An International Journal
computers & mathematics
with applications

PERGAMON Computers and Mathematics with Applications 41 (2001) 1085–1098
www.elsevier.nl/locate/camwa

Umbral Presentations for Polynomial Sequences

B. D. TAYLOR
Department of Mathematics, Wayne State University
Detroit, MI 48202, U.S.A.

Abstract—Using random variables as motivation, this paper presents an exposition of formalisms developed in [1,2] for the classical umbral calculus. A new construction and a variety of examples are presented, culminating in several descriptions of sequences of binomial type in terms of umbral polynomials. © 2001 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

The system of calculation now known as the "umbral calculus" originated with Blissard in the nineteenth century in informal calculations involving the "lowering" and "raising" of exponents. The work of Rota and his collaborators in [3–5] and other works formalized these methods in the modern language of linear operators and Hopf algebras. While this clarified the underlying theory, it rendered the original nineteenth century work no more easy to read or check. In [1,2,6], the original classical notation was revived and extended so as to be rigorous by modern standards. In [7], the first attempts were made to apply this newly revived classical umbral calculus to one of the most significant successes of the modern theory, namely the study of sequences of binomial type.

The purposes of this paper are twofold. Since much of this paper is expository, no prior knowledge of umbral calculus in any of its guises is assumed. To start, we develop the modern formulation of the classical umbral calculus in analogy with the idea of a random variable. This renders the definitions of [1,2] transparent. In the second part of this paper, we introduce a new operation on umbrae arising naturally from the analogy to random variables. We show that all sequences of binomial type and all umbral maps arise directly from the application of this operation. We further apply the tools of classical umbral calculus developed in the first part of the paper to provide several other compact presentations for sequences of binomial type.

The author has attempted to document at least the recent history of the main results and definitions contained herein.

2. RANDOM VARIABLES AND THE CLASSICAL UMBRAL CALCULUS

Fundamental to the classical umbral calculus is the idea of associating a sequence of numbers $a_0, a_1, a_2, \ldots$ to an "umbral variable" α, which is said to represent the sequence. To be slightly

I am greatly indebted to G.-C. Rota who first suggested that the formalization of umbrae developed in [1,2] was amenable to interpretation as a generalization of random variables. I am likewise indebted to N. Ray; the influence of my conversations with him can be traced throughout Sections 5.2 and 5.3.

0898-1221/01/$ - see front matter Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX
PII: S0898-1221(01)00083-9

more formal, the umbral calculus relies on associating the sequence $a_0, a_1, a_2, \ldots$ to the sequence $1, \alpha, \alpha^2, \alpha^3, \ldots$ of powers of α.

This kind of association is familiar in modern mathematics: to any random variable G, we associate a sequence of numbers $1, g_1, g_2, \ldots$ where g_i is the i^{th} moment of G. Specifically, we are defining the sequence $1, g_1, g_2, \ldots$ by applying the expectation operator, $\mathbf{E}$, componentwise to the sequence $1, G, G^2, \ldots$ consisting of powers of the random variable G.

We will proceed to carry this analogy further, calculating with random variables in precisely the way we will later be using umbral variables. We start by letting G be a random variable distributed uniformly over the interval $[0,1]$. The sequence of moments $1, g_1, g_2, \ldots$ associated to G is thus given by $g_n = \int_0^1 g^n\,dg = 1/(n+1)$.

If we let $p(t) \in \mathbb{C}[t]$ be any polynomial with complex coefficients, it is immediate that

$$\mathbf{E}\left[p'(G)\right] = \int_0^1 p'(g)\,dg = \Delta p(0), \tag{1}$$

where Δ is the forward difference operator $\Delta p(t) = p(t+1) - p(t)$. A comment on notation: since Δ is defined as an operator on the ring of polynomials in t, $\Delta p(0)$ can only be interpreted as $(\Delta p)(0)$ or as 0. We adopt the former reading. Since the calculation in equation (1) only required that $p(t)$ was differentiable, it could just as well have been carried out for a polynomial with coefficients in some larger integral domain or indeed for a polynomial in t whose coefficients contained various random variables *which were independent* of G. So suppose G' is a random variable independent of and identically distributed to G. Consider $p(G+G')$ as a polynomial $q(G)$ in G with coefficients in $\mathbb{C}[G']$. Using equation (1), the expected value, averaging over values of G, of $p''(G+G') = q''(G)$ is $\Delta q'(0) = \Delta p'(G')$. Applying equation (1) again, recalling that the derivative D and $\Delta = e^D - I$ commute and that G' is identically distributed to G, gives $\mathbf{E}[\Delta p'(G')] = \Delta^2 p(0)$. Somewhat more suggestively, this calculation can be written

$$\mathbf{E}\left[p''\left(G+G'\right)\right] = \mathbf{E}\left[\Delta p'\left(G'\right)\right] = \mathbf{E}\left[\Delta p(0)\right]. \tag{2}$$

The first property to observe here is that the independence of G and G' matters, as

$$\mathbf{E}\left[p''(G+G)\right] = \mathbf{E}\left[\Delta p''(2G)\right] = \frac{p'(2\cdot 1) - p'(0)}{2},$$

since $p''(2t) = D_t\left(p'(2t)/2\right)$, where D_t is the derivative with respect to t.

The second important property is the rather trivial observation that we can calculate the expectation of a polynomial in several random variables all independent and identically distributed to G simply by knowing the moments of G. For example, by equation (2), we know that $\mathbf{E}[20(G+G')^3] = \Delta^2 t^5|_{t=0}$. This could be evaluated directly. Alternatively, and denoting $E[G^k]$ by G_k, it could be evaluated as

$$\begin{aligned} 20 \cdot \mathbf{E}\left[G^3 + 3G^2G' + 3GG'^2 + G'^3\right] &= 20\left(G_3 + 3G_2G_1 + 3G_1G_2 + G_3\right) \\ &= 20\left(2G_3 + 6G_2G_1\right) \\ &= 20\left(\frac{2}{4} + \frac{6}{3\cdot 2}\right). \end{aligned}$$

Applying the above observations quickly reconstructs the moment generating function $E[e^{Gz}]$ for G. Since $D_t\left(e^{tz}/z\right) = e^{tz}$, equation (1) implies $E[e^{Gz}] = (e^z - e^0)/z$.

For the duration of the next calculation, we are going to make some assumptions that simply do not hold within the confines of probability theory. The remainder of this section will be devoted to describing how to replace random variables with "umbral variables" in a way that makes the following calculations legitimate. So, for the moment, let us assume that there is an object B

that behaves much like a random variable. Let us call this object an "umbral variable". We treat it just like a random variable, but stipulate both that it is independent of B and that $B+G=0$. With these stipulations, we find that by independence

$$\mathbf{E}\left[e^{(G+B)z}\right]=\mathbf{E}\left[e^{Gz}e^{Bz}\right]=\mathbf{E}\left[e^{Gz}\right]\mathbf{E}\left[e^{Bz}\right]=\frac{e^z-1}{z}\mathbf{E}\left[e^{Bz}\right].$$

But since $G+B=0$, the left-hand side above is just 1. We have just calculated that $\mathbf{E}[e^{Bz}]=t/(e^t-1)$. Since this is the exponential generating function, $\sum_{k\geq 0} B_k(z^k/k!)$, for the Bernoulli numbers, B_k, we find that if the above calculation can be made rigorous, then $\mathbf{E}[B^k]=B_k$.

Since calculations such as the previous are too useful to abandon (see [2] for a variety of examples involving the Bernoulli numbers) we define *umbral variables* or *umbrae* which formalize the roles of both the random variable G and the new object B in the preceding calculation.

Just as a random variable is usually capitalized, we will typically distinguish our umbrae by writing them as Greek letters, e.g., $\alpha,\beta,\alpha''',\ldots$. Let us denote the collection of whichever umbral variables we will be using by $\mathcal{A}$. See [1,2] for the relevant, and straightforward, technical details. In practice, when we introduce a new umbra, say α, we specify explicitly or implicitly how $\mathbf{E}$ acts on it, namely what values $\mathbf{E}[\alpha^k]$ takes for each k. Any two distinct umbrae in $\mathcal{A}$, say α and γ or α and α', will act like independent random variables, regardless of how $\mathbf{E}$ acts on them. Generalizing, any collection of distinct umbrae will behave as do independent random variables.

Formally, this can be accomplished by defining a linear *evaluation* map, $\mathbf{E}:\mathbf{F}[\mathcal{A}]\to\mathbf{F}$ on the polynomials $\mathbf{F}[\mathcal{A}]$ in the umbrae with coefficients in a suitably chosen commutative ring $\mathbf{F}$. We require that $\mathbf{E}$ is $\mathbf{F}$-linear, $\mathbf{E}[1]=1$, and $\mathbf{E}[M\cdot M']=\mathbf{E}[M]\mathbf{E}[M']$ for any two monomials, M and M', in $\mathbf{F}[\mathcal{A}]$ such that no umbra appears to nonzero power in both M and M'. This map was called **eval** in [1,2].

We call $p,q\in\mathbf{F}[\mathcal{A}]$ *umbrally equivalent*, written $p\simeq q$ when $\mathbf{E}[p]=\mathbf{E}[q]$. Analogous to the notion of identically distributed random variables, we define $p,q\in\mathbf{F}[\mathcal{A}]$ to be *umbrally exchangeable*, when $p^k\simeq q^k$ for all $k\geq 0$. If, for example, we have $\alpha\equiv 3$, then $\mathbf{E}[\alpha^k]=3^k$; this is consistent, in the analogy to random variables, with considering α analogous to a random variable which always takes on the value 3. We note that equality implies exchangeability which implies umbral equivalence. The converses are false.

We define $p,q\in\mathbf{F}[\mathcal{A}]$ to be *independent* when no umbra appears in both p and q. More formally, an umbra that appears to a nonzero power in some monomial with nonzero coefficient in p does not appear to a nonzero power in any monomial with nonzero coefficient in q. For example, $\alpha^2+\alpha\alpha'$ and $\beta\beta'^2-\beta+\alpha'''$ are independent, but $\alpha^2+\alpha\alpha'$ and $\beta\beta'^2-\beta+\alpha$ are not independent. Nor are the falling factorials $(\alpha)_{(n)}$ and $(\alpha)_{(n-1)}$ independent. On the other hand, $\alpha t^2+\beta$ and $\gamma-t$ are independent where t is some element of $\mathbf{F}$.

In random variables, we can usually substitute one identically distributed random variable for another (modulo independence constraints). Similarly, we can substitute exchangeable umbrae as per the following lemma.

LEMMA 1. *(See [1].) If a polynomial $p(t)$ is independent of two exchangeable umbrae α and α', then $p(\alpha)\equiv p(\alpha')$.* ∎

This substitution lemma holds equally well if α or α' is replaced by an umbral polynomial $q\in\mathbf{F}[\mathcal{A}]$.

For the duration of this paper, we let ε be an umbra such that $\varepsilon^k\simeq\delta_{0,k}$ where δ in the Kronecker delta. As long as we work with polynomials in the umbrae, there is no harm in defining $0^0=1$. Under this convention, we consider $0\equiv\varepsilon$. This is consistent with the convention that 1 is the 0^{th} moment of a random variable which always takes on value 0.

To pick up our earlier example, let γ be an umbra such that $\gamma^i\simeq 1/(n+1)$ and let β be an umbra such that $\gamma+\beta\equiv 0$. It is an easy exercise to see that given any umbra γ, such an *inverse* *umbra* can be found recursively. Here we have made formal in γ and β exactly the properties we had assumed for G and B. We have γ and β are independent and $\beta + \gamma \equiv 0$.

Extending our notions of independence, exchangeability, and equivalence coefficientwise to formal power series (see [6] for a general but technical treatment) we can duplicate the computation we performed for G and B. By the substitution lemma we have $e^{(\beta+\gamma)z} \simeq e^{0\cdot z} = 1$. Thus, we have $e^{\beta z}e^{\gamma z} \simeq 1$. By independence, this implies that $e^{\beta z}(e^z - 1)/z \simeq 1$, and hence (by linearity) that $e^{\beta z} \simeq z/(e^z - 1)$. From a technical viewpoint, there are a number of ways to justify the first step in the preceding sentence. The most direct solution is to apply the substitution lemma coefficientwise to the formal power series in z. A general approach which views multiplication by $e^{\gamma z}\, z/(e^z - 1)$ as a linear operator "equivalent" to the identity is given in [6]. The intuition behind both these proofs is that since $e^{\gamma z}$ and $e^{\beta z}$ are independent, $\mathbf{E}$ can be applied in two stages first to γ and then to β, analogously to finding the expectation by first averaging over one random variable and then over another independent random variable.

As a demonstration of these techniques, we rewrite in modern umbral notation the first example in [8], one in the series of papers in which Blissard during the 1860s introduced his "representative notation"—the umbral calculus. To point out just how closely the modern language captures Blissard's 19^{th} century original, we present most of this example in Blissard's own words.

> Blissard starts with the problem, "Required to expand $\{x/\log(1+x)\}^m$". He then lets, "$\{x/\log(1+x)\}^m = 1 + P_1x + P_2x^2 + \cdots + P_nx^n + \&c$" and defines U_n to be the coefficient of $\theta^n/n!$ in $((e^\theta - 1)/\theta)^m$ where θ is an ordinary variable. He observes that,
>
> $$\text{“}\left(\frac{e^\theta - 1}{\theta}\right)^m = 1 + U_1\theta + U_2\frac{\theta^2}{1\cdot 2} + \cdots + U_n\frac{\theta^n}{1\cdot 2\cdots n} + \&c$$
> $$= e^{U\theta}(\text{by representative notation})\text{.”}$$
>
> In modern language, he is letting U be an umbra such that $U^n \simeq U_n$, and his "$=$" would be replaced with "$\simeq$". The next operation takes place purely on the level of formal power series. Blissard substitutes $\log(1+x)$ for θ and finds that,
>
> $$\text{“}\left\{\frac{x}{\log(1+x)}\right\}^m = (1+x)^U\text{,”}$$
>
> where again the only change necessary to modernize his work is to replace "$=$" with "$\simeq$". If we "equate coefficients of x^n, then
>
> $$P_n = \frac{U(U-1)(U-2)\cdots(U-n+1)}{1\cdot 2\cdot 3\cdots n};\text{”}$$
>
> again we would replace "$=$" with "$\simeq$".

The preceding formula for P_n has the advantage of being extremely compact. Blissard concludes with an expansion of it, and we shall proceed likewise, though our precise techniques are somewhat more umbral than those Blissard used.

With γ as before, we have $(e^\theta - 1)/\theta \simeq e^{\gamma\theta}$. Thus,

$$\left(\frac{e^\theta - 1}{\theta}\right)^m \simeq e^{(\gamma' + \gamma'' + \cdots + \gamma''')\theta},$$

where $\gamma' + \gamma'' + \cdots + \gamma'''$ is a sum of m distinct (and thus independent) umbrae each exchangeable with γ. We conclude that $U \equiv \gamma' + \gamma'' + \cdots + \gamma'''$, and thus $P_n \simeq \binom{\gamma'+\gamma''+\cdots+\gamma'''}{n}$. We conclude with the following formula for evaluating the powers U^n. Since $D_t^m\, t^{m+n} = (m+n)_{(m)}t^n$, we have, generalizing calculation (2), that

$$(m+n)_{(m)}U^n \simeq (m+n)_{(m)}\left(\gamma' + \gamma'' + \cdots + \gamma'''\right)^n \simeq \Delta^m\, 0^{m+n}.$$

This last is better known as $m!\,S(m+n,m)$, where $S(n,k)$ is the Stirling number of the second kind counting the number of set partitions of an n-set into k parts. So $U^n \simeq S(m+n,m)/\binom{m+n}{m}$. We can use this, together with the expansion of the falling factorials in terms of Stirling numbers of the first kind, to derive

$$\binom{\gamma' + \cdots + \gamma'''}{n} = \frac{1}{n!}\sum_{k=0}^{n} s(n,k)\,(\gamma' + \cdots + \gamma''')^k \simeq \frac{1}{n!}\sum_{k=0}^{n} \frac{s(n,k)S(m+k,m)}{\binom{m+k}{m}}.$$

3. UMBRAL PRESENTATIONS OF APPELL SEQUENCES

Historically, the objects of interest in umbral computations were of course sequences of numbers or polynomials. For our present purposes, this means that we will primarily be studying the "moments" $\mathbf{E}[\alpha^k]$ of an umbra α. We say that the umbra α *represents* a sequence $a_0, a_1, a_2, \ldots$, $a_i \in \mathbf{F}$, when $\alpha^k \simeq a_k$ for all integers $k \geq 0$. Necessarily this implies that $a_0 = 1$. An *umbral presentation* of a sequence $a_1, a_2, \ldots$ of elements in $\mathbf{F}$ is any sequence $q_1, q_2, \ldots$ of polynomials in $\mathbf{F}[\mathcal{A}]$ such that $q_i \simeq a_i$ for $i \geq 0$. Throughout this paper we freely assume that, for any sequence in $\mathbf{F}$, we can find infinitely many umbrae representing the given sequence.

Now let $\mathbf{F}$ be $\mathbf{k}[x,y]$ where $\mathbf{k}$ is a commutative ring containing $\mathbb{Q}$. The remainder of this paper will focus on umbral presentations for sequences of polynomials. For example, for any umbra α, we can define a sequence of polynomials $s_n(x)$, $n = 0, 1, 2, \ldots$, by $s_n(x) \simeq (x+\alpha)^n$. This definition immediately yields the calculation

$$s_n(y+x) \simeq (y+x+\alpha)^n = \sum_i \binom{n}{i} y^i (x+\alpha)^{n-i} \simeq \sum_i \binom{n}{i} y^i s_{n-i}(x). \tag{3}$$

A sequence of polynomials $s_0(x), s_1(x), s_2(x), \ldots$ with $S_n(x)$ having degree n is said to be an *Appell sequence* when it satisfies the identity

$$s_n(y+x) = \sum_i \binom{n}{i} y^i s_{n-i}(x), \tag{4}$$

given by equation (3) for all $n \geq 0$. We shall call an Appell sequence $s_n(x)$ *normalized* when $s_1(x)$ is monic. Any Appell sequence may be rewritten as a normalized Appell sequence by replacing $s_n(x)$ with $s_n(x)/s_1'(0)$. Here, $s_1'(x)$ is the first derivative of $s(x)$. We hold with this notation for derivatives throughout this paper. In the literature, Appell sequences are frequently defined to be normalized.

PROPOSITION 2. *(See [2].) A sequence, $s_0(x), s_1(x), \ldots$, of polynomials in $\mathbf{k}[x]$ with $s_n(x)$ having degree n is a normalized Appell sequence iff there exists an umbra α such that $s_n \simeq (x+\alpha)^n$ for $n \geq 0$.*

PROOF. The if direction is given by calculation (3).

(only if): Replacing x with 0 in the defining equation (4) shows that in an Appell sequence, each polynomial $s_n(y)$ can be recovered from the sequence of values $s_0(0), s_1(0), \ldots$. Choosing an umbra α that represents this sequence guarantees $(x+\alpha)^n \simeq s_n(x)$. ∎

Similarly we have the standard result that a sequence $s_0(x), s_1(x), \ldots$ of polynomials, $s_n(x)$ of degree n, is an Appell sequence iff

$$s_n'(x) = n \cdot s_{n-1}(x), \tag{5}$$

for all $n \geq 0$. The only if direction follows since $D_x(x+\alpha)^n = n(x+\alpha)^{n-1}$. To show the if direction, we observe that any sequence of polynomials satisfying equation (5) is determined by the sequence of values of $s_0(0), s_1(0), s_2(0), \ldots$ and apply the argument in the preceding proof.

The sequences of polynomials with which we will be most concerned in this paper are those of "binomial type", i.e., sequences of polynomials which satisfy an analog of the binomial theorem. Before approaching this topic, however, we lift another tool from random variables to umbral calculus.

4. SUMS OF RANDOM VARIABLES AND THE "DOT" OPERATION ON UMBRAE

Suppose that X is a random variable. If n is a positive integer, one can of course run n trials of X and sum the results. Denote the sum by a new random variable $n_\bullet X$. Thus, $n_\bullet X$ has the same distribution as $X_1 + X_2 + \cdots + X_n$ where the X_i are all independent and identically distributed to X. In [1,2], the corresponding notion $n_\bullet \alpha$ was defined for an arbitrary umbra α. In particular, $n_\bullet \alpha$ is itself an umbra and it is defined to be exchangeable with $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ where $\alpha_i \equiv \alpha$ for each i. Similarly, for any umbral polynomial $p \in \mathbf{F}[\mathcal{A}]$, we define a new umbra $n_\bullet p$ which is exchangeable with $n_\bullet \gamma$ where γ is any umbra satisfying $\gamma \equiv p$. It is worth emphasizing that $n_\bullet p$ is itself an umbra. Thus, α $5_\bullet \alpha$, $3_\bullet \alpha$, and $5_\bullet(\alpha + \beta)$ are all distinct (hence, all independent). It is, however, clear from the definitions that $5_\bullet(\alpha + \beta) \equiv 5_\bullet \alpha + 5_\bullet \beta$.

We recall a technical consideration from [1,2]. The set of all umbrae $\mathcal{A}$ will be decomposed as a disjoint union $\mathcal{A} = \mathcal{A}_0 \uplus \mathcal{A}_1$; umbrae in $\mathcal{A}_1$ are called *auxiliary* umbrae. Umbrae of the form $n_\bullet p$ are auxiliary umbrae. This detail will be given more attention below.

It is an easy observation that if $g(z) \simeq e^{\alpha z}$ (this object is analogous to the moment generating function of a random variable), then

$$e^{n_\bullet \alpha z} \equiv e^{(\alpha_1 + \cdots + \alpha_n) z} = \prod_{i=1}^{n} e^{\alpha_i z} \simeq g(z)^n,$$

where the last equivalence uses the independence of the $e^{\alpha_i z}$s.

Even more directly, we see that

$$(m+n)_\bullet \alpha \equiv m_\bullet \alpha + n_\bullet \alpha', \tag{6}$$

where $\alpha' \equiv \alpha$. As a consequence, if for each positive integer n we define a sequence $f_0(n), f_1(n), f_2(n), \ldots$, by $f_i(n) \simeq (n_\bullet \alpha)^i$, then equation (6) implies that

$$((m+n)_\bullet \alpha)^k \simeq (m_\bullet \alpha + n_\bullet \alpha')^k = \sum_i \binom{k}{i} (m_\bullet \alpha)^i \, (n_\bullet \alpha')^{k-i},$$

and hence,

$$f_k(m+n) = \sum_i \binom{k}{i} f_i(m) \, f_{k-i}(n). \tag{7}$$

This kind of generalized binomial theorem will be explored further in the next section.

By way of introduction to the first new definition of this paper, we consider the following generalization of $n_\bullet X$. Let X be some random variable and let Y be a random variable which only takes positive integer values. Run one trial of Y, then run Y trials of X and sum the results. We define $Y_\bullet X$ to be a new random variable whose distribution is identical to $X_1 + X_2 + \cdots + X_Y$; for convenience, we are defining the X_is to be independent random variables identically distributed to X. Observe that if Z is another random variable taking only positive integer values, then according to the preceding definition, $(Y+Z)_\bullet X$ has the same distribution as $X_1 + X_2 + \cdots + X_Y + X_{Y+1} + \cdots + X_{Y+Z}$, and hence as $Y_\bullet X_1 + Z_\bullet X_2$. On the other hand, $X_\bullet(Y+Z)$ does not in general have the same distribution as $X_{1\bullet} Y + X_{2\bullet} Z$; in the first expression, only one trial of X is made, and in the second two trials of X are made. Of course, if X always returns the same value, say $X \equiv n$, this causes no trouble and $n_\bullet(X+Y)$ is identically distributed to $n_\bullet X + n_\bullet Y$.

Just as the definition $n_\bullet X$ for random variables generalizes to $n_\bullet \gamma$ for umbrae, we would like a generalization of the random variable $X_\bullet Y$ to umbrae. This generalization should satisfy results analogous to those recalled above for random variables.

The generalization relies on a simple observation first applied to the umbral calculus by Ray in [9].

PROPOSITION 3. *If $\gamma \in \mathcal{A}_0$ is an umbra, and n is a positive integer, then $\mathbf{E}[(n_{\bullet}\gamma)^k]$ is a polynomial in n.*

PROOF. This is equivalent to the observation that if $g(z)$ is in $\mathbf{k}[[z]]$, the ring of formal power series in z, and if $g(z) \simeq e^{\gamma z}$, then $e^{n_{\bullet}\gamma z} \simeq g(z)^n = e^{n \log(g(z))}$ and the coefficient of $z^k/k!$ in the last is a polynomial in n. ∎

Alternately, we could have observed that if $\gamma^i \simeq a_i$, and each $\gamma_i \equiv \gamma$ for $i = 1, \ldots, n$ then $a_1^{i_1} a_2^{i_2} \cdots a_k^{i_k}$ appears in $\mathbf{E}[(\gamma_1 + \cdots + \gamma_n)^k]$ as many times as there are monomials in the expansion of $(\gamma_1 + \cdots + \gamma_n)^k$ containing exactly i_j j^{th} powers. But this says that $(n_{\bullet}\gamma)^k$ is umbrally equivalent to

$$(\gamma_1 + \cdots + \gamma_n)^k \simeq \sum_{i_1,\ldots,i_k} \binom{n}{i_0, \ldots, i_k} \binom{k}{\underbrace{1, \ldots, 1}_{i_1 \text{ times}}, \ldots, \underbrace{k, \ldots, k}_{i_k \text{ times}}} a_1^{i_1} \cdots a_k^{i_k}, \tag{8}$$

where $i_0 = n - (i_1 + \cdots + i_k)$ and $\binom{n}{n-(i_1+\cdots+i_k), i_1, \ldots, i_k}$ is a polynomial of degree $(i_1 + \cdots + i_k)$ in n.

But this is precisely what we need to make sense of replacing n with α in $n_{\bullet}\gamma$.

DEFINITION 4. *Let $\alpha, \gamma \in \mathcal{A}_0$ be umbrae, and define $g(z) \in \mathbf{F}[[z]]$ by $g(z) \simeq e^{\gamma z}$. Let $q_{\gamma,k}(n)$ be the coefficient of $z^k/k!$ in $e^{n \log(g(z))}$. Define $\alpha_{\bullet}\gamma \in \mathcal{A}_1$ to be a new auxiliary umbra, such that $(\alpha_{\bullet}\gamma)^k \simeq q_{\gamma,k}(\alpha)$.*

In general, if $p, q \in \mathbf{F}[\mathcal{A}_0]$ are umbral polynomials, we define an auxiliary umbra $p_{\bullet}q \in \mathcal{A}_1$ by $p_{\bullet}q \equiv \alpha_{\bullet}\beta$, where $\alpha \equiv p$ and $\beta \equiv q$.

Equivalently, we could have defined $(\alpha_{\bullet}\gamma)$ by replacing n with α on the right-hand side of equation (8).

The definition immediately implies that $e^{\alpha_{\bullet}\gamma z} \simeq e^{\alpha \log(g(z))} = (g(z))^{\alpha}$. Similarly, if $a(z) \in \mathbf{k}[[z]]$ is defined by $a(z) \simeq e^{\alpha z}$, then $e^{\alpha_{\bullet}\gamma z} \simeq a(\log(g(z)))$. It is a straightforward exercise in probability theory to show that if $a(z)$ is the moment generating function of a random variable A taking only positive integer values, and if $g(z)$ is the moment generating function of a random variable B, then $B_1 + \cdots + B_A$ also has moment generating function $a(\log(g(z)))$.

It follows that under this definition $0_{\bullet}\gamma \equiv \varepsilon \equiv 0$, which is what one would expect from the analogy to random variables. We now state the promised analogues to the standard results on random variables.

PROPOSITION 5. *Let $p, q, r \in \mathbf{F}[\mathcal{A}_0]$ be umbral polynomials. If p, q are independent, then*

$$(p + q)_{\bullet}r \equiv p_{\bullet}r + q_{\bullet}r.$$

PROOF. By definition and the substitution lemma (Lemma 1), it suffices to prove that

$$(\alpha + \beta)_{\bullet}\gamma \equiv \alpha_{\bullet}\gamma + \beta_{\bullet}\gamma$$

for any distinct umbrae α, β, γ, i.e., that the k^{th} powers of each side of the displayed equation are umbrally equivalent for all $k \geq 0$. Letting $q_{\gamma,k}(n)$ be the polynomials from Definition 4, it suffices to show, for all $k \geq 0$, that the identity $q_{\gamma,k}(\alpha + \beta) = \sum_i \binom{k}{i} q_{\gamma,i}(\alpha) q_{\gamma,k-i}(\beta)$ holds purely on the level of polynomials in variables α, β. But this follows since equation (7) says this identity holds with α, β replaced by any pair of positive integers. ∎

As remarked above, we cannot expect that $p_{\bullet}(q + r) \equiv p_{\bullet}q + p_{\bullet}r$ will hold in general. However, we record the special case where p involves no umbrae.

PROPOSITION 6. *Let a be an element of $\mathbf{F}$. Let $q, r \in \mathbf{F}[\mathcal{A}_0]$ be umbral polynomials. If q, r are independent, then $a_{\bullet}(q + r) \equiv a_{\bullet}q + a_{\bullet}r$.*

PROOF. The result holds when a is any integer. Repeating the argument in the proof of Proposition 5 shows the identity holds when interpreted in terms of polynomials in a. ∎

The importance of independence is illuminated if we examine what fails on replacing a with α and trying to prove that $\alpha_\bullet(\beta+\gamma) \equiv \alpha_\bullet\beta + \alpha_\bullet\gamma$. Staying with the notation introduced in Definition 4, we would need to show that

$$q_{\beta+\gamma,k}(\alpha) \simeq \sum_i \binom{k}{i} q_{\beta,i}(\alpha) q_{\gamma,k-i}(\alpha'),$$

where $\alpha' \equiv \alpha$. This relation fails to hold. It only worked when $\alpha \equiv a$ for $a \in \mathbf{F}$ because the substitution lemma yielded $q_{\beta+\gamma,k}(\alpha) \simeq q_{\beta+\gamma,k}(a)$ giving $\sum_i \binom{k}{i} q_{\beta,i}(\alpha) q_{\gamma,k-i}(\alpha') \simeq \sum_i \binom{k}{i} q_{\beta,i}(a) q_{\gamma,k-i}(a)$ which *is* equal to $q_{\beta+\gamma,k}(a)$.

The special case of $\alpha_\bullet\gamma$ for $\alpha \equiv -n$ where n is a positive integer is Ray's definition in [9] of a "negative umbral integer". Let n be a positive integer. Since $-n_\bullet\gamma + n_\bullet\gamma \equiv (-n+n)_\bullet\gamma \equiv \varepsilon$, the umbra $-n_\bullet\gamma$ defined as above is the same as the umbral $-n_\bullet\gamma$ defined in [2].

The same techniques used in the preceding propositions prove the following.

PROPOSITION 7. *Let a, c be in $\mathbf{F}$. If $p \in \mathbf{F}[\mathcal{A}]$, then $a_\bullet(cp) \equiv c(a_\bullet p)$.* ∎

We observe that $-1_\bullet\alpha$ is *not* in general exchangeable with $-\alpha$. The latter is exchangeable with $-1(1_\bullet\alpha)$.

Our definition of $p_\bullet q$ does not allow for p or q to contain auxiliary umbrae. Nevertheless, we would like to be able to manipulate expressions that resemble $\alpha_\bullet(\beta_\bullet\gamma)$. Before we extend the notion of an auxiliary umbra to handle this kind of construct, we prove the following associativity result.

PROPOSITION 8. *Let α, β, γ be umbrae. Define an umbra ρ by $\rho \equiv \alpha_\bullet\beta$ and an umbra σ by $\sigma \equiv \beta_\bullet\gamma$. We have $\rho_\bullet\gamma \equiv \alpha_\bullet\sigma$.*

Before presenting the proof, which is a quick calculation with generating functions, we interpret the result probabilistically. $A_\bullet(B_\bullet C)$ can be viewed as finding A, then running A trials of $B_\bullet C$; i.e., A times we run a trial of B, and following each trial of B, we run that many trials of C. Then we add up all the trials of C. In this interpretation, $(A_\bullet B)_\bullet C$ differs only in that we run A trials of B and then run all the trials of C at once. We could extend this identity to umbrae α, β, γ by viewing each side as a polynomial in the variables $\mathbf{E}[\alpha^i]$, $\mathbf{E}[\beta^i]$, $\mathbf{E}[\gamma^i]$.

Alternately, we argue as follows.

PROOF. It suffices to check that $e^{\rho_\bullet\gamma\, z} \simeq e^{\alpha_\bullet\sigma\, z}$. If $a(z), b(z), c(z) \in \mathbf{k}[[z]]$ are given by $a(z) \simeq e^{\alpha\, z}$, $b(z) \simeq e^{\beta\, z}$, and $c(z) \simeq e^{\gamma\, z}$, then this amounts to observing that each side is umbrally equivalent to the composition $a(z) \circ \log(z) \circ b(z) \circ \log(z) \circ c(z)$. ∎

With this lemma in hand, the following definition makes sense.

DEFINITION 9. *Given umbral polynomials $p_1, \ldots, p_n \in \mathbf{k}[\mathcal{A}_0]$, inductively define the auxiliary umbra $p_{1\bullet}p_{2\bullet}\cdots_\bullet p_n \in \mathcal{A}_1$ by $p_{1\bullet}p_{2\bullet}\cdots_\bullet p_n \equiv p_{1\bullet}\rho$, where $\rho \equiv p_{2\bullet}\cdots_\bullet p_n$.*

5. PRESENTATIONS FOR SEQUENCES OF BINOMIAL TYPE

5.1. Sequences of Binomial Type and Sums of Umbrae

The notion of a sequence of binomial type is a direct generalization of equation (7).

DEFINITION 10. *A sequence of polynomials $p_0(x), p_1(x), p_2(x), \ldots$ with $p_n(x)$ of degree n is of binomial type when it satisfies*

$$p_k(x+y) = \sum_i \binom{k}{i} p_i(x)\, p_{k-i}(y). \tag{9}$$

Such a sequence is normalized when $p_1(x)$ is monic (equivalently $p_1(x) = x$).

Equation (7) arose directly as the umbral expansion of the identity (equation (6)) that $(m+n)_\bullet\alpha \equiv m_\bullet\alpha + n_\bullet\alpha'$. Recall that, for any umbra γ and any element $x \in \mathbf{F}$, $\mathbf{E}[(x_\bullet\gamma)^n]$ is a polynomial in x and that $(x+y)_\bullet\gamma \equiv x_\bullet\gamma + y_\bullet\gamma$ where y is also in $\mathbf{F}$. As in the proof of Proposition 5, raising both sides of the preceding equality to the n^{th} power and applying $\mathbf{E}$ implies the if direction of the following.

THEOREM 11. *Let $p_0(x), p_1(x), p_2(x), \ldots$ be a sequence of polynomials where $p_n(x)$ has degree n. This is a sequence of binomial type iff it is umbrally represented by $x_\bullet\gamma$ for some umbra γ.*

PROOF. By the remarks preceding the theorem, it suffices to show that any sequence of binomial type can be so represented. By standard results, which are briefly sketched below, it suffices to show that choosing γ appropriately allows us to choose the sequence $D_x\mathbf{E}[(x_\bullet\gamma)^1]|_{x=0}$, $D_x\mathbf{E}[(x_\bullet\gamma)^2]|_{x=0}$, $D_x\mathbf{E}[(x_\bullet\gamma)^3]|_{x=0}$, $\ldots$ arbitrarily.

It is enough to observe that equation (8) tells us that the coefficient of x in $(x_\bullet\gamma)^k$ is $\gamma^k + R$ where R is depends only on k and $\mathbf{E}[\gamma], \ldots, \mathbf{E}[\gamma^{k-1}]$. ∎

For completeness, we sketch the fact that knowing $p_1'(0), p_2'(0), \ldots$ determines a sequence $p_0(x), p_1(x), p_2(x), \ldots$ of binomial type. Replace y with 0 in equation (9) and recall that, by degree considerations, the $p_i(x)$ are linearly independent, which tells us that $p_0(0) = 1$ and that $p_i(0) = 0$ for $i > 0$. Taking the derivative of equation (9) with respect to y and setting y to 0 gives $p_k'(x) = \sum_{i=0}^{k-1} \binom{k}{i} p_i(x) p_{k-i}'(0)$. Since $p_k(0) = \delta_{k,0}$, this determines $p_k(x)$.

Following [4], the *umbral composition* $a(\mathbf{b}(x))$, of two polynomial sequences $a_0(x), a_1(x), \ldots$ and $b_0(x), b_1(x), \ldots$ is the sequence $T(a_0(x)), T(a_1(x)), \ldots$, where $T : \mathbf{k}[x] \to \mathbf{k}[x]$ is the linear operator defined by $T(x^i) = b_i(x)$ for all i. An *umbral operator* is defined to be a linear operator $U : \mathbf{k}[x] \to \mathbf{k}[x]$ such that the sequence $U(1), U(x), U(x^2), \ldots$ is of binomial type. The following corollaries are immediate.

COROLLARY 12. *A linear operator $U : \mathbf{k}[x] \to \mathbf{k}[x]$ is an umbral operator iff there exists an umbra γ such that $U(r(x)) = \mathbf{E}[(r(x))_\bullet\gamma]$ for all $r(x) \in \mathbb{Q}[x]$.* ∎

COROLLARY 13. *Let $p_0, p_1, \ldots$ and $q_0, q_1, \ldots$ be sequences of binomial type represented by $x_\bullet\alpha$ and $x_\bullet\beta$, respectively. The umbral composition of these sequences, $p_0(\mathbf{q}), p_1(\mathbf{q}), \ldots$ is represented by $x_\bullet\beta_\bullet\alpha$.* ∎

This makes obvious the fact from [4] that the umbral composition of two sequences of binomial type is also of binomial type.

5.2. Generalized Abel Polynomials

One of the best known sequences of binomial type has as its degree n polynomial the *Abel polynomial* $x(x+na)^{n-1}$ where a is a constant. Generalizing a to be an arbitrary umbra α and replacing na with $n_\bullet\alpha$ yields the following.

THEOREM 14. *(See [7].) Let $p_n(x) \in \mathbf{k}[x]$ be a sequence of polynomials with $p_1(x) = x$ and $p_n(x)$ of degree n.*

The sequence $p_n(x)$ is of binomial type iff there exists an umbra α such that

$$p_n(x) \simeq x(x + n_\bullet\alpha)^{n-1}.$$

The proof in [7] closely parallels the proof that the original Abel polynomials are of binomial type. Here we provide a combinatorial proof.

PROOF. To start with, assume that $\mathbf{E}[\alpha^i]$ is always an integer and that $\alpha_1, \ldots, \alpha_n$ are distinct umbrae all exchangeable with α. We start by interpreting $(x + \alpha_1 + \cdots + \alpha_n)^{n-1}$ as an ordinary generating function for sequences of length $n-1$ on $n+1$ symbols. By the Prüfer correspondence (see, for example, [10]) this is a generating function for the number of labeled free trees on $n+1$ vertices where each tree is counted with weight $x^{d_0} \prod_l \alpha_l^{d_l}$ where vertex l has degree $d_l + 1$. So $x(x + \alpha_1 + \cdots + \alpha_n)^{n-1}$ is the generating function for labeled trees with specified root on $n+1$ vertices where the same weight indicates that vertex l has outdegree d_l. This says that

$\mathbf{E}[x(x+\alpha_1+\cdots+\alpha_n)^{n-1}]$ is the generating function where the coefficient of x^k counts the number of labeled trees on $n+1$ vertices where the root has degree k and each nonroot vertex with outdegree i can be colored in any of a_i ways. Equivalently, $\mathbf{E}[x(x+\alpha_1+\cdots+\alpha_n)^{n-1}]$ counts the number of planted forests on n vertices where each vertex with outdegree i can be colored in any of a_i ways and where each tree in the forest can itself be colored in any of x ways. Let us call this structure a (x,α) degree-colored forest on n vertices.

So, counting the number of ways to form a $(x+y,\alpha)$ degree-colored forest on n vertex by the number of vertices, i, in the trees which were colored in one of the first x ways gives

$$(x+y)\,(x+y+n_{\bullet}\alpha)^{n-1} \simeq \sum_i \binom{n}{i}(x)\,(x+i_{\bullet}\alpha)^{i-1}\cdot(y)\,(y+(n-i)_{\bullet}\alpha)^{n-i-1}.$$

This fact for all positive integers $x, y, a_1, a_2, \ldots$ implies equation (9) as a polynomial identity.

To see that indeed all normalized sequences of binomial type arise in this fashion, it suffices, by the remarks after Theorem 11, to observe that the sequence $p_2'(0), p_3'(0), \ldots$ can be chosen arbitrarily. Indeed, $p_n'(0) \simeq (n_{\bullet}\alpha)^{n-1} \simeq n\alpha^{n-1}+R$ where R is a function of $\alpha_1, \ldots, \alpha_{n-2}$. ∎

The interpretation of $x(x+n_{\bullet}\alpha)^{n-1}$ as a generating function for colored forests was suggested to the author by Ray. It generalizes the notion of reluctant functions developed by Mullin and Rota in [3] and is closely related to the chromatic polynomials in [11].

The calculations used in [7] to prove Theorem 14 show the following.

PROPOSITION 15. *For any umbra α and any $n \geq 1$, we find that*

$$D_x e^{-1_{\bullet}\alpha D_x}\left(x(x+n_{\bullet}\alpha)^{n-1}\right) \simeq nx\,(x+(n-1)_{\bullet}\alpha)^{n-2}.$$ ∎

COROLLARY 16. *A sequence $p_0(x), p_1(x), \ldots$ of polynomials, $p_n(x)$ of degree n is a sequence of binomial type iff there exists a formal power series $g(t) \in \mathbf{k}[[t]]$ with $g(0)=0$ and $g'(0) \neq 0$ such that $g(D_x)(p_n(x)) = np_{n-1}(x)$ for all $n \geq 1$.*

PROOF. If $p_1(x)$ is monic or $g'(0)=1$, the result follows immediately from the preceding proposition and the remarks after Theorem 11.

If the sequence is of binomial type and $p_1(x)=ax$, then so is the sequence whose n^{th} term is $p_n(x)/a^n$; if $g(t)$ is the series associated to this new sequence, then $g(t)/a$ works for the original. The converse follows similarly. ∎

Formal power series of the sort described above are called *delta operators* and the correspondence between them and their associated sequences of binomial type was established in [4].

The transfer formula also arises as an immediate corollary. Since we now know that the operator $Q \simeq De^{-1_{\bullet}\alpha D_x}$ is associated to the sequence presented as $p_n(x) \simeq x(x+\alpha_1+\cdots+\alpha_n)^{n-1}$, it follows that

$$p_n(x) \simeq xe^{(\alpha_1+\cdots+\alpha_n)D_x}\left(x^{n-1}\right) = xe^{\alpha_1 D_x}\cdots e^{\alpha_n D_x}\left(x^{n-1}\right) \simeq x\left(\frac{Q}{D}\right)^{-n} x^{n-1}.$$

With the definition of a delta operator in hand, we recall the first expansion theorem from [4]. If Q is a delta operator associated to a sequence $p_0(x), p_1(x), \ldots$ of binomial type and if $T=f(D_x)$ for some formal power series $f(t) \in \mathbf{k}[[t]]$, then

$$T = \sum_{k\geq 0} Tp_k(0)\frac{Q^k}{k!}. \tag{10}$$

The usual proof employs the binomial expansion for $p_n(x+y)$ to verify the identity $f(D_y)p_n(x+y) = \sum_{k\geq 0} f(D_y)p_k(y)(Q^k/k!)p_n(x)$ for all $n \geq 0$. Setting $y=0$ gives the desired result.

We have recalled the expansion theorem in order to derive the identity

$$D_x = \sum_{k\geq 0} p_k'(0)\, \frac{Q^k}{k!}, \tag{11}$$

where $p_0(x), p_1(x), \ldots$ is the sequence of binomial type associated to a delta operator, Q. Thus if $Q = f(D)$, we recover the fact that $p_k'(0)$ is the coefficient of $t^k/k!$ in $f^{\langle -1\rangle}(t)$ the power series inverse, under composition, to $f(t)$. This relationship together with the transfer formula is used to prove the Lagrange inversion formula. See [3,7] for such derivations.

5.3. Generalized Rising Factorials

Our next presentation generalizes the binomial type sequence of rising factorials, $x(x+1)\cdots(x+n-1)$. More generally, it is well known that the sequence $p_n(x) = x(x+a)\cdots(x+(n-1)a)$ is of binomial type for all constants and that its associated delta operator for $a \neq 0$ is $(I - e^{-aD})/a$. For the rising factorials, this is the backwards difference operator $f(x) \mapsto f(x) - f(x-1)$. Our proofs closely follow those for the usual rising factorials.

THEOREM 17. *Let $p_n(x) \in \mathbf{k}[x]$ be a sequence of polynomials with $p_1(x) = x$ and $p_n(x)$ of degree n.*

The sequence $p_n(x)$ is of binomial type iff there exists an umbra μ such that

$$p_n(x) \simeq x\,(x+\mu_1)\,(x+\mu_1+\mu_2)\,(x+\mu_1+\mu_2+\mu_3)\cdots(x+\mu_1+\cdots+\mu_{n-1}),$$

where $\mu_1, \ldots, \mu_{n-1}$ are distinct umbrae exchangeable with μ.

PROOF. We start by showing that all such presentations are of binomial type.

By induction, we have that

$$\begin{aligned}
&(x+y)\,(x+y+\mu_1)\,(x+y+\mu_1+\mu_2)\cdots(x+y+\mu_1+\cdots+\mu_{n-1})\\
&\quad\simeq y(y+x+\mu)\,(y+x+\mu+\mu_1)\,(y+x+\mu+\mu_1+\mu_2)\cdots(y+x+\mu+\mu_1+\cdots+\mu_{n-2})\\
&\qquad + x(x+y+\mu)\,(x+y+\mu+\mu_1)\,(x+y+\mu+\mu_1+\mu_2)\cdots(x+y+\mu+\mu_1+\cdots+\mu_{n-2})\\
&\quad\simeq y p_{n-1}\,(x+(y+\mu)) + x p_{n-1}\,((x+\mu)+y)\\
&\quad= y\sum_i \binom{n-1}{i} p_{n-1-i}(x)p_i(y+\mu) + x\sum_i \binom{n-1}{i} p_{n-1-i}(x+\mu)p_i(y)\\
&\quad\simeq \sum_i \binom{n-1}{i} p_{n-1-i}(x) y(y+\mu)\,(y+\mu+\mu_1)\cdots(y+\mu+\mu_1+\cdots+\mu_{i-1})\\
&\qquad + \sum_i \binom{n-1}{i} x(x+\mu)\,(x+\mu+\mu_1)\cdots(x+\mu+\mu_1+\cdots+\mu_{n-2-i})\,p_i(y)\\
&\quad\simeq \sum_i \binom{n-1}{i} p_{n-1-i}(x)p_{i+1}(y) + \sum_i \binom{n-1}{i} p_{n-i}(x)p_i(y)\\
&\quad\simeq \sum_{i=1}^{n} \binom{n-1}{i-1} p_{n-i}(x)p_i(y) + \sum_i \binom{n-1}{i} p_{n-i}(x)p_i(y)\\
&\quad\simeq \sum_i \binom{n}{i} p_{n-i}(x)p_i(y).
\end{aligned}$$

As before, all normalized sequences of binomial type arise in this fashion, since the sequence $p_2'(0), p_3'(0), \ldots$ can be chosen arbitrarily. Observe that $p_n'(0) \simeq \mu_1(\mu_1+\mu_2)\cdots(\mu_1+\cdots+\mu_{n-1}) \simeq \mu_1^{n-1} + R$ where each term of R has degree less than $n-1$ in each μ_i. ∎

PROPOSITION 18. *Let μ be an umbra and let $\mu_1, \mu_2, \ldots$ be distinct umbrae exchangeable with μ. Consider the sequence of binomial type $p_n(x) \simeq x(x+\mu)\cdots(x+\mu_1+\cdots+\mu_{n-1})$ presented by the generalized rising factorials. The corresponding delta operator is $q(D_x)$ where $q(t) \in \mathbf{k}[[t]]$ is given by*

$$q(t) \simeq \frac{e^{-1_\bullet\mu\, t}-1}{-1_\bullet\mu}.$$

PROOF. Evaluating $q(D_x)$ on $p_n(x)$, we find

$$\begin{aligned}
\frac{e^{-1_\bullet\mu\, D_x}-I}{-1_\bullet\mu}p_n(x) &= \frac{p_n\left(x+-1_\bullet\mu\right)-p_n(x)}{-1_\bullet\mu}\\
&= \sum_{i\geq 1}\binom{n}{i}p_{n-i}(x)\frac{p_i\left(-1_\bullet\mu\right)}{-1_\bullet\mu}\\
&\simeq \sum_{i\geq 1}\binom{n}{i}p_{n-i}(x)\left(-1_\bullet\mu+\mu_1\right)\left(-1_\bullet\mu+\mu_1+\mu_2\right)\cdots\left(-1_\bullet\mu+\mu_1+\cdots+\mu_{i-1}\right)\\
&\simeq \binom{n}{1}p_{n-1}(x)\cdot 1.
\end{aligned}$$

Here, the last line follows from the substitution lemma by replacing $-1_\bullet\mu+\mu_1$ with $\varepsilon \equiv 0$. ∎

Just as the transfer formula arose from the generalized Abel presentation, the Rodrigues formula arises from the presentation by generalized rising factorials. Preserve the notation from the preceding proposition. We have

$$\begin{aligned}
p_n(x) &\simeq x\left(x+\mu_1\right)+\cdots\left(x+\mu_1+\cdots+\mu_{n-1}\right)\\
&\simeq x\cdot e^{\mu D_x}\left(x\left(x+\mu_2\right)\cdots\left(x+\mu_2+\cdots+\mu_{n-2}\right)\right)\\
&\simeq x\cdot e^{\mu D_x}p_{n-1}(x)\\
&\simeq x\cdot\left(q'(D)\right)^{-1}p_{n-1}(x),
\end{aligned}$$

where the last line follows since $q'(t) \simeq e^{-1_\bullet\mu\, t}$.

We close our consideration of generalized rising factorials by observing that the presentation $p_n(x) \simeq x(x+\mu_1)\cdots(x+\mu_1+\cdots+\mu_{n-1})$ immediately suggests a combinatorial interpretation along the lines of reluctant functions.

PROPOSITION 19. *Let x be a nonnegative integer and let μ be an umbra such that $\mu^i \simeq m_i$ where each m_i is a nonnegative integer. The value of $p_n(x) \simeq x(x+\mu_1)\cdots(x+\mu_1+\cdots+\mu_{n-1})$ is the number of labeled forests on n vertices where each tree is assigned one of x colors, each vertex of outdegree j is assigned one of m_j colors, and where each parent vertex has a smaller label than each of its children.*

PROOF. It suffices to observe that we can construct such a tree by specifying a function mapping each vertex to its parent and then choosing colors. Choosing μ_j from the i^{th} multiplicand corresponds to requiring the function to map vertex i to vertex j. Choosing x of course indicates that vertex i is a root. ∎

6. PRESENTATIONS FOR SHEFFER SEQUENCES

Recall that a sequence of polynomials $s_0(x), s_1(x), s_2(x), \ldots$ with $s_n(x)$ of degree n is said to be a *Sheffer sequence* with respect to a delta operator Q (or with respect to the associated sequence of binomial type) when $Qs_n(x) = ns_{n-1}(x)$ for all $n \geq 0$. We will call a Sheffer sequence *normalized* when $s_1(x)$ is monic.

Let $Q = f(D_x)$ be a delta operator. Let $p_0(x), p_1(x), \ldots$ be the associated sequence of binomial type. Since Q and $e^{\beta D_x}$ commute, we have $Qp_n(x+\beta) \simeq np_{n-1}(x+\beta)$, and thus, $p_n(x+\beta)$ is a Sheffer sequence for Q. Since Q/D is invertible, $s_n(0)$, and $Qs_n(x)$ determine $s_n(x)$. Hence, $s_1(0), s_2(0), \ldots$ determines any Sheffer sequence with respect to Q. But because the $p_i(x)$ have different degrees, any such sequence arises from the umbral presentation $s_n(x) \simeq p_n(x+\beta)$ for suitable choice of umbra β. We have proved the following.

PROPOSITION 20. *If $p_0(x), p_1(x), \ldots$ is a sequence of binomial type, then all associated Sheffer sequences are umbrally presented by the sequence $p_0(x+\beta)$, $p_1(x+\beta)$, $p_2(x+\beta)$, $\ldots$ for some $a \in \mathbf{k}$ and some umbra β.* ∎

The following is immediate from the preceding and the presentation results in the preceding section.

COROLLARY 21. *Let $s_0(x), s_1(x), \ldots$ be a sequence of polynomials in $\mathbf{k}[x]$, and let $p_0(x), p_1(x)$, $p_2(x), \ldots$ in $\mathbf{k}[x]$ be a sequence of binomial type. The following are equivalent.*

1. *The sequence $s_0(x), s_1(x), \ldots$ is Sheffer with respect to $p_0(x), p_1(x), \ldots$.*
2. *There exist umbrae β, γ such that $p_n(x) \simeq (x_\bullet\gamma)^n$ and $s_n(x) \simeq ((x+\beta)_\bullet\gamma)^n$.*
3. *There exist umbrae β, α such that $p_n(x) \simeq x(x+n_\bullet\alpha)^{n-1}$ and $s_n(x) \simeq (x+\beta)(x+\beta+n_\bullet\alpha)^{n-1}$.*
4. *There exist umbrae β, μ such that $p_n(x) \simeq x(x+\mu_1)\cdots(x+\mu_1+\cdots+\mu_{n-1})$ and $s_n(x) \simeq (x+\beta)(x+\beta+\mu_1)\cdots(x+\beta+\mu_1+\cdots+\mu_{n-1})$.* ∎

Since $(x+y+\beta)_\bullet\gamma \equiv x_\bullet\gamma + (y+\beta_\bullet\gamma)$, the standard expansion result for Sheffer sequences,

$$s_n(x+y) = \sum_{i=1}^{n} \binom{n}{i} p_i(x) s_{n-i}(y),$$

follows immediately from Part 2 of this corollary.

7. MULTIPLICATIVE SEQUENCES

We present our final results as easy applications of the preceding constructions. We have relied on the umbral relation $(x+y)_\bullet\gamma \equiv x_\bullet\gamma + y_\bullet\gamma$ to find a presentation for sequences of binomial type. It is natural to ask what happens if we replace x and y themselves by umbrae. Fix umbrae α, β such that $\alpha^n \simeq a_n$ and $\beta^n \simeq b_n$. We will consider sequences of polynomials in multiple variables $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$.

Fix an umbra γ and define $K_m(a_1, a_2, \ldots) \simeq (\alpha_\bullet\gamma)^m$. By equation (8) $K_m(a_1, a_2, \ldots)$ has degree m, is linear in the variables a_i and only depends on the variables $a_1, \ldots, a_m$.

From the point of view of generating functions, the fairly simple umbral relation $(\alpha+\beta)_\bullet\gamma \equiv \alpha_\bullet\gamma + \beta_\bullet\gamma$ becomes the following result.

PROPOSITION 22. *Define polynomials $K_m \in \mathbf{k}[a_1, \ldots, a_m]$ by $K_m(a_1, a_2, \ldots, a_m) \simeq (\alpha_\bullet\gamma)^m$. If*

$$\sum_{k\geq 0} c_k \frac{z^k}{k!} = \left(\sum_{i\geq 0} a_i \frac{z^i}{i!}\right)\left(\sum_{j\geq 0} b_j \frac{z^j}{j!}\right),$$

then

$$\sum_{k\geq 0} K_k\left(c_1, \ldots, c_k\right) \frac{z^k}{k!} = \left(\sum_{i\geq 0} K_i\left(a_1, \ldots, a_i\right) \frac{z^i}{i!}\right)\left(\sum_{j\geq 0} K_j\left(b_1, \ldots, b_j\right) \frac{z^j}{j!}\right).$$ ∎

We formalize this "nice" behavior of generating functions under multiplication as follows. Suppose that $K_0, K_1, \ldots$ is a sequence of polynomials in the variables $t_1, t_2, \ldots$. For $r \in \mathbf{k}[\mathcal{A}]$, denote by $K_m(r)$ the evaluation, $K_m(\mathbf{E}[r], \mathbf{E}[r^2], \mathbf{E}[r^3], \ldots)$, of the polynomial K_m. Define the sequence $K_0, K_1, \ldots$ to be *multiplicative* if whenever α, β, χ are umbrae such that $\alpha+\beta \equiv \chi$, we have $\rho+\sigma \equiv \tau$ where $\rho^m \simeq K_m(\alpha)$, $\sigma^m \simeq K_m(\beta)$, and $\tau^m \simeq K_m(\gamma)$.

Since $n_\bullet(\alpha+\beta) \equiv n_\bullet\alpha + n_\bullet\beta$, the sequence of polynomials K_m in $a_1, a_2, \ldots$ defined by $K_m(a_1, \ldots, a_m) \simeq (n_\bullet\alpha)^m$ is multiplicative.

These constructions can be generalized as follows.

PROPOSITION 23. *Let α be an umbra with $\alpha^i \simeq a_i$. Let l be a positive integer and γ an umbra. For $c_1, \dots, c_l \in \mathbf{k}$, define the polynomial $K_m(a_1, \dots, a_m)$ in $\mathbf{k}[a_1, \dots, a_m]$ by $K_m(a_1, \dots, a_m) \simeq (\sum_{i=1}^{l} i_{\bullet}(c_i\alpha)_{\bullet}\gamma)^m$. The sequence $1, K_1, K_2, \dots$ is multiplicative and K_m has total degree m when a_i is given degree i.*

If $\gamma \equiv 1$, then K_m is homogeneous in the above grading. ∎

If $K_0, K_1, \dots$ is multiplicative and homogeneous, then $L_0, L_1, \dots$ where $L_i(a_1, a_2, \dots) = K_i(a_1, 2a_2, 3!a_3, \dots)/i!$ is an m-sequence or multiplicative sequence in the sense of Hirzebruch [12]; namely, if

$$\sum_{k \geq 0} c_k z^k = \left(\sum_{i \geq 0} a_i z^i \right) \left(\sum_{j \geq 0} b_j z^j \right),$$

then

$$\sum_{k \geq 0} L_k(c_1, c_2, \dots, c_k) z^k = \left(\sum_{i \geq 0} L_i(a_1, a_2, \dots, a_i) z^i \right) \left(\sum_{j \geq 0} L_j(b_1, b_2, \dots, b_j) z^j \right).$$

8. OPEN PROBLEMS

We close with a brief list of open problems and areas for future work.

1. Generalize Theorems 14 and 17 and their corollaries by finding umbral presentations corresponding to other well-known sequences of binomial type.
2. Determine which sequences of binomial type over the integers and which sequences of integral type (see [13]) may be presented in the above fashions. Find general presentation formulae for these situations.
3. Find umbral presentation theorems for the various generalizations of the umbral calculus (see, for instance, [14,15]).
4. Give conditions for a sequence of binomial type to be presentable as $x_{\bullet}G$ where G is a random variable rather than an arbitrary umbra.

REFERENCES

1. G.-C. Rota and B.D. Taylor, An introduction to the umbral calculus, In *Analysis, Geometry and Groups: A Riemann Legacy Volume*, pp. 513–525, Hadronic Press, Palm Harbor, FL, (1993).
2. G.-C. Rota and B.D. Taylor, The classical umbral calculus, *SIAM J. Math. Anal.* **25**, 694–711, (1994).
3. R. Mullin and G.-C. Rota, Theory of binomial enumeration, In *Graph Theory and Its Applications*, pp. 167–213, Academic Press, New York, (1970).
4. G.-C. Rota, D. Kahaner and A. Odlyzko, Finite operator calculus, *J. Math. Anal. Appl.* **42**, 685–760, (1973).
5. S. Roman and G.-C. Rota, The umbral calculus, *Adv. Math.* **27**, 95–188, (1978).
6. B.D. Taylor, Difference equations via the classical umbral calculus, In *Mathematical Essays in Honor of Gian-Carlo Rota*, pp. 397–411, Birkhäuser, Boston, (1998).
7. G.-C. Rota, J.H. Shen and B.D. Taylor, All polynomials of binomial type are represented by Abel polynomials, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **25**, 731–738, (1998).
8. J. Blissard, Examples of the use and application of representative notation, *Quart. J. Pure. Appl. Math.* **4**, 49–64, (1864).
9. N. Ray, Universal constructions in umbral calculus, In *Mathematical Essays in Honor of Gian-Carlo Rota*, pp. 343–357, Birkhäuser, Boston, (1998).
10. R. Stanley, *Enumerative Combinatorics, Volume 2*, Cambridge University Press, Cambridge, (1999).
11. N. Ray, W. Schmitt and C. Wright, Colouring simplicial complexes, *Ars Combinatorica* **29A**, 161–169, (1990).
12. F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Springer-Verlag, Berlin, (1978).
13. M. Barnabei, A. Brini and G. Nicoletti, Polynomial sequences of integral type, *J. Math. Anal. Appl.* **78**, 598–617, (1980).
14. D. Loeb and G.-C. Rota, Formal power series of logarithmic type, *Adv. Math.* **75**, 1–118, (1989).
15. D. Loeb, The iterated logarithmic algebra, *Adv. Math.* **86**, 155–234, (1991).
16. N. Ray, All binomial sequences are factorial, Unpublished manuscript.