

ON GENERALISED LAH-NUMBERS

by SELMO TAUBER

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1. Introduction

In (1) numbers related to the Stirling numbers are defined. Later in (2) these numbers were called Lah-numbers (cf. 2, p. 43, Ex. 16). According to (1) these numbers are of importance in Mathematical Statistics. In this paper we shall generalise the method and apply it to generalised Stirling numbers as defined in (3).

In (3) the polynomials  $Q$  and the numbers  $A_n^s$  and  $B_n^s$  were defined by the relations

$$Q(x; M, N, n) = Q(x, n) = \prod_{m=1}^n [M(m) + N(m)x] = \sum_{s=0}^n A_n^s x^s, \quad n \geq 1,$$

$$Q(x, 0) = M(0), \tag{1}$$

$$x^n = \sum_{m=0}^n B_n^m Q(x, m),$$

$$1 = B_0^0 Q(x, 0) = B_0^0 M(0), \tag{2}$$

where  $M$  and  $N$  are two functions such that  $M(0) \neq 0$ ,  $M(n)$  is defined for all positive integers  $n$ , and  $N(n)$  is defined for all positive integers  $n$  and  $N(n) \neq 0$ . The numbers  $A_n^s$  and  $B_n^s$  are called generalised Stirling numbers.

2. Relationship between different Q-polynomials

Let  $Q_1(x, n) = Q(x; M_1, N_1, n)$  and  $Q_2(x, n) = Q(x; M_2, N_2, n)$ , then

$$Q_1(x, n) = \sum_{s=0}^n A_{1,n}^s x^s, \quad x^n = \sum_{m=0}^n B_{1,n}^m Q_1(x, m), \tag{3}$$

$$Q_1(x, 0) = M_1(0), \quad 1 = B_{1,0}^0 Q_1(x, 0) = B_{1,0}^0 M_1(0), \tag{3a}$$

and

$$Q_2(x, n) = \sum_{s=0}^n A_{2,n}^s x^s, \quad x^n = \sum_{m=0}^n B_{2,n}^m Q_2(x, m), \tag{4}$$

$$Q_2(x, 0) = M_2(0), \quad 1 = B_{2,0}^0 Q_2(x, 0) = B_{2,0}^0 M_2(0), \tag{4a}$$

where  $A_n^m$  and  $B_n^m$  are zero for  $m > n$ ,  $m < 0$ ,  $n < 0$ .

Let us express  $Q_2(x, n)$  in terms of  $Q_1(x, m)$ . We can write

$$Q_2(x, n) = \sum_{s=0}^n A_{2,n}^s x^s = \sum_{s=0}^n A_{2,n}^s \left[ \sum_{m=0}^s B_{1,s}^m Q_1(x, m) \right],$$

$$Q_2(x, n) = \sum_{s=0}^n \sum_{m=0}^s A_{2,n}^s B_{1,s}^m Q_1(x, m). \tag{5}$$

Considering the conditions on the numbers  $A$  and  $B$  we can extend the second summation to  $n$  and then change the order of summation, thus,

$$Q_2(x, n) = \sum_{s=0}^n \sum_{m=0}^n A_{2,n}^s B_{1,s}^m Q_1(x, m) = \sum_{m=0}^n Q_1(x, m) \sum_{s=0}^n A_{2,n}^s B_{1,s}^m \quad (6)$$

and, using again the conditions on the  $A$  and  $B$  numbers, we have,

$$Q_2(x, n) = \sum_{m=0}^n Q_1(x, m) \sum_{s=m}^n A_{2,n}^s B_{1,s}^m \quad (7)$$

Let

$$\sum_{s=m}^n A_{2,n}^s B_{1,s}^m = L_{2,1,n}^m \quad (8)$$

where  $L_{2,1,n}^m = 0$  for  $m < 0$ ,  $n < 0$ ,  $m > n$ .

Similarly, if we express  $Q_1(x, n)$  in terms of  $Q_2$ -polynomials we obtain

$$Q_1(x, n) = \sum_{m=0}^n Q_2(x, m) \sum_{s=m}^n A_{1,n}^s B_{2,s}^m \quad (9)$$

with

$$\sum_{s=m}^n A_{1,n}^s B_{2,s}^m = L_{1,2,n}^m \quad (10)$$

and where  $L_{1,2,n}^m = 0$ , for  $n < 0$ ,  $m < 0$ ,  $n < m$ . It is clear that in the special case where  $Q_1 = Q_2$ ,  $L_{1,2,n}^m = L_{2,1,n}^m = \delta_n^m$ , where  $\delta_n^m$  is the Kronecker delta.

### 3. Quasi-orthogonality of the $L$ -numbers

For obvious reasons we shall call the  $L$ -numbers generalised Lah-numbers (cf. Ex. 2 hereafter). Using (7) and (8) we can write

$$Q_2(x, n) = \sum_{m=0}^n Q_1(x, m) L_{2,1,n}^m \quad (11)$$

and using (9) and (10)

$$Q_1(x, n) = \sum_{m=0}^n Q_2(x, m) L_{1,2,n}^m \quad (12)$$

Substituting (12) into (11) we obtain

$$Q_2(x, n) = \sum_{m=0}^n L_{2,1,n}^m \sum_{s=0}^n L_{1,2,m}^s Q_2(x, s),$$

and using the conditions on the  $L$ -numbers we obtain

$$Q_2(x, n) = \sum_{s=0}^n Q_2(x, s) \sum_{m=s}^n L_{2,1,n}^m L_{1,2,m}^s,$$

thus

$$\sum_{m=s}^n L_{2,1,n}^m L_{1,2,m}^s = \delta_n^s \quad (13)$$

(13) expresses the quasi-orthogonality of  $L$ -numbers for reversed indices.

4. Recurrence relations

According to (1) we can write

$$Q_1(x, n+1) = \sum_{m=0}^{n+1} A_{1,n+1}^m x^m = [M_1(n+1) + N_1(n+1)x] \sum_{m=0}^n A_{1,n}^m x^m,$$

and,

$$Q_2(x, n+1) = \sum_{m=0}^{n+1} A_{2,n+1}^m x^m = [M_2(n+1) + N_2(n+1)x] \sum_{m=0}^n A_{2,n}^m x^m,$$

which according to (11) yields

$$Q_2(x, n+1) = \sum_{m=0}^{n+1} Q_1(x, m) L_{2,1,n+1}^m = [M_2(n+1) + N_2(n+1)x] \sum_{m=0}^n Q_1(x, m) L_{2,1,n}^m. \tag{14}$$

But

$$Q_1(x, m+1) = [M_1(m+1) + N_1(m+1)x] Q_1(x, m),$$

so that

$$x Q_1(x, m) = [Q_1(x, m+1) - M_1(m+1) Q_1(x, m)] / N_1(m+1). \tag{15}$$

Substituting (15) into (14) we obtain

$$\begin{aligned} \sum_{m=0}^{n+1} Q_1(x, m) L_{2,1,n+1}^m &= M_2(n+1) \sum_{m=0}^n Q_1(x, m) L_{2,1,n}^m \\ &+ N_2(n+1) \sum_{m=0}^n [Q_1(x, m+1) - M_1(m+1) Q_1(x, m)] L_{2,1,n}^m / N_1(m+1). \end{aligned}$$

By equating the coefficients of  $Q_1(x, m)$  we obtain

$$L_{2,1,n+1}^m = M_2(n+1) L_{2,1,n}^m + \frac{N_2(n+1)}{N_1(m)} L_{2,1,n}^{m-1} - \frac{N_2(n+1) M_1(m+1)}{N_1(m+1)} L_{2,1,n}^m,$$

or,

$$L_{2,1,n}^m = \left[ M_2(n) - \frac{N_2(n) M_1(m+1)}{N_1(m+1)} \right] L_{2,1,n-1}^m + \frac{N_2(n)}{N_1(m)} L_{2,1,n-1}^{m-1} \tag{16}$$

and by inverting the indices 2 and 1,

$$L_{1,2,n}^m = \left[ M_1(n) - \frac{N_1(n) M_2(m+1)}{N_2(m+1)} \right] L_{1,2,n-1}^m + \frac{N_1(n)}{N_2(m)} L_{1,2,n-1}^{m-1}. \tag{17}$$

5. Examples

1. Let  $M_1(\alpha) = \alpha$ ,  $N_1(\alpha) = \alpha - 1$ ,  $M_2(\alpha) = \alpha + 1$ ,  $N_2(\alpha) = 1/\alpha$ . We obtain, taking  $L_{2,1,1}^1 = L_{1,2,1}^1 = 1$ , and  $\alpha > 0$ ,

$$\begin{aligned} L_{2,1,n}^m &= \left[ (n+1) - \frac{m+1}{nm} \right] L_{2,1,n-1}^m + \frac{1}{n(m-1)} L_{2,1,n-1}^{m-1}, \\ L_{1,2,n}^m &= [n - (n-1)(m+1)(m+2)] L_{1,2,n-1}^m + m(n-1) L_{1,2,n-1}^{m-1}, \end{aligned}$$

and the following numerical values:

$L_{2,1,n}^m$ :	$m =$	1	2	3	4
$n:$					
1		1			
2		2	$1/2$		
3		$20/3$	$29/12$	$1/12$	
4		30	$1233/96$	$199/288$	$1/144$
$L_{1,2,n}^m$ :	$m =$	1	2	3	4
$n:$					
1		1			
2		-4	2		
3		36	-58	12	
4		-504	2072	-1194	144

2. Let  $M_1(\alpha) = 1 - \alpha$ ,  $N_1(\alpha) = 1$ ,  $M_2(\alpha) = \alpha - 1$ ,  $N_2(\alpha) = 1$ , then the numbers obtained are the Lah-numbers as studied in (1).

#### REFERENCES

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