k=0}^{\infty}\frac{t^k}{[k]_q!}B_k\left(x|q\right)=\prod_{j=0}^{\infty}V_q\left(x|q^jt\right),\ \sum_{k=0}^{\infty}\frac{t^k}{(q)_k}b_k\left(x|q\right)=\prod_{j=0}^{\infty}v(x|q^jt).$$

Comparing the above mentioned formulae, we see that

$$\sum_{k=0}^{\infty}\frac{t^k}{[k]_q!}B_k\left(x|q\right)=1/\sum_{k=0}^{\infty}\frac{t^k}{[k]_q!}H_k(x|q),$$

and similarly for polynomials $\{b_n\}$.

3. Connection coefficients and other useful finite expansions

3.1. Connection coefficients. We consider $n \geq 0$.

T&U

$$T_n(x) = \left(U_n(x) - U_{n-2}(x)\right)/2,$$
$$U_n(x) = 2\sum_{k=0}^{\lfloor n/2\rfloor} T_{n-2k}(x) - \left(1+(-1)^n\right)/2.$$

These expansions belong to common knowledge of the special functions theory.

H&T

$$H_n\left(x|q\right) = (1-q)^{-n/2}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q T_{n-2k}\Big(x\sqrt{1-q}/2\Big),$$

if one sets $T_{-n}(x) = T_n\left(x\right)$.

First notice that (2.40) is equivalent to $h_n(x) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \cos\left(2k-n\right)\theta$ where $x = \cos\theta$. Next we use (2.8).

H&H

$$H_n\left(x|p\right) = \sum_{k=0}^{\lfloor n/2\rfloor}\tilde{C}_{n,n-2k}\left(p,q\right)H_{n-2k}(x|q),$$

where

$$\tilde{C}_{n,n-2k}(p,q) = \frac{(1-q)^{n/2-k}}{(1-p)^{n/2}}$$
$$\times\sum_{j=0}^{k}(-1)^j p^{k-j}q^{j(j+1)/2}\begin{bmatrix} n-2k+j \\ j \end{bmatrix}_q\left(\begin{bmatrix} n \\ k-j \end{bmatrix}_p - p^{n-2k+2j+1}\begin{bmatrix} n \\ k-j-1 \end{bmatrix}_p\right).$$

This formula follows the 'change of base' formula for the continuous q-Hermite polynomials (i.e. polynomials h_n) in e.g. [17], [2] or [14] (formula 7.2) that states that:

$$h_n\left(x|p\right) = \sum_{k=0}^{\lfloor n/2\rfloor}c_{n,n-2k}\left(p,q\right)h_{n-2k}(x|q),$$

where

$$c_{n,n-2k}\left(p,q\right) = \frac{(1-p)^{n/2}}{(1-q)^{n/2-k}}\tilde{C}_{n,n-2k}(p,q).$$

U&H

$$U_n\left(x\sqrt{1-q}/2\right) = \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\left(1-q\right)^{n/2-j}q^{j(j+1)/2}\begin{bmatrix} n-j \\ j \end{bmatrix}_q H_{n-2j}(x|q),$$

$$H_n\left(y|q\right)=\sum_{k=0}^{\lfloor n/2\rfloor}(1-q)^{-n/2}\frac{q^k-q^{n-k+1}}{1-q^{n-k+1}}\begin{bmatrix}n\\k\end{bmatrix}_q U_{n-2k}\Big(y\sqrt{1-q}/2\Big).$$

These expansion follow the previous one setting once $p=0$ and then secondly $q=0$ and then $p=q$.

H&bH

$$h_n\left(x|a,q\right)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^kq^{\binom{k}{2}}a^kh_{n-k}(x|q), \tag{3.1}$$

$$H_n\left(x|a,q\right)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(-1)^kq^{\binom{k}{2}}a^kH_{n-k}(x|q). \tag{3.2}$$

(3.1) is a formula (19) of [11] (see also [12]). (3.2) is a simple consequence of (3.1).

H&P

$$P_n\left(x|y,\rho,q\right)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}\rho^{n-j}B_{n-j}\left(y|q\right)H_j(x|q), \tag{3.3}$$

$$H_n(x|q)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}\rho^{n-j}H_{n-j}\left(y|q\right)P_j(x|y,\rho,q). \tag{3.4}$$

For the proof of (3.3), see Remark 1 following Theorem 1 in [5]. For the proof of (3.4), we start with formula (4.7) in [19] which gives connection coefficients of h_n with respect to Q_n. Then we pass to the polynomials H_n&P_n using formulae $h_n\left(x|q\right)=(1-q)^{n/2}H_n\left(\frac{2x}{\sqrt{1-q}}|q\right)$, $n\geq1$ and $p_n(x|a,b,q)=(1-q)^{n/2}P_n\left(\frac{2x}{\sqrt{1-q}}|\frac{2a}{\sqrt{(1-q)b}},\sqrt{b},q\right)$. By the way, notice that this formula can be easily derived from assertions iv) and (3.18) with $m=0$ presented below and the standard change of order of summation. Now it remains to return to polynomials H_n.

As a corollary of (3.4) and (2.30), we get a nice formula given in [5]: For $\forall n\geq1,|\rho|<1,\ y\in S\left(q\right)$

$$\int\limits_{S(q)}H_n(x|q)f_{CN}\left(x|y,\rho,q\right)dx=\rho^nH_n\left(y|q\right).$$

bH&P

$$H_n\left(x|a,q\right)=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_qP_j\left(x|y,\frac{a}{b},q\right)\left(\frac{a}{b}\right)^{n-j}H_{n-j}(y|b,q), \tag{3.5}$$

$$P_n\left(x|y,\rho,q\right)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\rho^{n-k}B_{n-k}\left(x|a/\rho,q\right)H_k(x|a,q), \tag{3.6}$$

where we denoted $B_m(x|b,q) \overset{df}{=} \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q b^{m-j} B_j(x|q)$. Strict proof of (3.5) and (3.6) is presented in [36]. It is easy and is based on (3.3) and (3.2).

P&P

$$P_n(x|y,\rho,q) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q r^{n-j} P_j(x|z,r,q)\, P_{n-j}(z|y,\rho/r,q), \tag{3.7}$$

$$\frac{P_n(y|z,t,q)}{(t^2)_n} = \sum_{j=0}^{n} (-t)^j q^{j(j-1)/2} \begin{bmatrix} n \\ j \end{bmatrix}_q t^j H_{n-j}(y|q) \frac{P_j(z|y,t,q)}{(t^2)_j}, \tag{3.8}$$

if one extends definition of polynomials P_n for $|\rho| > 1$ by (3.3). (3.7) has been proved in [36], while (3.8) is given in [35] Corollary 2. Besides, it follows directly from one of the infinite expansions that will be presented in section 4.

As a corollary of (3.8) and of course (2.30), we get the following formula: For $\forall n \geq 1, |\rho| < 1, x \in S(q)$

$$\int_{S(q)} P_n(x|y,\rho,q)\, f_{CN}(y|x,\rho,q)\, dy = \left(\rho^2\right)_n H_n(x|q).$$

R&R

For $|\beta|, |\gamma| < 1$:

$$R_n(x|\gamma,q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{[n]_q! \beta^k (\gamma/\beta)_k (\gamma)_{n-k} \left(1 - \beta q^{n-2k}\right)}{[k]_q! [n-2k]_q! (\beta q)_{n-k} (1-\beta)} R_{n-2k}(x|\beta,q). \tag{3.9}$$

(3.9) is in fact celebrated connection coefficient formula for the Rogers polynomials which was in fact expressed in terms of polynomials C_n. For details see [16] (13.3.1).

R&H

For $|\beta|, |\gamma| < 1$:

$$R_n(x|\gamma,q) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-\gamma)^k \frac{q^{k(k-1)/2} [n]_q! (\gamma)_{n-k}}{[k]_q! [n-2k]_q!} H_{n-2k}(x|q), \tag{3.10}$$

$$H_n(x|q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{[n]_q!}{[k]_q! [n-2k]_q!} \frac{\beta^k \left(1 - \beta q^{n-2k}\right)}{(1-\beta)(\beta q)_{n-k}} R_{n-2k}(x|\beta,q). \tag{3.11}$$

(3.10) and (3.11) are particular cases of (3.9), the first for $\beta = 0$ and the second for $\gamma = 0$.

B&H

$$B_n(x|q) = (-1)^n q^{\binom{n}{2}} \sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ k \end{bmatrix}_q [k]_q! q^{k(k-n)} H_{n-2k}(x|q). \tag{3.12}$$

(3.12) was proved in [35] Lemma 2 assertion i).

As an immediate observation, we have the following expansion of the ASC polynomials in the q-Hermite polynomials.

Proposition 2.

$$P_n(x|y,\rho,q) = \sum_{k=0}^{\lfloor n/2\rfloor} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ k \end{bmatrix}_q [k]_q! q^{k(k-1)} \rho^{2k} \times \sum_{s=0}^{n-2k} (-1)^s \begin{bmatrix} n-2k \\ s \end{bmatrix}_q q^{\binom{s}{2}} (q^k\rho)^s H_{n-2k-s}(x|q) H_s(y|q).$$

Proof. First we use (3.3) and then (3.12) obtaining: $P_n(x|y,\rho,q) = \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix}_q H_{n-s}(x|q)\rho^s(-1)^s q^{\binom{s}{2}} \times \sum_{k=0}^{\lfloor s/2\rfloor} \begin{bmatrix} s \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ k \end{bmatrix}_q [k]_q! q^{k(k-s)} H_{s-2k}(y|q)$. Now we change the order of summation. ∎

3.2. Useful finite expansions. We start with the so called 'linearization formulae'. These are formulae expressing the product of two or more polynomials of some type as linear combinations of polynomials of this very type. We will extend the name 'linearization formulae' by relaxing the requirement of polynomials involved to be of the same type. Generally obtaining 'linearization formula' is not simple and requires a lot of tedious calculations.

3.2.1. Linearization formulae. We assume $n, m, k \geq 0$.

H&H

The formulae below can be found in e.g. [16] (Thm. 13.1.5) and also in [1] and originally were formulated for polynomials h_n. Below, they are presented for polynomials H_n using (2.13):

$$H_n(x|q)H_m(x|q) = \sum_{j=0}^{\min(n,m)} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q [j]_q! H_{n+m-2j}(x|q), \tag{3.13}$$

$$\begin{aligned} &H_n(x|q)H_m(x|q)H_k(x|q) \\ &= \sum_{r,s} \begin{bmatrix} m \\ r \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q \begin{bmatrix} k \\ s \end{bmatrix}_q \begin{bmatrix} m+n-2r \\ s \end{bmatrix}_q [s]_q! [r]_q! H_{n+m+k-2r-2s}(x|q) \\ &= \sum_{j=0}^{\lfloor (k+m+n)/2\rfloor} \left(\sum_{r=\max(j-k,0)}^{\min(m,n,m+n-j)} \begin{bmatrix} m \\ r \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q \begin{bmatrix} k \\ j-r \end{bmatrix}_q \begin{bmatrix} m+n-2r \\ j-r \end{bmatrix}_q [r]_q [j-r]_q \right) \\ &\qquad \times H_{n+m+k-2j}(x|q). \end{aligned} \tag{3.14}$$

In fact, (3.13) can be easily derived (by re-scaling and changing of variables) from an old result of Carlitz ([10]) that was formulated in terms of the Rogers–Szegö $\{s_n(x|q)\}_{n\geq -1}$ polynomials. Carlitz proved in the same

paper another useful identity concerning polynomials s_n that can be easily reformulated in terms of the polynomials H_n. The formula below is in a sense an inverse of (3.13). Namely we have:
(3.15)

$$H_{n+m}(x|q)=\sum_{k=0}^{\min(n,m)}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}m\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q[k]_q!H_{n-k}(x|q)H_{m-k}(x|q).$$

H&B

$$\begin{aligned}(3.16)\qquad & H_m(x|q)B_n(x|q)\\ &=(-1)^n q^{\binom{n}{2}}\sum_{k=0}^{\lfloor(n+m)/2\rfloor}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n+m-k\\k\end{bmatrix}_q[k]_q!q^{-k(n-k)}H_{n+m-2k}(x|q).\end{aligned}$$

This formula, having technical importance, has been proved in [35], Lemma 2, assertion ii).

H&R

We have also useful formula:

$$\begin{aligned}(3.17)\qquad & H_m(x|q)R_n\left(x|\beta,q\right)\\ &=\sum_{k,j}\begin{bmatrix}m\\j\end{bmatrix}_q\begin{bmatrix}n\\k+j\end{bmatrix}_q\begin{bmatrix}n-k-j\\k\end{bmatrix}[k+j]_q!\left(-\beta\right)^k q^{\binom{k}{2}}\left(\beta\right)_{n-k}H_{n+m-2k-2j}(x|q).\end{aligned}$$

Which was proved in [1] (1.9) for h_n and C_n and then modified using (2.13) and (2.33).

Q&Q

For completeness, let us mention that in [31] there is given a very complicated linearization formula for Al-Salam–Chihara polynomials given in Theorem 1.

3.2.2. Useful finite sums and identities. We have also the following a very useful generalization of formula (1.12) of [5] which was proved in [35] (Lemma 2, assertion i)).

For all $n\geq 0$:

$$(3.18)\qquad \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q B_{n-k}(x|q)H_{k+m}\left(x|q\right)=\begin{cases}0 & \text{if } n>m,\\ (-1)^n q^{\binom{n}{2}}\frac{[m]_q!}{[m-n]_q!}H_{m-n}(x|q) & \text{if } m\geq n.\end{cases}$$

Let us remark that for $q=0$, (3.18) reduces to 3-term recurrence of polynomials $U_n\left(x/2\right)$.

For $q=1$, we get

$$\sum_{k=0}^{n}\binom{n}{k}i^{n-k}H_{n-k}\left(ix\right)H_{k+m}\left(x\right)=\begin{cases}0 & \text{if } n>m,\\ (-1)^{n}\frac{m!}{(m-n)!}H_{m-n}(x) & \text{if } m\geq n.\end{cases}$$

Recently in [38], the following identities involving ASC polynomials p_n were given:

i) $\forall n\geq 1, 0\leq k<n, z,y,t\in\mathbb{R}$:

$$\sum_{j=0}^{n-k}\begin{bmatrix}n-k\\ j\end{bmatrix}_{q}\frac{p_{j}\left(z|y,tq^{k},q\right)}{\left(t^{2}q^{2k}\right)_{j}}\frac{g_{n-k-j}\left(z|y,tq^{n-1},q\right)}{\left(t^{2}q^{n+j+k-1}\right)_{n-k-j}}=0,$$

ii) $\forall n\geq 1, 0\leq k<n, z,y,t\in\mathbb{R}$:

$$\sum_{m=0}^{n-k}\begin{bmatrix}n-k\\ m\end{bmatrix}_{q}\frac{p_{n-k-m}\left(z|y,tq^{m+k},q\right)g_{m}(z|y,tq^{m+k-1},q)}{\left(t^{2}q^{2m+2k}\right)_{n-k-m}\left(t^{2}q^{m+2k-1}\right)_{m}}=0,$$

where polynomials g_n are somewhat analogous to polynomials b_n and are defined by the formula:

$$g_{n}\left(x|y,\rho,q\right)=\begin{cases}\rho^{n}p_{n}\left(y|x,\rho^{-1},q\right) & \text{if } \rho\neq 0,\\ b_{n}(x|q) & \text{if } \rho=0.\end{cases} \tag{3.19}$$

Similar ones, involving polynomials P_n and appropriately modified polynomials g_n, were also presented in [38].

Let us mention that polynomials g_n play, with respect to polynomials p_n, similar rôle as polynomials b_n with respect to polynomials h_n. Namely we have:

for all $|t|,|q|,|\rho|<1$, $|x|,|y|\leq 1$:

$$\sum_{n=0}^{\infty}\frac{t^{n}}{\left(q\right)_{n}}g_{n}\left(x|y,\rho,q\right)=1/\varphi_{p}(x,t|y,\rho,q),$$

for all $n\geq 1$, $x,y,\rho\in\mathbb{R}$:

$$\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}_{q}p_{j}\left(x|y,\rho,q\right)g_{n-j}\left(x|y,\rho,q\right)=0.$$

4. Infinite expansions

4.1. Kernels. We start with the famous Poisson–Mehler expansion of $f_{CN}\left(x|y,\rho,q\right)/f_{N}(x|q)$ in an infinite series of Mercier's type (compare e.g. [25]). Namely the following fact is true:

Theorem 1. $\forall |q|, |\rho| < 1; x, y \in S(q):$

$$\frac{(\rho^2)_\infty}{\prod_{k=0}^{\infty} W_q(x, y|\rho q^k)} = \sum_{n=0}^{\infty} \frac{\rho^n}{[n]_q!} H_n(x|q) H_n(y|q). \tag{4.1}$$

For $q = 1$, $x, y \in \mathbb{R}$ we have

$$\frac{\exp\left(\frac{x^2+y^2}{2}\right)}{\sqrt{1-\rho^2}} \exp\left(-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} H_n(x) H_n(y). \tag{4.2}$$

Proof. There exist many proofs of both formulae (see e.g. [16], [2]). One of the shortest, exploiting connection coefficients, given in (3.3) is given in [33]. ∎

Corollary 1. $\forall |q|, |\rho| < 1;\ x \in S(q):$

$$\sum_{k \geq 0} \frac{\rho^k \left(\rho q^{k-1}\right)_\infty}{[k]_q!} H_{2k}(x|q) = \frac{\left(\rho^2\right)_\infty}{(\rho)_\infty} \prod_{k=0}^{\infty} L_q^{-1}\left(x|\rho q^k\right).$$

Proof. We put $y = x$ in (4.1), then we apply (3.13), change order of summation and finally apply formulae $\frac{1}{(\rho)_{j+1}} = \sum_{k\geq 0} \begin{bmatrix} j+k \\ k \end{bmatrix}_q \rho^k$ and $\frac{(\rho)_\infty}{(\rho)_{j+1}} = \left(q^{j-1}\rho\right)_\infty$ ∎

We will call expression of the form of the right hand side of (4.1) the kernel expansion while the expressions from the left hand side of (4.1) kernels. The name refers to Mercier's theorem and the fact that for example

$$\int_{S(q)} k(x, y|\rho, q) H_n(x|q) f_N(x|q) dx = \rho^n H_n(y|q) f_N(y|q),$$

where we denoted by $k(x, y|\rho, q)$ the left hand side of (4.1). Hence, we see that k is a kernel, while function $H_n(x|q) f_N(x|q)$ are eigenfunctions of kernel k with ρ^n being an eigenvalue related to an eigenfunction $H_n(x|q) f_N(x|q)$. Such kernels and kernel expansions are very important in analysis or quantum physics in the analysis of different models of harmonic oscillators.

In the literature however there is some confusion concerning terminology. Sometimes expression of the form $\sum_{n\geq 0} a_n p_n(x) p_n(y)$ where $\{p_n\}$ is a family of polynomials are also called kernels (like in [41])) or even sometimes 'bilinear generating function' (see e.g. [26])) or also Poisson kernels. If $p_n(y)$ is replaced with $q_n(y)$ then we deal with the non-symmetric kernel.

The process of expressing these sums in a closed form is then called 'summing of kernels'.

Summing the kernel expansions is difficult. Proving positivity of the kernels is another difficult problem. Only some are known and have relatively simple forms. In most cases sums are in the form of a complex finite sum of the so called basic hypergeometric functions. Below, we will present several of them. Mostly the ones involving the big q-Hermite, Al-Salam–Chihara and q-ultraspherical polynomials.

To present more complicated sums we will need the following definition of the basic hypergeometric function namely

$$ {}_j\phi_k\left[\begin{matrix} a_1\ a_2\ \ldots\ a_j \\ b_1\ b_2\ \ldots\ b_k \end{matrix};q,x\right]=\sum_{n=0}^{\infty}\frac{(a_1,\ldots,a_j)}{(b_1,\ldots,b_k)}\left((-1)^n q^{\binom{n}{2}}\right)^{1+k-j}x^n, \tag{4.3}$$

$$\begin{aligned} &{}_{2m}W_{2m-1}\left(a,a_1,\ldots,a_{2m-3};q,x\right) \\ &\quad={}_{2m}\phi_{2m-1}\left[\begin{matrix} a & q\sqrt{a} & -q\sqrt{a} & a_1 & a_2 & \ldots & a_{2m-3} \\ \sqrt{a} & -\sqrt{a} & \frac{qa}{a_1} & \frac{qa}{a_2} & \ldots & \frac{qa}{a_{2m-3}} & \end{matrix};q,x\right]. \end{aligned} \tag{4.4}$$

We will now present the kernels built of families of polynomials that are discussed here and their sums.

Theorem 2. i) *For all* $|t|<1,\ |x|,|y|<2$:

$$\sum_{n=0}^{\infty}t^nU_n\left(x/2\right)U_n\left(y/2\right)=\frac{\left(1-t^2\right)}{\left(\left(1-t^2\right)^2-t\left(1+t^2\right)xy+t^2(x^2+y^2)\right)}.$$

ii) *For all* $|t|<1,\ |x|,|y|<1$:

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{(1-\beta q^n)\,(q)_n}{(1-\beta)\,(\beta^2)_n}t^nC_n\left(x|\beta,q\right)C_n\left(y|\beta,q\right)&=\frac{(\beta q)_\infty^2}{(\beta^2)_\infty\,(\beta t^2)_\infty}\prod_{n=0}^{\infty}\frac{w\left(x,y|t\beta q^n\right)}{w\left(x,y|tq^n\right)} \\ &\times{}_8W_7\left(\frac{\beta t^2}{q},\frac{\beta}{q},te^{i(\theta+\phi)},te^{-i(\theta+\phi)},te^{i(\theta-\phi)},te^{-i(\theta-\phi)};q,\beta q\right), \end{aligned}$$

where $x=\cos\theta,\ y=\cos\phi$.
iii) *For all* $|x|,|y|,|t|,|tb/a|\leq 1$:

$$\begin{aligned} \sum_{n\geq 0}\frac{(tb/a)^n}{(q)_n}h_n\left(x|a,q\right)h_n\left(y|b,q\right)&=\left(\frac{b^2t^2}{a^2}\right)_\infty\prod_{k=0}^{\infty}\frac{v\left(x|tbq^k\right)}{w\left(x,y|t\frac{b}{a}q^k\right)} \\ &\times{}_3\phi_2\left(\begin{matrix} t & bte^{i(\theta+\phi)}/a & bte^{i(-\theta+\phi)}/a \\ b^2t^2/a^2 & bte^{i\phi} & \end{matrix};q,be^{-i\phi}\right), \end{aligned} \tag{4.5}$$

with $x=\cos\theta$ *and* $y=\cos\phi$.

iv) *For all* $|t|<1,\ x,y\in S(q),\ ab=\alpha\beta$:

$$\sum_{n\geq 0}\frac{(t\alpha/a)^{n}}{(q)_{n}(ab)_{n}}Q_{n}(x|a,b,q)\,Q_{n}(y|\alpha,\beta,q)$$
$$=\frac{\left(\frac{\alpha^{2}t^{2}}{a},\frac{\alpha^{2}t}{a}e^{i\theta},be^{-i\theta},bte^{i\theta},\alpha te^{-i\phi},\alpha te^{i\phi}\right)_{\infty}}{\left(ab,\frac{\alpha^{2}t^{2}}{a}e^{i\theta}\right)_{\infty}\prod_{k=0}^{\infty}w\left(x,y|\frac{\alpha t}{a}q^{k}\right)}$$
$$\times{}_{8}W_{7}\left(\frac{\alpha^{2}t^{2}e^{i\theta}}{aq},t,\frac{\alpha t}{\beta},ae^{i\theta},\frac{\alpha t}{a}e^{i(\theta+\phi)},\frac{\alpha t}{a}e^{i(\theta-\phi)};q,be^{-i\theta}\right),$$

where as before $x=\cos\theta$ *and* $y=\cos\phi$ *and*

$$\sum_{n\geq 0}\frac{t^{n}}{(q)_{n}(ab)_{n}}Q_{n}(x|a,b,q)\,Q_{n}(y|\alpha,\beta,q)$$
$$=\frac{\left(\frac{\beta t}{a}\right)_{\infty}}{(\alpha at)_{\infty}}\prod_{k=0}^{\infty}\frac{(1+\alpha^{2}t^{2}q^{2k})^{2}-2\alpha tq^{k}(x+y)\left(1+\alpha^{2}t^{2}q^{2k}\right)+4\alpha^{2}xyt^{2}q^{2k}}{w\left(x,y|tq^{k}\right)}$$
$$\times\ {}_{8}W_{7}\left(\frac{\alpha at}{q},\frac{\alpha t}{b},ae^{i\theta},ae^{-i\theta},\alpha e^{i\phi},\alpha e^{-i\phi};q;\frac{\beta t}{a}\right).$$

v) *For all* $|\rho_1|,|\rho_2|,|q|<1,\ x,y\in S(q)$

$$0\leq\sum_{n\geq 0}\frac{\rho_{1}^{n}}{[n]_{q}!\left(\rho_{2}^{2}\right)_{n}}P_{n}(x|y,\rho_{2},q)\,P_{n}\left(z|y,\frac{\rho_{2}}{\rho_{1}},q\right)=\frac{\left(\rho_{1}^{2}\right)_{\infty}}{\left(\rho_{2}^{2}\right)_{\infty}}\prod_{k=0}^{\infty}\frac{W_{q}\left(x,z|\rho_{2}q^{k}\right)}{W_{q}\left(x,y|\rho_{1}q^{k}\right)}.\tag{4.6}$$

Remarks concerning the proof. i) We set $q=0$ in (4.1) and use (2.15). ii) It is formula (1.7) in [26] based on [15]. iii) It is formula (14.14) in [41]. iv) These are formulae (14.5) and (14.8) of [41]. v) Notice that it cannot be derived from assertion iv) since the condition $ab=\alpha\beta$ is not satisfied. Recall that (see (2.26)) $ab=\rho_2^2$ while $\alpha\beta=\rho_1^2$. For the proof recall the idea of expansion of ratio of densities presented in [33], use formulae (3.7) and (2.30) and finally notice that $f_{CN}(x|y,\rho_1,q)/f_{CN}(x|z,\rho_2,q)=\frac{\left(\rho_1^2\right)_\infty}{\left(\rho_2^2\right)_\infty}\prod_{k=0}^{\infty}\frac{W_q\left(x,z|\rho_2q^k\right)}{W_q\left(x,y|\rho_1q^k\right)}$. ∎

Corollary 2. *For all* $|a|>|b|,\ x,y\in S(q)$:

$$0\leq\sum_{n\geq 0}\frac{b^{n}}{[n]_{q}!a^{n}}H_{n}(x|a,q)\,H_{n}(y|b,q)=\left(\frac{b^{2}}{a^{2}}\right)_{\infty}\prod_{k=0}^{\infty}\frac{V_{q}\left(x|bq^{k}\right)}{W_{q}\left(x,y|\frac{b}{a}q^{k}\right)}.$$

Proof. We set $t=0$ in (4.5) and assume $|b|<|a|$. For an alternative simple proof see [36]. ∎

4.2. Other infinite expansions. In this subsection, we will present some expansions that can be viewed as reciprocals of some presented above expansions and also some generalizations of so called Kibble–Slepian formula.

We start with some reciprocals of the expansions obtained above.

4.2.1. Expansions of kernel's reciprocals

Theorem 3. i) *For* $|q|,|\rho|<1,\ x,y\in S(q)$:

$$1/\sum_{n=0}^{\infty}\frac{\rho^n}{[n]_q!}H_n(x|q)\,H_n(y|q)=\sum_{n=0}^{\infty}\frac{\rho^n}{(\rho^2)_n\,[n]_q!}B_n(y|q)\,P_n(x|y,\rho,q).$$

ii) *For* $x,y\in\mathbb{R}$ *and* $\rho^2<1/2$

$$1/\sum_{n=0}^{\infty}\frac{\rho^n}{n!}H_n(x)H_n(y)=\sum_{n=0}^{\infty}\frac{\rho^n i^n}{n!\,(1-\rho^2)^{n/2}}H_n(ix)\,H_n\left(\frac{(x-\rho y)}{\sqrt{1-\rho^2}}\right).$$

iii) *For* $|q|<1,\ |a|<|b|,\ x,y\in S(q)$:

$$1/\sum_{n\geq 0}\frac{a^n}{[n]_q!b^n}H_n(x|a,q)\,H_n(y|b,q)$$
$$=\sum_{n\geq 0}\frac{a^n}{[n]_q!b^n\,(a^2/b^2)_n}B_n(y|b,q)\,P_n(x|y,a/b,q).$$

iv) *For* $|\rho_1|,|\rho_2|,|q|<1,\ x,y\in S(q)$:

$$1/\sum_{n\geq 0}\frac{\rho_1^n}{[n]_q!\,(\rho_2^2)_n}P_n(x|y,\rho_2,q)\,P_n\left(z|y,\frac{\rho_2}{\rho_1},q\right)$$
$$=\sum_{n\geq 0}\frac{\rho_2^n}{[n]_q!\,(\rho_1^2)_n}P_n(x|z,\rho_1,q)\,P_n\left(y|z,\frac{\rho_1}{\rho_2},q\right).$$

Remarks concerning the proof. i) and ii) are proved in [33]. iii) is proved in [36]. iv) directly follows from (4.6) ∎

4.2.2. Some auxiliary infinite expansions. The result below can be viewed as summing certain non-symmetric kernel.

Lemma 1. *For* $x,y\in S(q)$, $|\rho|<1$ *let us denote*

$$\gamma_{m,k}(x,y|\rho,q)=\sum_{k=0}^{\infty}\frac{\rho^k}{[k]_q!}H_{k+m}(x|q)\,H_{k+k}(y|q).$$

Then

$$\gamma_{m,k}(x,y|\rho,q)=\gamma_{0,0}(x,y|\rho,q)\,\Xi_{m,k}(x,y|\rho,q), \tag{4.7}$$

where $\Xi_{m,k}$ *is a polynomial in* x *and* y *of order at most* $m+k$.

Further denote $C_n(x,y|\rho_1,\rho_2,\rho_3,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\rho_1^{n-k}\rho_2^k\Xi_{n-k,k}\left(x,y|\rho_{3,}q\right)$. *Then we have in particular*

i) $\Xi_{m,k}\left(x,y|\rho,q\right)=\Xi_{k,m}(y,x|\rho,q)$,

$$\Xi_{m,k}\left(x,y|\rho,q\right)=\sum_{s=0}^{k}(-\rho)^sq^{\binom{s}{2}}\begin{bmatrix}k\\s\end{bmatrix}_qH_{k-s}\left(y|q\right)P_{m+s}(x|y,\rho,q)/(\rho^2)_{m+s},$$

ii) *and*

$$C_n\left(x,y|\rho_1,\rho_2,\rho_3,q\right) =\sum_{s=0}^{n}\begin{bmatrix}n\\s\end{bmatrix}_qH_{n-s}\left(y|q\right)P_s\left(x|y,\rho_3,q\right)\rho_1^{n-s}\rho_2^s\left(\rho_1\rho_3/\rho_2\right)_s/\left(\rho_3^2\right)_s. \tag{4.8}$$

Proof. Proof that $\gamma_{m,k}\left(x,y|\rho,q\right)/\gamma_{0,0}\left(x,y|\rho,q\right)$ is a polynomial can be deduced from [8] (formula 1.4) where the result was formulated for Rogers–Szegö polynomials. To get the ii) from this result of Carlitz using (2.40) is not easy. For the alternative, simple although lengthy proof of the general case and other assertions we refer the reader to [34] and [35]. ∎

Exploring Carlitz paper [8] and confronting it with above Lemma 1, we arrive at the following conversion lemma.

Lemma 2. $\forall n,m\geq 0$, $|t|<1$, $\theta\in(-\pi,\pi]$:

$$\sum_{k=0}^{m}\sum_{l=0}^{n}\begin{bmatrix}m\\k\end{bmatrix}_q\begin{bmatrix}n\\l\end{bmatrix}_q\frac{\left(te^{i(-\theta+\eta)}\right)_k\left(te^{i(\theta-\eta)}\right)_l\left(te^{-i(\theta+\eta)}\right)_{k+l}}{(t^2)_{k+l}}e^{-i(m-2k)\theta}e^{-i(n-2l)\eta} =\sum_{j=0}^{n}(-1)^jq^{\binom{j}{2}}\begin{bmatrix}n\\j\end{bmatrix}_qt^jh_{n-j}(y|q)p_{m+j}\left(x|y,t,q\right)/\left(t^2\right)_{j+m}, \tag{4.9}$$

with $x=\cos\theta$ *and* $y=\cos\eta$.

Proof. See [38] Proposition 6. ∎

4.2.3. Generalization of Kibble–Slepian formula. Recall that Kibble in 1949 [21] and independently Slepian in 1972 [30] extended the Poisson–Mehler formula to higher dimensions, expanding ratio of the standardized multidimensional Gaussian density divided by the product of one dimensional marginal densities in the multiple sum involving only constants (correlation coefficients) and the Hermite polynomials. The formula in its generality can be found in [16] (4.7.2 p. 107). Since we are going to generalize its 3-dimensional version, only this version will be presented here.

Namely let us consider 3-dimensional density $f_{3D}\left(x_1,x_2,x_3;\rho_{12},\rho_{13},\rho_{23}\right)$ of Normal random vector $N\left(\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}1&\rho_{12}&\rho_{13}\\\rho_{12}&1&\rho_{23}\\\rho_{13}&\rho_{23}&1\end{bmatrix}\right)$. Of course we must assume that the parameters ρ_{12}, ρ_{13}, ρ_{23} are such that the variance covariance matrix is positive definite i.e. such that

$$1+2\rho_{12}\rho_{13}\rho_{23}-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2>0. \tag{4.10}$$

Then Kibble–Slepian formula reads that

$$\exp\left(\frac{x_1^2+x_2^2+x_3^2}{2}\right)f_{3D}\left(x_1,x_2,x_3;\rho_{12},\rho_{13},\rho_{23}\right) = \sum_{k,m,m=0}^{\infty}\frac{\rho_{12}^k\rho_{13}^m\rho_{23}^n}{k!m!n!}H_{k+m}\left(x_1\right)H_{k+n}\left(x_2\right)H_{m+n}(x_3).$$

Thus immediate generalization of this formula would be to substitute the Hermite polynomials by the q-Hermite ones and factorials by the q-factorials.

The question is if such sum is positive. It turns out that not in general i.e. not for all ρ_{12}, ρ_{13}, ρ_{23} satisfying (4.10). Nevertheless, it is interesting to compute the sum

$$\sum_{k,m,m=0}^{\infty}\frac{\rho_{12}^k\rho_{13}^m\rho_{23}^n}{[k]_q!\,[m]_q!\,[n]_q!}H_{k+m}\left(x_1|q\right)H_{k+n}\left(x_2|q\right)H_{m+n}(x_3|q). \tag{4.11}$$

For simplicity, let us denote this sum by $g\left(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q\right)$.

In [37], the following result have been formulated and proved.

Theorem 4. i)

$$g\left(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q\right) = \frac{\left(\rho_{13}^2\right)_\infty}{\prod_{k=0}^{\infty}W_q\left(x_1,x_3|\rho_{13}q^k\right)}\sum_{s\geq0}\frac{1}{[s]_q!}H_s\left(x_2|q\right)C_s\left(x_1,x_3|\rho_{12},\rho_{23},\rho_{13},q\right) \tag{4.12}$$

where $C_n\left(x_1,x_3|\rho_{12},\rho_{23},\rho_{13},q\right)$ *is given by either* (4.8) *or can be expressed in terms of polynomials* H_n *in the following form:*

$$C_n\left(x_1,x_3|\rho_{12},\rho_{23},\rho_{13},q\right) = \frac{1}{\left(\rho_{13}^2\right)_n}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^kq^{\binom{k}{2}}\begin{bmatrix}n\\2k\end{bmatrix}_q\begin{bmatrix}2k\\k\end{bmatrix}_q[k]_q!\rho_{12}^k\rho_{13}^k\rho_{23}^k\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_k\left(\frac{\rho_{13}\rho_{23}}{\rho_{12}}\right)_k \sum_{j=0}^{n-2k}\begin{bmatrix}n-2k\\j\end{bmatrix}_q\rho_{23}^j\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}q^k\right)_k\rho_{12}^{n-j-2k}\left(\frac{\rho_{13}\rho_{23}}{\rho_{12}}q^k\right)_{n-2k-j} H_j(x_1|q)H_{n-2k-j}(x_3|q),$$

similarly for other pairs $(1,3)$ *and* $(2,3)$*,* ii)

$$g\left(x_{1},x_{2},x_{3}|\rho_{12},\rho_{13},\rho_{23},q\right) = \frac{\left(\rho_{13}^{2},\rho_{23}^{2}\right)_{\infty}}{\prod_{k=0}^{\infty}W_{q}\left(x_{1},x_{3}|\rho_{13}q^{k}\right)W_{q}(x_{3},x_{2}|\rho_{23}q^{k})} \times\sum_{s=0}^{\infty}\frac{\rho_{12}^{s}\left(\rho_{13}\rho_{23}/\rho_{12}\right)_{s}}{\left[s\right]_{q}!\left(\rho_{13}^{2}\right)_{s}\left(\rho_{23}^{2}\right)_{s}}P_{s}\left(x_{1}|x_{3},\rho_{13},q\right)P_{s}(x_{2}|x_{3},\rho_{23},q), \tag{4.13}$$

similarly for other pairs $(1,3)$ *and* $(2,3)$.

Unfortunately, as shown in [37], one can find such ρ_{12}, ρ_{13}, ρ_{23} that the function g with these parameters assumes negative values for some $x_{j}\in S\left(q\right)$, $j=1,2,3$ hence $g\left(x_{1},x_{2},x_{3}|\rho_{12},\rho_{13},\rho_{23},q\right)\prod_{j=0}^{3}f_{N}\left(x_{j}|q\right)$ with these values of parameters is not a density of a probability distribution.

Remark 2. Notice that if $\rho_{12}=q^{m}\rho_{13}\rho_{23}$ then the sum in 4.13 is finite, having only m summands.

References

[1] W. A. Al-Salam, M. E. H. Ismail, *q-beta integrals and the q-Hermite polynomials*, Pacific J. Math. 135(2) (1988), 209–221. MR0968609 (90c:33001)

[2] D. M. Bressoud, *A simple proof of Mehler's formula for q-Hermite polynomials*, Indiana Univ. Math. J. 29(4) (1980), 577–580. MR0578207 (81f:33009)

[3] M. Bożejko, B. Kümmerer, R. Speicher, *q-Gaussian processes: non-commutative and classical aspects*, Comm. Math. Phys. 185(1) (1997), 129–154. MR1463036 (98h:81053)

[4] W. Bryc, *Stationary random fields with linear regressions*, Ann. Probab. 29(1) (2001), 504–519. MR1825162 (2002d:60014)

[5] W. Bryc, W. Matysiak, P. J. Szabłowski, *J. Probabilistic aspects of Al-Salam–Chihara polynomials*, Proc. Amer. Math. Soc. 133(4) (2005), 1127–1134 (electronic). MR2117214 (2005m:33033)

[6] W. Bryc, J. Wesołowski, *Askey–Wilson polynomials, quadratic harnesses and martingales*, Ann. Probab. 38(3) (2010), 1221–1262.

[7] W. Bryc, W. Matysiak, J. Wesołowski, *The bi-Poisson process: a quadratic harness*, Ann. Probab. 36(2) (2008), 623–646. MR2393992 (2009d:60103)

[8] L. Carlitz, *Generating functions for certain Q-orthogonal polynomials*, Collect. Math. 23 (1972), 91–104. MR0316773 (47 #5321)

[9] L. Carlitz, *Some polynomials related to theta functions*, Ann. Mat. Pura Appl. 41(4) (1956), 359–373. MR0078510 (17,1205e)

[10] L. Carlitz, *Some polynomials related to Theta functions*, Duke Math. J. 24 (1957), 521–527. MR0090672 (19,849e)

[11] W. Y. C. Chen, H. L. Saad, L. H. Sun, *The bivariate Rogers–Szegő polynomials*, J. Phys. A 40(23) (2007), 6071–6084. MR2343510 (2008k:33064)

[12] R. Floreanini, J. LeTourneux, L. Vinet, *More on the q-oscillator algebra and q-orthogonal polynomials*, J. Phys. A 28(10) (1995), L287–L293. MR1343867 (96e:33043)

[13] R. Floreanini, J. LeTourneux, L. Vinet, *Symmetry techniques for the Al-Salam–Chihara polynomials*, J. Phys. A 30(9) (1997), 3107–3114. MR1456902 (98k:33036)

[14] K. Garrett, M. E. H. Ismail, D. Stanton, *Variants of the Rogers–Ramanujan identities*, Adv. Appl. Math. 23 (1999), 274–299.

[15] G. Gasper, M. Rahman, *Positivity of the Poisson kernel for the continuous q-ultraspherical polynomials*, SIAM J. Math. Anal. 14(2) (1983), 409–420. MR0688587 (84f:33008)

[16] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, with two chapters by W. Van Assche, with a foreword by R. A. Askey, Encyclopedia of Mathematics and its Applications, 98, Cambridge University Press, Cambridge, 2005. xviii+706 pp. ISBN: 978-0-521-78201-2; 0-521-78201-5 MR2191786 (2007f:33001)

[17] M. E. H. Ismail, D. Stanton, *Tribasic integrals and identities of Rogers–Ramanujan type*, Trans. Amer. Math. Soc. 355(10) (2003), 4061–4091.

[18] M. E. H. Ismail, D. Stanton, G. Viennot, *The combinatorics of q-Hermite polynomials and the Askey–Wilson integral*, European J. Combin. 8(4) (1987), 379–392. MR0930175 (89h:33015)

[19] M. E. H. Ismail, M. Rahman, D. Stanton, *Quadratic q-exponentials and connection coefficient problems*, Proc. Amer. Math. Soc. 127(10) (1999), 2931–2941. MR1621949 (2000a:33027)

[20] M. E. H. Ismail, D. R. Masson, *q-Hermite polynomials, biorthogonal rational functions, and q-beta integrals*, Trans. Amer. Math. Soc. 346(1) (1994), 63–116. MR1264148 (96a:33022)

[21] W. F. Kibble, *An extension of a theorem of Mehler's on Hermite polynomials*, Proc. Cambridge Philos. Soc. 41 (1945), 12–15. MR0012728 (7,65f)

[22] R. Koekoek, P. A. Lesky, R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and their q-Analogues*, with a foreword by T. H. Koornwinder, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2010. xx+578 pp. ISBN: 978-3-642-05013-8 MR2656096 (2011e:33029)

[23] W. Matysiak, P. J. Szabłowski, *A few remarks on Bryc's paper on random fields with linear regressions*, Ann. Probab. 30(3) (2002), 1486–1491. MR1920274 (2003e:60111)

[24] W. Matysiak, P. J. Szabłowski, (2005), *Bryc's random fields: the existence and distributions analysis.* ArXiv:math.PR/math/0507296

[25] J. Mercer, *Functions of positive and negative type and their connection with the theory of integral equations*, Philosophical Transactions of the Royal Society A 209 (1909), 415–446.

[26] M. Rahman, Q. M. Tariq, *Poisson kernel for the associated continuous q-ultraspherical polynomials*, Methods Appl. Anal. 4(1) (1997), 77–90. MR1457206 (98k:33038)

[27] L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. 25 (1894), 318–343.

[28] L. J. Rogers, *On the expansion of certain infinite products*, Proc. London Math. Soc. 24 (1893), 337–352.

[29] L. J. Rogers, *Third memoir on the expansion of certain infinite products*, Proc. London Math. Soc. 26 (1895), 15–32.

[30] D. Slepian, *On the symmetrized Kronecker power of a matrix and extensions of Mehler's formula for Hermite polynomials*, SIAM J. Math. Anal. 3 (1972), 606–616. MR0315173 (47 #3722)

[31] D. Kim, D. Stanton, J. Zeng, *The combinatorics of the Al-Salam–Chihara q-Charlier polynomials*, Sém. Lothar. Combin. 54 (2005/07), Art. B54i, 15 pp. (electronic). MR2223031 (2007b:05024)

[32] P. J. Szabłowski, *Probabilistic implications of symmetries of q-Hermite and Al-Salam–Chihara polynomials*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 11(4) (2008), 513–522. MR2483794 (2010g:60125)

[33] P. J. Szabłowski, *Expansions of one density via polynomials orthogonal with respect to the other*, J. Math. Anal. Appl. 383 (2011), 35–54. http://arxiv.org/abs/1011.1492

[34] P. J. Szabłowski, *Multidimensional q-normal and related distributions - Markov case*, Electron. J. Probab. 15(40) (2010), 1296–1318. MR2678392

[35] P. J. Szabłowski, *On the structure and probabilistic interpretation of Askey–Wilson densities and polynomials with complex parameters*, J. Funct. Anal. 262 (2011), 635–659. http://arxiv.org/abs/1011.1541

[36] P. J. Szabłowski, *On summable form of Poisson–Mehler kernel for big q-Hermite and Al-Salam–Chihara polynomials*, Infinite Dimensional Analysis, Quantum Probability and Related Topics 15(3) (2012). http://arxiv.org/abs/1011.1848

[37] P. J. Szabłowski, *Towards a q-analogue of the Kibble–Slepian formula in 3 dimensions*, J. Funct. Anal. 262 (2012), 210–233. http://arxiv.org/abs/1011.4929

[38] P. J. Szabłowski, *Befriending Askey–Wilson polynomials*, submitted. http://arxiv.org /abs/1111.0601

[39] P. J. Szabłowski, (2009), *q-Gaussian distributions: simplifications and simulations*, Journal of Probability and Statistics, 2009, (article ID 752430).

[40] G. Szegö, *Beitrag zur theorie der thetafunctionen*, Sitz. Preuss. Akad. Wiss. Phys. Math. KL 19 (1926), 242–252.

[41] R. A. Askey, M. Rahman, S. K. Suslov, *On a general q-Fourier transformation with nonsymmetric kernels*, J. Comput. Appl. Math. 68(1–2) (1996), 25–55. MR1418749 (98m:42033)

[42] D. Voiculescu, (2000), *Lectures on free probability theory, Lectures on probability theory and statistics* (Saint-Flour, 1998), 279–349, Lecture Notes in Math., 1738, Springer, Berlin, 2000. MR1775641 (2001g:46121)

[43] D. V. Voiculescu, K. J. Dykema, A. Nica, (1992), *Free Random Variables: A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups*, CRM Monograph Series, 1, American Mathematical Society, Providence, RI, 1992. vi+70 pp. ISBN: 0-8218-6999-X MR1217253

DEPARTMENT OF MATHEMATICS AND INFORMATION SCIENCES
WARSAW UNIVERSITY OF TECHNOLOGY
ul. Koszykowa 75
00-662 WARSAW, POLAND
E-mail: pawel.szablowski@gmail.com

Received November 4, 2011; revised version May 15, 2012.