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EIGENVECTORS OF BLOCK CIRCULANT AND ALTERNATING CIRCULANT MATRICES

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Abstract. The eigenvectors and eigenvalues of symmetric block circulant matrices had been found, and that method is extended to general block circulant matrices. That analysis is applied to Stephen J. Watson's alternating circulant matrices, which reduce to block circulant matrices with square submatrices of order 2.

1. Circulant Matrix

A square matrix in which each row (after the first) has the elements of the previous row shifted cyclically one place right, is called a *circulant matrix*. Philip R. Davis (1979, p.69) denotes it as

$$\mathbf{B} \;=\; \mathrm{circ}(b_0, b_1, \cdots, b_{n-1}) \;\stackrel{\mathrm{def}}{=}\; \begin{bmatrix} b_0 & b_1 & b_2 & \ddots & b_{n-2} & b_{n-1} \\ b_{n-1} & b_0 & b_1 & \ddots & b_{n-3} & b_{n-2} \\ b_{n-2} & b_{n-1} & b_0 & \ddots & b_{n-4} & b_{n-3} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ b_2 & b_3 & b_4 & \ddots & b_0 & b_1 \\ b_1 & b_2 & b_3 & \ddots & b_{n-1} & b_0 \end{bmatrix}. \tag{1}$$

Thomas Muir (1911, Volume 2, Chapter 14) had denoted its determinant as $\mathrm{C}(b_0, b_1, \cdots, b_{n-1})$.

1.1. Complex nth roots of 1.

Denote

$$\vartheta \;\stackrel{\mathrm{def}}{=}\; \frac{2\pi}{n}. \tag{2}$$

Then the nth roots of 1 are:

$$\rho_j \;=\; \mathrm{e}^{\mathrm{i}2\pi j/n} \;=\; \mathrm{e}^{\mathrm{i}j\vartheta} \;=\; \cos j\vartheta + \mathrm{i}\sin j\vartheta \qquad (j = 0, 1, 2, \cdots, n-1). \tag{3}$$

For all integers f and j,

$$\rho_j^f + \overline{\rho_j^f} \;=\; \mathrm{e}^{\mathrm{i}fj\vartheta} \;+\; \mathrm{e}^{-\mathrm{i}fj\vartheta} \;=\; 2\cos jf\vartheta, \tag{4}$$

$$\rho_{n-j}^f \;=\; \mathrm{e}^{\mathrm{i}2\pi(n-j)f/n} \;=\; \mathrm{e}^{\mathrm{i}2\pi f - \mathrm{i}2\pi jf/n} \;=\; \mathrm{e}^{-\mathrm{i}2\pi jf/n} \;=\; \overline{\rho_j^f}, \tag{5}$$

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and

$$\cos(n-j)f\vartheta \;=\; \cos 2\pi(n-j)f/n \;=\; \cos(2f\pi - 2\pi fj/n) \;=\; \cos 2\pi fj/n \;=\; \cos jf\vartheta. \qquad (6)$$

For even $n = 2h$ and all integers j

$$\rho_h = \mathrm{e}^{\mathrm{i}\pi} = -1, \qquad \rho_j^h = \mathrm{e}^{\mathrm{i}2\pi jh/(2h)} = \mathrm{e}^{\mathrm{i}\pi j} = (-1)^j. \qquad (7)$$

1.2. Eigenvalues of circulant matrices. It is readily verified that $\mathbf{B}$ has n orthogonal eigenvectors and n eigenvalues

$$\mathbf{w}^{(j)} \;=\; \begin{bmatrix} 1 \\ \rho_j \\ \rho_j^2 \\ \vdots \\ \rho_j^{n-1} \end{bmatrix}, \qquad \lambda_j \;=\; b_0 + b_1\rho_j + b_2\rho_j^2 + b_3\rho_j^3 + \cdots + b_{n-1}\rho_j^{n-1} \qquad (8)$$

for $j = 0, 1, 2, \cdots, n-1$.

1.2.1. *Duplicated eigenvalues.* From this, it follows that the circulant matrix $\mathbf{B}$ is real and has eigenvalues λ_j, if and only if [Davis (1979), p.80, Problem 15][1]

$$\lambda_j = \overline{\lambda_{n-j}}, \quad (j = 1, 2, \cdots, n-1). \qquad (9)$$

This does not relate to λ_0 (which is real for real $\mathbf{B}$), and for even $n = 2h$ it implies that λ_h is real.

If $\mathbf{B}$ is real symmetric then its eigenvalues are real, and (9) reduces to

$$\lambda_j = \lambda_{n-j}, \quad (j = 1, 2, \cdots, n-1). \qquad (10)$$

The entire set of eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{(n-1)\div 2}$ is duplicated by (10).

With odd $n = 2h-1$, $\mathbf{B}$ has also the eigenvalue λ_0 which is a simple eigenvalue with multiplicity 1; unless λ_0 equals any λ_j for $1 \le j \le h-1$, in which case λ_0 occurs with odd multiplicity greater than or equal to 3. With even $n = 2h$, $\mathbf{B}$ has also the eigenvalues λ_0 and λ_h. If $\lambda_0 = \lambda_h$ then they have even multiplicity, whether or not they equal any λ_j for $1 \le j \le h-1$. Otherwise λ_0 either has multiplicity 1 or else it equals some λ_j for $1 \le j \le h-1$, in which case λ_0 occurs with odd multiplicity greater than or equal to 3. And similarly for λ_h.

Hence, for real symmetric $\mathbf{B}$, if n is odd then there is one eigenvalue with odd multiplicity, and if n is even then there are either two eigenvalues or none with odd multiplicity. All other eigenvalues have even multiplicity.

1.2.2. *Eigenvalues of complex symmetric circulant matrices.* A square matrix $\mathbf{A}$ is called Hermitian, if and only if $\mathbf{A}^* = \mathbf{A}$. If the matrix is real, this reduces to $\mathbf{A}^T = \mathbf{A}$. Each Hermitian matrix has a full set of orthogonal eigenvectors, each with real eigenvalue.

The complex circulant matrix $\mathbf{B}$ in (1) is symmetric if and only if $b_j = b_{n-j}$ for $j = 1, 2, \cdots, (n-1) \div 2$. Complex symmetric matrices are usually regarded as being less interesting than Hermitian matrices. Symmetric circulant matrices are

[1]Actually, Davis denotes our λ_{j-1} by λ_j and his result is printed as $\lambda_j = \overline{\lambda}_{n+1-j}$ for $j = 1, 2, \cdots, n$; whereas it should be (in Davis's notation) $\lambda_j = \overline{\lambda}_{n+2-j}$ for $j = 2, 3, \cdots, n$.

mentioned only briefly by Philip R. Davis (1979, p.67, Problem 1 & p.81, Problem 16), and the following result seems to be new.

Theorem 1: Every complex symmetric circulant matrix of order n has a single eigenvalue with odd multiplicity if n is odd, but it has either two eigenvalues or none with odd multiplicity if n is even. All other eigenvalues occur with even multiplicity.

Proof: For $j = 0, 1, \cdots, n-1$, the formula (8) gives the eigenvalue

$$\begin{aligned}\lambda_j &= b_0 + b_1\rho_j + b_2\rho_j^2 + \cdots + b_{n-2}\rho_j^{n-2} + b_{n-1}\rho_j^{n-1} \\ &= b_0 + b_1\rho_j + b_2\rho_j^2 + \cdots + b_2\rho_j^{-2} + b_1\rho_j^{-1} \\ &= b_0 + b_1\left[\rho_j + \overline{\rho_j}\right] + b_2\left[\rho_j^2 + \overline{\rho_j^2}\right] + \cdots \\ &\quad + b_{h-1}\left[\rho_j^{h-1} + \overline{\rho_j^{h-1}}\right] + \left\{\begin{array}{ll} 0 & \text{if } n = 2h-1 \\ b_h\rho_j^h & \text{if } n = 2h\ . \end{array}\right\} \end{aligned} \tag{11}$$

In view of (4) and (7), this reduces to

$$\lambda_j = b_0 + 2\sum_{f=1}^{h-1} b_f \cos jf\vartheta + \left\{\begin{array}{ll} 0 & \text{if } n = 2h-1 \\ b_h(-1)^j & \text{if } n = 2h\ . \end{array}\right. \tag{12}$$

Replacing j by $n-j$, we get

$$\begin{aligned}\lambda_{n-j} &= b_0 + 2\sum_{f=1}^{h-1} b_f \cos(n-j)f\vartheta + \left\{\begin{array}{ll} 0 & \text{if } n = 2h-1 \\ b_h(-1)^{2h-j} & \text{if } n = 2h \end{array}\right. \\ &= b_0 + 2\sum_{f=1}^{h-1} b_f \cos jf\vartheta + \left\{\begin{array}{ll} 0 & \text{if } n = 2h-1 \\ b_h(-1)^j & \text{if } n = 2h \end{array}\right. \\ &= \lambda_j, \end{aligned} \tag{13}$$

in view of (6).

Thus, with $h = (n+1) \div 2$, the sequence of eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{h-2}, \lambda_{h-1}$ is duplicated as the sequence $\lambda_{n-1}, \lambda_{n-2}, \cdots, \lambda_{n-h+2}, \lambda_{n-h+1}$. Every distinct eigenvalue in the first sequence occurs an even number of times within that pair of sequences.

With odd $n = 2h-1$, $\mathbf{B}$ has the eigenvalue

$$\lambda_0 = b_0 + 2(b_1 + b_2 + \cdots + b_{h-1}) \tag{14}$$

which is a simple eigenvalue with odd multiplicity 1; unless λ_0 equals any λ_j for $1 \le j \le h-1$, in which case λ_0 occurs with odd multiplicity greater than or equal to 3.

With even $n = 2h$, $\mathbf{B}$ has the eigenvalues

$$\begin{aligned}\lambda_0 &= b_0 + 2(b_1 + b_2 + \cdots + b_{h-1}) + b_h, \\ \lambda_h &= b_0 + 2\big(-b_1 + b_2 - \cdots + (-1)^{h-1}b_{h-1}\big) + (-1)^h b_h. \end{aligned} \tag{15}$$

If $\lambda_0 = \lambda_h$ then they have even multiplicity, whether or not they equal any λ_j for $1 \le j \le h-1$. Otherwise λ_0 either has multiplicity 1 or else it equals some λ_j for $1 \le j \le h-1$, in which case λ_0 occurs with odd multiplicity greater than or equal to 3. And similarly for λ_h.

Therefore, if n is odd then $\mathbf{B}$ has 1 eigenvalue with odd multiplicity, and if n is even then $\mathbf{B}$ has either 2 eigenvalues or none with odd multiplicity.

2. Block Circulant Matrices $\mathcal{BC}_{n,\kappa}$

Thomas Muir (1920,Volume 3, Chapter 15) defined a circulant determinant whose elements are determinants to be a *block circulant* determinant.

The eigenvalues of block circulant matrices (as in (19) below) were stated by Bernard Friedman (1961, Theorem 6a). He declared that those eigenvalues were well known — but for a proof he gave an erroneous citation of MUIR, T. AND METZLER, W. *Theory of determinants*, Chapter XII. That appears to refer to THOMAS MUIR (1930), *A Treatise On The Theory Of Determinants*, (revised and enlarged by William H. Metzler), Albany NY, Privately published. Metzler's Chapter 12 contains a section on block circulant determinants, but says nothing about eigenvalues. Chapter 15 on Determinantal Equations discusses characteristic equations of matrices, but says nothing about block circulants. Metzler's text is obscured by many misprints[2]; but nonetheless several results in his Chapter 12 could be inverted, modified and rewritten for matrices, to form a basis for proving that "well known" expression for the eigenvalues of a block circulant matrix. The London edition [Muir, 1933] is said [Farebrother, Jensen & Styan, p.6] to be "Apparently identical to Muir (1930)", and the Dover edition (1960) is described as "Unabridged and corrected (paperback) republication of Muir (1933)".

For real symmetric block circulant matrices with symmetric submatrices, the eigenvectors and eigenvalues (with their multiplicities) have been found [Tee, 1963].

The class of complex block circulant matrices $\mathbf{B} = \text{bcirc}(\mathbf{b}_0, \mathbf{b}_1, \cdots, \mathbf{b}_{n-1})$, where all square submatrices are of order $\kappa > 1$, is denoted by Philip R. Davis as $\mathcal{BC}_{n,\kappa}$ (1979, p.177). The eigenvectors of $\mathbf{B}$ had been found for the case where all submatrices $\mathbf{b}_j$ are themselves circulant [Trapp], [Chao] and Davis (1979, p.185, Theorem 5.8.1).

The analysis of the eigenvectors for real symmetric block circulant matrices with symmetric submatrices [Tee, 1963] is here extended to general complex block circulant matrices.

2.1. Eigenvectors and eigenvalues. Consider a compound vector $\mathbf{w}$, of the form:

$$\mathbf{w} = \begin{bmatrix} \mathbf{v} \\ \rho\mathbf{v} \\ \rho^2\mathbf{v} \\ \vdots \\ \rho^{n-1}\mathbf{v} \end{bmatrix}, \tag{16}$$

where $\mathbf{v}$ is a non-null κ-vector and ρ is any nth root (3) of 1.

[2]The Forder Collection in Auckland University Library has a copy of that book by Muir & Metzler, with a letter from Metzler to Henry George Forder (1930 June 5) pasted into it. Metzler explained that, in revising and enlarging Muir's *Treatise*, "My object was to give the mathematicians of the world a chance to benefit by more than thirty years of labor in this particular field without having to pay twice as much for the book if put up by a regular publisher. I am sorry that there are so many misprints but more than 50% occurred after the last proof went back".
That copy includes Metzler's Errata sheet listing about 150 misprints: but the book contains very many further misprints.

The vector $\mathbf{w}$ is an eigenvector of $\mathbf{B}$ with eigenvalue λ, if and only if

$$\mathbf{B}\mathbf{w} = \mathbf{w}\lambda. \tag{17}$$

The first few of the n compound rows of this equation are:

$$\begin{aligned}
(\mathbf{b}_0 + \mathbf{b}_1\rho + \mathbf{b}_2\rho^2 + \mathbf{b}_3\rho^3 + \cdots + \mathbf{b}_{n-1}\rho^{n-1})\mathbf{v} &= \mathbf{v}\lambda, \\
(\mathbf{b}_{n-1} + \mathbf{b}_0\rho + \mathbf{b}_1\rho^2 + \mathbf{b}_2\rho^3 + \cdots + \mathbf{b}_{n-2}\rho^{n-1})\mathbf{v} &= \rho\mathbf{v}\lambda, \\
(\mathbf{b}_{n-2} + \mathbf{b}_{n-1}\rho + \mathbf{b}_0\rho^2 + \mathbf{b}_1\rho^3 + \cdots + \mathbf{b}_{n-3}\rho^{n-1})\mathbf{v} &= \rho^2\mathbf{v}\lambda,
\end{aligned} \tag{18}$$

and so forth.

Each of the n compound rows in (18) reduces to the first of them, and that can be rewritten as the eigenvector equation

$$\mathbf{H}\mathbf{v} \;=\; \mathbf{v}\lambda, \tag{19}$$

where the square matrix $\mathbf{H}$ (of order κ) is:

$$\mathbf{H} \;=\; \mathbf{b}_0 + \mathbf{b}_1\rho + \mathbf{b}_2\rho^2 + \mathbf{b}_3\rho^3 + \cdots + \mathbf{b}_{n-1}\rho^{n-1}. \tag{20}$$

For vectors of the form (16) the set (20) of n eigenvector equations of order κ, for the n values of ρ, is equivalent to the single eigenvector equation (17), of order $n\kappa$. For each of the n values of ρ, the equation (19) can be solved to give κ eigenvectors $\mathbf{v}$ (or fewer if the eigenvectors of $\mathbf{H}$ happen to be defective), each with its eigenvalue λ. Each eigenvector $\mathbf{v}$ of $\mathbf{H}$, with its corresponding ρ, gives an eigenvector $\mathbf{w}$ of the block circulant matrix $\mathbf{B}$, with that eigenvalue λ.

For any ρ, if $\mathbf{H}$ has m linearly independent eigenvectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \cdots, \mathbf{v}^{(m)}$, then it follows from the structure of (16) that the corresponding eigenvectors $\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \cdots, \mathbf{w}^{(m)}$ of $\mathbf{B}$ are linearly independent. Conversely, if $\mathbf{B}$ has linearly independent eigenvectors $\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \cdots, \mathbf{w}^{(m)}$ of the form (16), each with the same ρ, then the first elements of those compound vectors are a set of linearly independent eigenvectors of $\mathbf{H}$ (for that ρ).

2.2. Orthogonal eigenvectors. Consider any κ-vectors $\mathbf{s}$ and $\mathbf{t}$, and the compound vectors

$$\mathbf{x} \;=\; \begin{bmatrix} \mathbf{s} \\ \rho_j\mathbf{s} \\ \rho_j^2\mathbf{s} \\ \vdots \\ \rho_j^{n-1}\mathbf{s} \end{bmatrix}, \qquad \mathbf{y} \;=\; \begin{bmatrix} \mathbf{t} \\ \rho_k\mathbf{t} \\ \rho_k^2\mathbf{t} \\ \vdots \\ \rho_k^{n-1}\mathbf{t} \end{bmatrix}. \tag{21}$$

The scalar product of $\mathbf{x}$ and $\mathbf{y}$ is

$$\begin{aligned}
\mathbf{x}^*\mathbf{y} &= \mathbf{s}^*\mathbf{t} + \mathbf{s}^*(\overline{\rho_j}\rho_k)\mathbf{t} + \mathbf{s}^*(\overline{\rho_j}\rho_k)^2\mathbf{t} + \cdots + \mathbf{s}^*(\overline{\rho_j}\rho_k)^{n-1}\mathbf{t} \\
&= \Big(1 + (\overline{\rho_j}\rho_k) + (\overline{\rho_j}\rho_k)^2 + \cdots + (\overline{\rho_j}\rho_k)^{n-1}\Big)(\mathbf{s}^*\mathbf{t}) \\
&= \Big(1 + u + u^2 + \cdots + u^{n-1}\Big)(\mathbf{s}^*\mathbf{t}),
\end{aligned} \tag{22}$$

where $u = \overline{\rho_j}\rho_k = \mathrm{e}^{\mathrm{i}2\pi(k-j)/n}$, and so $u^n = \mathrm{e}^{\mathrm{i}2\pi(k-j)} = 1$.

If $j \neq k$ then $u \neq 1$ and $\mathbf{x}^*\mathbf{y} = (\mathbf{s}^*\mathbf{t})(1-u^n)/(1-u)$, and so $\mathbf{x}^*\mathbf{y} = \mathbf{0}$.

Therefore, if $j \neq k$ then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

In particular, if $\mathbf{s}$ is an eigenvector of $\mathbf{H}_j$ (with $\rho = \rho_j$) and $\mathbf{t}$ is an eigenvector of $\mathbf{H}_k$ (with $\rho = \rho_k$) and $j \neq k$, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal eigenvectors of $\mathbf{B}$.

For each ρ, $\mathbf{H}$ has at least 1 eigenvector, and hence the eigenvectors of $\mathbf{B}$ span at least n dimensions.

2.3. Hermitian block circulant matrices. The block circulant matrix $\mathbf{B}$ in (17) is Hermitian if and only if $\mathbf{b}_j^* = \mathbf{b}_{n-j}$ for $j = 1, 2, \cdots, n \div 2$ and $\mathbf{b}_0$ is Hermitian. Note that, if $n = 2h$ then $\mathbf{b}_h = \mathbf{b}_h^*$, so that both $\mathbf{b}_0$ and $\mathbf{b}_h$ must be Hermitian.

The matrix $\mathbf{H}$ then (20) reduces to

$$\begin{aligned} \mathbf{H} &= \mathbf{b}_0 + \mathbf{b}_1\rho + \mathbf{b}_2\rho^2 + \cdots + \mathbf{b}_{n-2}\rho^{n-2} + \mathbf{b}_{n-1}\rho^{n-1} \\ &= \mathbf{b}_0 + \mathbf{b}_1\rho + \mathbf{b}_2\rho^2 + \cdots + \mathbf{b}_2^*\rho^{-2} + \mathbf{b}_1^*\rho^{-1} \\ &= \mathbf{b}_0 + \Big[\mathbf{b}_1\rho + \mathbf{b}_1^*\overline{\rho}\Big] + \Big[\mathbf{b}_2\rho^2 + \mathbf{b}_2^*\overline{\rho^2}\Big] + \cdots \\ &\quad + \Big[\mathbf{b}_{h-1}\rho^{h-1} + \mathbf{b}_{h-1}^*\overline{\rho^{h-1}}\Big] + \begin{cases} \mathbf{0} & \text{if } n = 2h-1 \\ \mathbf{b}_h\rho^h & \text{if } n = 2h\ . \end{cases} \end{aligned} \tag{23}$$

Hence, for $\rho = \rho_j$, in view of (7) the expression (23) for $\mathbf{H}$ reduces to

$$\mathbf{H}_j = \mathbf{b}_0 + \sum_{f=1}^{h-1}\Big[\mathbf{b}_f\rho_j^f + \mathbf{b}_f^*\overline{\rho_j^f}\Big] + \begin{cases} \mathbf{0} & \text{if } n = 2h-1 \\ \mathbf{b}_h(-1)^j & \text{if } n = 2h\ . \end{cases} \tag{24}$$

Taking the conjugate transpose of (24), it follows that for all j, $\mathbf{H}_j = \mathbf{H}_j^*$.

Therefore, if $\mathbf{B}$ is Hermitian then each $\mathbf{H}_j$ in (20) is Hermitian, with κ real eigenvalues and κ orthogonal complex eigenvectors, and those give $n\kappa$ orthogonal complex eigenvectors (16) of $\mathbf{B}$.

Hermitian submatrices. Whether or not the entire matrix $\mathbf{B}$ is Hermitian, if each submatrix $\mathbf{b}_0, \cdots, \mathbf{b}_{n-1}$ is Hermitian (including real-symmetric), then it follows from (20) that the matrix $\mathbf{H}_0$ with $\rho_0 = 1$ is Hermitian. Hence, $\mathbf{H}_0$ has κ orthogonal eigenvectors and κ corresponding real eigenvalues.

Accordingly, every matrix of type $\mathcal{BC}_{n,\kappa}$ with Hermitian submatrices has κ orthogonal eigenvectors (16) with $\rho = 1$ and real eigenvalues. And if n is even then similarly for $\rho = -1$.

2.3.1. *Real symmetric block circulant matrices.* Equation (24) for Hermitian $\mathbf{H}_j$ now reduces to

$$\mathbf{H}_j = \mathbf{b}_0 + \sum_{f=1}^{h-1}\Big[\mathbf{b}_f{\rho_j}^f + \mathbf{b}_f^T\overline{\rho_j^f}\Big] + \begin{cases} \mathbf{0} & \text{if } n = 2h-1 \\ \mathbf{b}_h(-1)^j & \text{if } n = 2h\ , \end{cases} \tag{25}$$

and so $\mathbf{H}_0$ and (for even $n = 2h$) $\mathbf{H}_h$ are real-symmetric, with real orthogonal eigenvectors with real eigenvalues. But the other $\mathbf{H}_j$ are not real-symmetric in general, and so their eigenvectors are complex in general. But each $\mathbf{H}_j$ is Hermitian with all eigenvalues real. And the eigenvalues of $\mathbf{B}$ are the union of the n sets of κ eigenvalues for all the $\mathbf{H}_j$.

Replacing j by $n-j$ in (25), we get that

$$\begin{aligned}\mathbf{H}_{n-j} &= \mathbf{b}_0 + \sum_{f=1}^{h-1}\left[\mathbf{b}_f \rho_{n-j}^{\,f} + \mathbf{b}_f^T \overline{\rho_{n-j}^{\,f}}\right] + \begin{cases}\mathbf{0} & \text{if } n = 2h-1\\ \mathbf{b}_h(-1)^{2h-j} & \text{if } n = 2h\end{cases}\\ &= \mathbf{b}_0 + \sum_{f=1}^{h-1}\left[\mathbf{b}_f \overline{\rho_j^{\,f}} + \mathbf{b}_f^T {\rho_j}^{f}\right] + \begin{cases}\mathbf{0} & \text{if } n = 2h-1\\ \mathbf{b}_h(-1)^{j} & \text{if } n = 2h\end{cases}\\ &= \overline{\mathbf{H}_j} \qquad (0 < j < n).\end{aligned} \tag{26}$$

Let $\mathbf{v}$ be an eigenvector of $\mathbf{H}_j$ with eigenvalue λ, i.e. $\mathbf{H}_j\mathbf{v} = \mathbf{v}\lambda$ (17), for $1 \le j \le (n-1) \div 2$. Taking the complex conjugate of this equation, we get $\overline{\mathbf{H}_j\mathbf{v}} = \overline{\mathbf{v}\lambda}$, and hence $\overline{\mathbf{H}_j}\,\overline{\mathbf{v}} = \overline{\mathbf{v}}\lambda$, since λ is real. Therefore,

$$\mathbf{H}_j\mathbf{v} = \mathbf{v}\lambda \quad \Leftrightarrow \quad \mathbf{H}_{n-j}\overline{\mathbf{v}} = \overline{\mathbf{v}}\lambda \qquad (j = 1, 2, \cdots, (n-1) \div 2). \tag{27}$$

Thus, $\mathbf{H}_{n-j}$ also has eigenvalue λ, with eigenvector $\overline{\mathbf{v}}$. The entire set of eigenvalues of $\mathbf{H}_1,\ \mathbf{H}_2, \cdots, \mathbf{H}_{(n-1)\div 2}$ is duplicated by the relation (27).

But that duplication does not apply to $\mathbf{H}_0$, nor (for even $n = 2h$) to $\mathbf{H}_h$.

Let μ be any of the κ eigenvalues of $\mathbf{H}_0$, or of the 2κ eigenvalues of $\mathbf{H}_0$ and of $\mathbf{H}_h$ when $n = 2h$. If the sum of the multiplicities of μ in those 2 submatrices is odd (e.g. if μ is a simple eigenvalue of $\mathbf{H}_0$ but not an eigenvalue of $\mathbf{H}_h$), then $\mathbf{B}$ has μ as an eigenvalue of odd multiplicity, whether or not $\mu = \lambda_j$ for some $j = 1, 2, \cdots, (n-1)\div 2$. But if the sum of the multiplicities of μ in those submatrices is even, then $\mathbf{B}$ has μ as an eigenvalue of even multiplicity, whether or not $\mu = \lambda_j$ for some $j = 1, 2, \cdots, (n-1) \div 2$.

Thus, we have proved the following generalization of Theorem 1:

Theorem 2: Every real symmetric block circulant matrix of type $\mathcal{BC}_{n,\kappa}$ has no more than 2κ eigenvalues with odd multiplicity if n is even, but it has no more than κ eigenvalues with odd multiplicity if n is odd. All other eigenvalues occur with even multiplicity.

Non-extension from real-symmetric to Hermitian case. Does the relation (27) generalize from block circulant matrices which are real-symmetric to Hermitian block circulant matrices?

No, for a counter-example is given by Hermitian $\mathbf{B} \in \mathcal{BC}_{4,2}$ with

$$\mathbf{b}_0 = \begin{bmatrix} 0 & 2+\mathrm{i} \\ 2-\mathrm{i} & 0 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 0 & \mathrm{i} \\ 0 & 0 \end{bmatrix}, \quad \mathbf{b}_2 = \mathbf{0}, \quad \mathbf{b}_3 = \mathbf{b}_1^*. \tag{28}$$

Since $\rho_1 = \mathrm{i}$,

$$\mathbf{H}_1 = \mathbf{b}_0 + \mathrm{i}\,\mathbf{b}_1 - \mathbf{b}_2 - \mathrm{i}\,\mathbf{b}_3 = \begin{bmatrix} 0 & 2+2\mathrm{i} \\ 2-2\mathrm{i} & 0 \end{bmatrix}, \tag{29}$$

with the eigenvalues $\lambda = \mp\sqrt{8}$. But $\rho_3 = -\mathrm{i}$, and hence

$$\mathbf{H}_3 = \mathbf{b}_0 - \mathrm{i}\,\mathbf{b}_1 - \mathbf{b}_2 + \mathrm{i}\,\mathbf{b}_3 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \tag{30}$$

with the eigenvalues $\lambda = \mp 2$.

Thus, the relation (27) does not apply for Hermitian $\mathbf{B}$.

Real orthogonal eigenvectors of $\boldsymbol{B}$. From the eigenvector $\mathbf{v}$ of $\mathbf{H}_j$ and the eigenvector $\overline{\mathbf{v}}$ of $\mathbf{H}_{n-j}$, both with real eigenvalue λ, we can construct complex orthogonal eigenvectors $\mathbf{w}$ and $\overline{\mathbf{w}}$ of $\mathbf{B}$ with eigenvalue λ as in (16). In $\mathbf{w}$ we use $\rho = \rho_j$ and vector $\mathbf{v}$, and in $\overline{\mathbf{w}}$ we use $\rho = \rho_{n-j} = \overline{\rho_j}$ and vector $\overline{\mathbf{v}}$.

But since $\mathbf{B}$ is real symmetric it has a full set of real orthogonal eigenvectors, whereas $\mathbf{w}$ and $\overline{\mathbf{w}}$ are complex. Those may be replaced by their linear combinations $\Re(\mathbf{w}) = (\overline{\mathbf{w}} + \mathbf{w})/2$ and $\Im(\mathbf{w}) = \mathrm{i}(\overline{\mathbf{w}} - \mathbf{w})/2$, which are an orthogonal pair of real eigenvectors of $\mathbf{B}$ for the double real eigenvalue λ.

2.3.2. *Real symmetric matrix and submatrices.* If all submatrices $\mathbf{b}_j$ are also real-symmetric, then the expression (25) for $\mathbf{H}$ simplifies, as in [Tee, 1963].

In (25) now $\mathbf{b}_f = \mathbf{b}_f^T$, and so

$$\mathbf{b}_f {\rho_j}^f + \mathbf{b}_f^T \overline{\rho_j^f} = \mathbf{b}_f \left(\rho_j^f + \overline{\rho_j^f} \right) = 2\mathbf{b}_f \cos fj\vartheta, \tag{31}$$

which is a real symmetric submatrix. Accordingly, the expression (25) simplifies to

$$\mathbf{H}_j \;=\; \mathbf{b}_0 + 2\sum_{f=1}^{h-1} \mathbf{b}_f \cos fj\vartheta + \begin{cases} \mathbf{0} & \text{if } n = 2h-1 \\ \mathbf{b}_h(-1)^j & \text{if } n = 2h\ , \end{cases} \tag{32}$$

which is a real-symmetric submatrix. Therefore, each $\mathbf{H}_j$ has κ real orthogonal eigenvectors $\mathbf{v}$ with real eigenvalues.

However, each corresponding (16) eigenvector $\mathbf{w}$ of $\mathbf{B}$ is complex, except for $\rho_0 = 1$ and also (with even $n = 2h$) for $\rho_h = -1$. But, for even $n = 2h$, $(-1)^{n-j} = (-1)^{2h-j} = (-1)^{2h}(-1)^{-j} = (-1)^j$; and so it follows from (6) and (32) that for all n,

$$\mathbf{H}_{n-j} \;=\; \mathbf{H}_j, \quad (1 \le j \le (n-1) \div 2). \tag{33}$$

Hence, if $\mathbf{H}_j$ has a real eigenvector $\mathbf{v}$ with real eigenvalue λ, then so does $\mathbf{H}_{n-j}$. For $1 < j < n$ those subscripts are distinct, except for $j = h$ with even $n = 2h$. Accordingly, Theorem 2 applies in this case.

It follows from (33) that $\mathbf{B}$ has double real eigenvalue λ with a complex conjugate pair (16) of eigenvectors

$$\mathbf{w} \;=\; \begin{bmatrix} \mathbf{v} \\ \rho_j \mathbf{v} \\ \rho_j^2 \mathbf{v} \\ \vdots \\ \rho_j^{n-1} \mathbf{v} \end{bmatrix}, \qquad \overline{\mathbf{w}} \;=\; \begin{bmatrix} \mathbf{v} \\ \overline{\rho_j} \mathbf{v} \\ \overline{\rho_j^2} \mathbf{v} \\ \vdots \\ \overline{\rho_j^{n-1}} \mathbf{v} \end{bmatrix}, \tag{34}$$

which are orthogonal by (22). Those may be replaced by their linear combinations $(\overline{\mathbf{w}} + \mathbf{w})/2$ and $\mathrm{i}(\overline{\mathbf{w}} - \mathbf{w})/2$:

$$\Re(\mathbf{w}) \;=\; \begin{bmatrix} \mathbf{v} \\ \cos j\vartheta\, \mathbf{v} \\ \cos 2j\vartheta\, \mathbf{v} \\ \vdots \\ \cos(n-1)j\vartheta\, \mathbf{v} \end{bmatrix}, \qquad \Im(\mathbf{w}) \;=\; \begin{bmatrix} \mathbf{0} \\ \sin j\vartheta\, \mathbf{v} \\ \sin 2j\vartheta\, \mathbf{v} \\ \vdots \\ \sin(n-1)j\vartheta\, \mathbf{v} \end{bmatrix}, \tag{35}$$

which are an orthogonal pair of real eigenvectors of $\mathbf{B}$ for the double real eigenvalue λ.

2.4. Computation of all eigenvectors. If the n eigenvector equations (19) give $n\kappa$ distinct eigenvalues, as is almost always the case, then those are the complete set of eigenvalues of $\mathbf{B}$. The corresponding vectors (16) give all eigenvectors of $\mathbf{B}$, which are unique (apart from arbitrary nonzero scaling factors for each).

But if $\mathbf{B}$ has multiple eigenvalues, this is not necessarily the case. For example, if $\mathbf{B} = \mathbf{I}$, then *every* non-null vector is an eigenvector of $\mathbf{B}$, with eigenvalue 1.

If, for any ρ, $\mathbf{H}$ has multiple eigenvalue λ with linearly independent eigenvectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \cdots, \mathbf{v}^{(m)}$, then the corresponding $\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \cdots, \mathbf{w}^{(m)}$ are linearly independent eigenvectors of $\mathbf{B}$ with eigenvalue λ, and conversely. Any non-null linear combination $\mathbf{v}^{(1)}\alpha_1 + \cdots + \mathbf{v}^{(m)}\alpha_m$ is an eigenvector of $\mathbf{H}$ with eigenvalue λ, and that corresponds to $\mathbf{w}^{(1)}\alpha_1 + \cdots + \mathbf{w}^{(m)}\alpha_m$, which is an eigenvector of $\mathbf{B}$ with eigenvalue λ.

However, if the eigenvectors $\mathbf{x}$ and $\mathbf{y}$ of $\mathbf{B}$ in (21), with $\rho_j \neq \rho_k$, have the same eigenvalue λ, then every non-null linear combination $\mathbf{z} = \mathbf{x}\alpha + \mathbf{b}\beta$ is also an eigenvector of $\mathbf{B}$ with eigenvalue λ. But, except in the trivial cases $\alpha = 0$ or $\beta = 0$, $\mathbf{z}$ is *not* of the form (16), as in (35).

2.4.1. *Jordan Canonical Form.* For the general case of $\mathbf{H}$ with multiple eigenvalues, with eigenvectors which might be defective, consider the Jordan Canonical Form

$$\mathbf{L} \;=\; \mathbf{U}^{-1}\mathbf{H}\mathbf{U}, \tag{36}$$

so that

$$\mathbf{H}\mathbf{U} \;=\; \mathbf{U}\mathbf{L}. \tag{37}$$

Here, $\mathbf{U}$ is non-singular and the block-diagonal matrix $\mathbf{L}$ is a direct sum of Jordan blocks

$$\mathbf{L} = \lceil \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3, \cdots, \mathbf{J}_m \rfloor \;\stackrel{\text{def}}{=}\; \begin{bmatrix} \mathbf{J}_1 & & & & \\ & \mathbf{J}_2 & & & \\ & & \mathbf{J}_3 & & \\ & & & \ddots & \\ & & & & \mathbf{J}_m \end{bmatrix} \qquad (1 \le m \le \kappa). \tag{38}$$

Each Jordan block is either of the form $[\lambda_q]$, where the corresponding column of $\mathbf{U}$ is an eigenvector with eigenvalue λ_q; or else it is of the form $\mathbf{J}_q = \lambda_q\mathbf{I} + \mathbf{R}$, where $\mathbf{R}$ and the unit matrix $\mathbf{I}$ are of the same order as $\mathbf{J}_q$, and

$$\mathbf{R} \;=\; \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}. \tag{39}$$

In this case, the first of the corresponding sequence of columns of $\mathbf{U}$ is an eigenvector with eigenvalue λ_q, and the other columns of $\mathbf{U}$ in that sequence are generalized eigenvectors with eigenvalue λ_q. $\mathbf{L}$ is unique, apart from permutation of the Jordan blocks. The eigenvectors (and generalized eigenvectors) in $\mathbf{U}$ are not uniquely specified — any eigenvector (and its associated generalized eigenvectors) can be scaled by any nonzero factor; and if $\mathbf{H}$ has multiple eigenvalue λ with linearly independent eigenvectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \cdots, \mathbf{v}^{(g)}$, and if $\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \cdots, \mathbf{z}^{(g)}$ are a linearly independent set of linear combinations of $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \cdots, \mathbf{v}^{(g)}$, then those g **v**s could be replaced by the g **z**s.

H has defective eigenvectors, if and only if **L** has at least one Jordan block of order greater than 1. For any eigenvalue λ_q, its eigenvectors span space whose dimension equals the number of Jordan blocks with λ_q. The eigenvalue λ_q has multiplicity equal to the number of times that it occurs on the diagonal of **L**, which is the sum of the orders of the Jordan blocks with λ_q.

Consider the compound vector **T**, whose n elements are square matrices of order κ (cf. (16)):

$$\mathbf{T} \; = \; \begin{bmatrix} \mathbf{U} \\ \rho\mathbf{U} \\ \rho^2\mathbf{U} \\ \vdots \\ \rho^{n-1}\mathbf{U} \end{bmatrix} . \tag{40}$$

The κ columns of **U** are linearly independent (since **U** is non-singular), and hence the columns of **T** are linearly independent.

The first few compound rows of **BT** are:

$$\begin{aligned} (\mathbf{b}_0 + \mathbf{b}_1\rho + \mathbf{b}_2\rho^2 + \mathbf{b}_3\rho^3 + \cdots + \mathbf{b}_{n-1}\rho^{n-1})\mathbf{U} &= \mathbf{HU} \; = \mathbf{UL}, \\ (\mathbf{b}_{n-1} + \mathbf{b}_0\rho + \mathbf{b}_1\rho^2 + \mathbf{b}_2\rho^3 + \cdots + \mathbf{b}_{n-2}\rho^{n-1})\mathbf{U} &= \rho\mathbf{HU} \; = \; (\rho\mathbf{U})\mathbf{L}, \\ (\mathbf{b}_{n-2} + \mathbf{b}_{n-1}\rho + \mathbf{b}_0\rho^2 + \mathbf{b}_1\rho^3 + \cdots + \mathbf{b}_{n-3}\rho^{n-1})\mathbf{U} &= \rho^2\mathbf{HU} \; = \; (\rho^2\mathbf{U})\mathbf{L}, \end{aligned} \tag{41}$$

and so forth (cf. (18)). Therefore,

$$\mathbf{BT} \; = \; \mathbf{TL} \, . \tag{42}$$

For **H** with $\rho = \rho_j$, rewrite (42) as

$$\mathbf{BT}_j \; = \; \mathbf{T}_j\mathbf{L}_j. \tag{43}$$

Define the square matrices of order $n\kappa$:

$$\mathbf{W} \; \stackrel{\text{def}}{=} \; [\mathbf{T}_0 \; \mathbf{T}_1 \; \ldots \; \mathbf{T}_{n-1}] \tag{44}$$

and

$$\mathbf{\Lambda} \; \stackrel{\text{def}}{=} \; \lceil \mathbf{L}_0, \mathbf{L}_1, \mathbf{L}_2, \cdots, \mathbf{L}_{n-1} \rfloor \; = \; \begin{bmatrix} \mathbf{L}_0 & & & & \\ & \mathbf{L}_1 & & & \\ & & \mathbf{L}_2 & & \\ & & & \ddots & \\ & & & & \mathbf{L}_{n-1} \end{bmatrix} . \tag{45}$$

Within each $\mathbf{T}_k$ the columns are linearly independent, and (22) each column of $\mathbf{T}_k$ is orthogonal to every column of every $\mathbf{T}_j$, for $j \neq k$. Therefore the columns of **W** are linearly independent, and so **W** is non-singular.

For $k = 0, 1, \cdots, n-1$, the eigenvector equations (43) can be represented as

$$\mathbf{BW} = \mathbf{W\Lambda}. \tag{46}$$

The matrix $\mathbf{\Lambda}$ is a direct sum of Jordan blocks (of order $\leq \kappa$), and $\mathbf{W}$ is non-singular. Hence (46) gives the Jordan Canonical form of $\mathbf{B}$.

$$\mathbf{\Lambda} \;=\; \mathbf{W}^{-1}\mathbf{B}\mathbf{W}. \tag{47}$$

Thus, we have proved the following theorem:

Theorem 3: The Jordan Canonical Form of any block circulant matrix bcirc$(\mathbf{b}_0, \mathbf{b}_1, \cdots, \mathbf{b}_{n-1})$ of type $\mathcal{BC}_{n,\kappa}$ is the direct sum of the Jordan Canonical Forms of the n matrices $\mathbf{H}_j = \sum_{m=0}^{n-1} \mathbf{b}_m \rho_j^m$ of order κ, for $j = 0, 1, \cdots, n-1$.

Radka Turcajová (1997, Theorem 3.1) has proved the related result that "Each block circulant matrix is unitarily similar to a block diagonal matrix".

2.5. Block circulant matrix with eigenvector defect. As an example, consider $\mathbf{B}$ with submatrices

$$\mathbf{b}_0 = \mathbf{0}, \qquad \mathbf{b}_j = j\mathbf{I} - \mathbf{R} \quad (j = 1, 2, \cdots, n-1), \tag{48}$$

where $\mathbf{R}$ is a Jordan block of order $\kappa > 1$, with a single eigenvector $\mathbf{v}$ and all κ eigenvalues are zero:

$$\mathbf{R} \;=\; \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}, \qquad \mathbf{v} \;=\; \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{49}$$

For each ρ,

$$\begin{aligned} \mathbf{H} \;&=\; \mathbf{b}_0 + \mathbf{b}_1\rho + \mathbf{b}_2\rho^2 + \mathbf{b}_3\rho^3 + \cdots + \mathbf{b}_{n-1}\rho^{n-1} \\ &=\; \mathbf{I}(\rho + 2\rho^2 + \cdots + (n-1)\rho^{n-1}) - \mathbf{R}(\rho + \rho^2 + \cdots + \rho^{n-1}). \end{aligned} \tag{50}$$

For $j = 1, 2, \cdots, n-1$ the root $\rho_j \neq 1$, and the coefficient of $\mathbf{R}$ is 1, since $1 + \rho_j + \rho_j^2 + \cdots + \rho_j^{n-1} = (1 - \rho_j^n)/(1 - \rho_j) = 0$. The coefficient s of $\mathbf{I}$ is found thus:

$$\begin{aligned} s(1-\rho) = \rho &+ 2\rho^2 + 3\rho^3 + \cdots + (n-1)\rho^{n-1} \\ &- \rho^2 - 2\rho^3 - \cdots - (n-2)\rho^{n-1} - (n-1)\rho^n \\ = \rho &+ \rho^2 + \rho^3 + \cdots + \rho^{n-1} - (n-1)\rho^n. \end{aligned} \tag{51}$$

Therefore $s(1-\rho_j) = -1 - (n-1)$, and so $s = n/(\rho_j - 1)$. Hence,

$$\mathbf{H}_j \;=\; \frac{n}{\rho_j - 1}\mathbf{I} + \mathbf{R}, \quad (j = 1, 2, \cdots, n-1). \tag{52}$$

Thus, for $j > 0$, $\mathbf{H}_j$ is a Jordan block with the single eigenvector $\mathbf{v}$ as in (49), with all κ eigenvalues equal to $n/(\rho_j - 1)$. Indeed, since $\mathbf{H}_j$ is its own Jordan Canonical Form $\mathbf{L}_j$ with $\mathbf{H}_j\mathbf{I} = \mathbf{I}\mathbf{L}_j$, the unit matrix $\mathbf{I}$ is an orthogonal matrix of the single eigenvector and the $\kappa - 1$ generalized eigenvectors of $\mathbf{H}_j$.

With $\rho_0 = 1$, we get $\mathbf{H}_0 = \mathbf{I}(n(n-1)/2) - \mathbf{R}(n-1)$, so that $\mathbf{H}_0$ has the single eigenvector $\mathbf{v}$ as in (49), with all κ eigenvalues equal to $n(n-1)/2$.

Thus, the matrix $\mathbf{B}$ (as in (48)) has an orthogonal set of n eigenvectors, each with an eigenvalue of multiplicity κ.

To get the Jordan Canonical Form of $\mathbf{H}_0$, consider the diagonal matrix $\mathbf{D} = \lceil 1, (1-n)^{-1}, (1-n)^{-2}, \cdots, (1-n)^{1-\kappa} \rfloor$. Then $\mathbf{D}^{-1}\mathbf{R}\mathbf{D} = \frac{1}{1-n}\mathbf{R}$, and hence

$$\mathbf{D}^{-1}\mathbf{H}_0\mathbf{D} \;=\; \mathbf{D}^{-1}\left(\tfrac{1}{2}n(n-1)\mathbf{I} + (1-n)\mathbf{R}\right)\mathbf{D} \;=\; \tfrac{1}{2}n(n-1)\mathbf{I} + \mathbf{R}, \tag{53}$$

which is the Jordan Canonical Form of $\mathbf{H}_0$. Thus $\mathbf{H}_0\mathbf{D} \;=\; \mathbf{D}\left(\frac{1}{2}n(n-1)\mathbf{I} + \mathbf{R}\right)$, with $\mathbf{D}$ as an orthogonal matrix of the single eigenvector and the $\kappa - 1$ generalized eigenvectors of $\mathbf{H}_0$.

Therefore, we have found the complete Jordan Canonical Form (45) $\boldsymbol{\Lambda} = \lceil \mathbf{L}_0, \mathbf{L}_1, \cdots, \mathbf{L}_{n-1} \rfloor$ of $\mathbf{B}$, with

$$\mathbf{L}_0 \;=\; \tfrac{1}{2}n(n-1)\mathbf{I} + \mathbf{R}, \qquad \mathbf{L}_j \;=\; \frac{n}{\rho_j - 1}\mathbf{I} + \mathbf{R} \quad (j = 1, 2, \cdots, n-1), \tag{54}$$

and the complete orthogonal matrix $\mathbf{W}$ (44) of n eigenvectors, each with $\kappa - 1$ associated generalized eigenvectors, where (cf. (40))

$$\mathbf{W} \;=\; \begin{bmatrix} \mathbf{D} & \mathbf{I} & \mathbf{I} & \dots & \mathbf{I} \\ \mathbf{D} & \rho_1\mathbf{I} & \rho_2\mathbf{I} & \dots & \rho_{n-1}\mathbf{I} \\ \mathbf{D} & \rho_1^2\mathbf{I} & \rho_2^2\mathbf{I} & \dots & \rho_{n-1}^2\mathbf{I} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{D} & \rho_1^{n-1}\mathbf{I} & \rho_2^{n-1}\mathbf{I} & \dots & \rho_{n-1}^{n-1}\mathbf{I} \end{bmatrix}. \tag{55}$$

This class (48) of matrices of type $\mathcal{BC}_{n,\kappa}$ could be used as a test for procedures for computing eigenvectors and eigenvalues of matrices with defective eigenvectors. As with any test matrix, it could also be used with any symmetric permutation of the rows and columns.

3. Alternating Circulant Matrices

Stephen J. Watson considered a matrix which he calls an alternating circulant matrix. "Specifically, consider a $2n$ by $2n$ matrix where each row is obtained from the preceding row by the simple cyclic permutation $(1\ 2\ 3\ \dots\ 2n)$ followed by multiplication by -1" [Watson]. He reported that he has discovered a way to characterize the spectrum of such matrices, and he wondered whether that was a known fact.

Watson's alternating circulant matrix has rows 1 and 2 of the form:

$$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 & \cdots & c_{2n-2} & c_{2n-1} \\ -c_{2n-1} & -c_0 & -c_1 & -c_2 & \cdots & -c_{2n-3} & -c_{2n-2} \end{bmatrix}. \tag{56}$$

That represents a matrix $\mathbf{B}$ of type $\mathcal{BC}_{n,2}$ where the square submatrices are

$$\mathbf{b}_0 = \begin{bmatrix} c_0 & c_1 \\ -c_{2n-1} & -c_0 \end{bmatrix}, \; \mathbf{b}_j = \begin{bmatrix} c_{2j} & c_{2j+1} \\ -c_{2j-1} & -c_{2j} \end{bmatrix}, \; (j = 1, 2, \cdots, n-1). \tag{57}$$

In this case, for each of the n values of ρ, the matrix $\mathbf{H}$ is

$$\mathbf{H} \;=\; \begin{bmatrix} d & e \\ -e\rho & -d \end{bmatrix}, \tag{58}$$

where

$$\begin{aligned} d &= c_0 + c_2\rho + c_4\rho^2 + c_6\rho^3 + \cdots + c_{2n-2}\rho^{n-1}, \\ e &= c_1 + c_3\rho + c_5\rho^2 + c_7\rho^3 + \cdots + c_{2n-1}\rho^{n-1}. \end{aligned} \tag{59}$$

If any value of ρ gives $d = e = 0$ then $\mathbf{H}$ is a null matrix with both eigenvalues equal to 0, and every non-null 2-vector $\mathbf{v}$ is an eigenvector of $\mathbf{H}$. The vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ could be used as an orthogonal basis for the eigenspace of that null matrix.

Otherwise, for each of the n values of ρ, the two eigenvalues of $\mathbf{H}$ are

$$\lambda_1 \;=\; \sqrt{d^2 - \rho e^2}, \quad \lambda_2 = -\lambda_1, \tag{60}$$

with eigenvectors

$$\mathbf{v}^{(1)} \;=\; \begin{bmatrix} e \\ -d + \lambda_1 \end{bmatrix}, \quad \mathbf{v}^{(2)} \;=\; \begin{bmatrix} e \\ -d - \lambda_1 \end{bmatrix}. \tag{61}$$

If the eigenvectors of $\mathbf{H}$ are defective then the 2 eigenvalues of $\mathbf{H}$ are equal, which happens if and only if they are both 0, with $d^2 = \rho e^2$. In that case, both eigenvectors of $\mathbf{H}$ are of the form

$$\mathbf{v} \;=\; \begin{bmatrix} e \\ -d \end{bmatrix}, \tag{62}$$

so that the eigenspace of $\mathbf{H}$ has dimension 1 rather than 2.

For each ρ and $\mathbf{v}$ we construct (16) an eigenvector $\mathbf{w}$ of $\mathbf{B}$, with corresponding eigenvalue λ. Thus, the eigenvectors and eigenvalues of Watson's alternating circulant matrix have been found. The eigenvectors span $2n$ dimensions, except that for each value of ρ such that $d^2 = \rho e^2 \neq 0$ the eigenspace loses 1 dimension.

3.1. Bound for eigenvalues. For any square matrix $\mathbf{A}$ and any norm $\|\mathbf{A}\|$, each eigenvalue μ has modulus $|\mu| \leq \|\mathbf{A}\|$. The spectral radius $\varsigma(\mathbf{A})$ is defined as the maximum modulus of any eigenvalue, and so $0 \leq \varsigma(\mathbf{A}) \leq \|\mathbf{A}\|$.

For the alternating circulant matrix $\mathbf{B}$ in (56), using the rowsum norm we get that

$$\varsigma(\mathbf{B}) \leq \|\mathbf{B}\|_{row} = \sum_{k=0}^{2n-1} |c_k| \,. \tag{63}$$

For each ρ and its corresponding eigenvalue λ of $\mathbf{B}$ (cf. (60)),

$$|\lambda|^2 \;=\; \left|d^2 - \rho e^2\right| \leq |d|^2 + |-\rho|\,|e|^2 = |d|^2 + |e|^2 \,. \tag{64}$$

From (59),

$$|d| = \left|\sum_{j=0}^{n-1} c_{2j}\rho^j\right| \leq \sum_{j=0}^{n-1} |c_{2j}|\,|\rho|^j = \sum_{j=0}^{n-1} |c_{2j}| \,, \tag{65}$$

and similarly

$$|e| \leq \sum_{j=0}^{n-1} |c_{2j+1}| \,. \tag{66}$$

Therefore, for each eigenvalue λ,

$$|\lambda|^2 \;\leq\; |d|^2 + |e|^2 \;\leq\; \left(\sum_{j=0}^{n-1} |c_{2j}|\right)^2 + \left(\sum_{j=0}^{n-1} |c_{2j+1}|\right)^2 . \tag{67}$$

This inequality holds for all λ, and hence

$$\varsigma(\mathbf{B}) \ \leq \ \sqrt{\left(\sum_{j=0}^{n-1}|c_{2j}|\right)^2 + \left(\sum_{j=0}^{n-1}|c_{2j+1}|\right)^2}\,. \tag{68}$$

3.1.1. *Sharp bound for spectral radius.* The inequality (68) may be rewritten as:

$$\begin{aligned}\varsigma(\mathbf{B})^2 \ &\leq \ \left(\sum_{j=0}^{n-1}|c_{2j}|\right)^2 + \left(\sum_{j=0}^{n-1}|c_{2j+1}|\right)^2 \\ &= \left(\sum_{j=0}^{n-1}|c_{2j}| + \sum_{j=0}^{n-1}|c_{2j+1}|\right)^2 - 2\left(\sum_{j=0}^{n-1}|c_{2j}|\right)\left(\sum_{j=0}^{n-1}|c_{2j+1}|\right) \\ &= \ \|\mathbf{B}\|_{row}^2 - 2\sum_{j=0}^{n-1}|c_{2j}|\sum_{h=0}^{n-1}|c_{2h-1}|\,. \end{aligned} \tag{69}$$

This gives an upper bound for $\varsigma(\mathbf{B})$ less than or equal to (63), with those bounds being equal if and only if $c_k = 0$ either for all even k, or for all odd k.

Indeed, we shall shew that (68) gives the smallest possible bound for $\varsigma(\mathbf{B})$, in terms of the moduli $|c_k|$.

For alternating circulant matrix $\mathbf{B}$ with first row $[c_0, c_1, \ldots, c_{2n-1}]$, consider the complex alternating circulant matrix $\mathbf{B}'$ with first row $[c'_0, c'_1, \ldots, c'_{2n-1}]$, where $c'_k = |c_k| \geq 0$ for even k and $c'_k = \mathrm{i}\,|c_k|$ for odd k. Hence, each element of $\mathbf{B}'$ has the same modulus as the corresponding element of $\mathbf{B}$.

For $\mathbf{B}'$ with $\rho = 1$, we get that $d = c'_0 + c'_2 + c'_4 + \cdots + c'_{2n-2} = \sum |c'_k| = \sum |c_k|$ for even k, and $e = c'_1 + c'_3 + \cdots + c'_{2n-1} = \mathrm{i}\sum|c'_k| = \mathrm{i}\sum|c_k|$ for odd k. Hence, for the real eigenvalues $\mp\lambda$ with $\rho = 1$,

$$\lambda^2 = d^2 - \rho e^2 = d^2 - e^2 = \left(\sum_{j=0}^{n-1}\left|c'_{2j}\right|\right)^2 + \left(\sum_{j=0}^{n-1}\left|c'_{2j+1}\right|\right)^2, \tag{70}$$

and so the inequality (67) for eigenvalues reduces to an equality when $\rho = 1$ for such $\mathbf{B}'$. Hence, the inequality (68) reduces to an equality for such $\mathbf{B}'$.

Therefore the inequality (68) is the sharpest bound for $\varsigma(\mathbf{B})$, in terms of the moduli of the elements of $\mathbf{B}$.

3.1.2. *Real alternating circulant matrices.* If $\mathbf{B}$ is real then with $\rho = 1$ it follows from (59) and (58) that $\mathbf{H} = \begin{bmatrix} d & e \\ -e & -d \end{bmatrix}$ is real, and hence its eigenvalues $\mp\sqrt{d^2 - e^2}$ are either both real or both imaginary. Therefore, every real alternating circulant matrix $\mathbf{B}$ has at least one pair of eigenvalues $\mp\lambda$ which are either both real or both imaginary (or both zero), with eigenvectors (or eigenvector) of the form (16) with $\rho = 1$. If $d = \pm e \neq 0$ then the double zero eigenvalue has a single real eigenvector (16), with (cf. (62)) $\mathbf{v}= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ if $d = e$, but $\mathbf{v}= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ if $d = -e$.

If $\mathbf{B}$ is real with even n, then with $\rho = -1$ it follows from (59) and (58) that $\mathbf{H} = \begin{bmatrix} d & e \\ e & -d \end{bmatrix}$ is real-symmetric. Hence $\mathbf{H}$ has a pair of real and orthogonal eigenvectors, with real eigenvalues $\mp\sqrt{d^2 + e^2}$. Therefore, every real alternating circulant matrix $\mathbf{B}$ with even n has at least one pair of real orthogonal eigenvectors (16) with $\rho = -1$, and with real eigenvalues $\mp\lambda$.

3.1.3. *Sharp bound for spectral radius with even n.* For any complex alternating circulant matrix $\mathbf{B}$ with first row $[c_0, c_1, \ldots, c_{2n-1}]$ with even n, consider the real alternating circulant matrix $\mathbf{B}'$ with first row $[c'_0, c'_1, \ldots, c'_{2n-1}]$. Here

$$c'_k \;=\; \begin{cases} \;\;\,|c_k| & \text{for} \quad (k = 0, 4, 8, \cdots, 2n-4), \\ -|c_k| & \text{for} \quad (k = 1, 5, 9, \cdots, 2n-3), \\ -|c_k| & \text{for} \quad (k = 2, 6, 10, \cdots, 2n-2), \\ \;\;\,|c_k| & \text{for} \quad (k = 3, 7, 11, \cdots, 2n-1). \end{cases} \tag{71}$$

Hence, each element of $\mathbf{B}'$ has the same modulus as the corresponding element of $\mathbf{B}$.

With $\rho = -1$ the inequality in (65) then reduces to equality, since for each j, $c_{2j}(-1)^j \geq 0$; and hence

$$d \;=\; \sum_{j=0}^{n-1} \left| c'_{2j} \right| . \tag{72}$$

Similarly, with $\rho = -1$ the inequality in (66) then reduces to equality, since for each j, $c_{2j-1}(-1)^j \leq 0$; and hence

$$e \;=\; -\sum_{j=0}^{n-1} \left| c'_{2j+1} \right| . \tag{73}$$

Thus, if the coefficients of $\mathbf{B}'$ satisfy (71), then the inequalities in (67) and (68) reduce to equalities. In that case, the pair of real eigenvalues $\mp\lambda$ for $\rho = -1$ each attain the bound (67):

$$\begin{aligned} \lambda^2 \;&=\; d^2 - \rho e^2 \;=\; d^2 + e^2 \\ &=\; (c'_0 - c'_2 + c'_4 - c'_7 + \cdots + c'_{2n-4} - c'_{2n-2})^2 \\ &\quad\;+ (c'_1 - c'_3 + c'_5 - c'_7 + \cdots + c'_{2n-3} - c'_{2n-1})^2 \\ &=\; \left(\sum_{j=0}^{n-1} \left| c'_{2j} \right| \right)^2 + \left(\sum_{j=0}^{n-1} \left| c'_{2j+1} \right| \right)^2 . \end{aligned} \tag{74}$$

It was shewn in (70) that, for any n, the eigenvalue bound (67) can also be attained with $\rho = 1$ by complex $\mathbf{B}'$.

3.2. Alternating circulant matrix with defective eigenvectors. As a real example with defective eigenvectors, consider

$$\mathbf{B} \;=\; \begin{bmatrix} 1 & 0 & -3 & 2 \\ -2 & -1 & 0 & 3 \\ -3 & 2 & 1 & 0 \\ 0 & 3 & -2 & -1 \end{bmatrix} \tag{75}$$

with $n = 2$, so that the values of ρ are 1 and -1. For $\rho = 1$ the pair of eigenvalues must be both real or both imaginary or both zero: in this case (78) they prove to be both zero.

For this real alternating circulant matrix $\mathbf{B}$, $n = 2$ is even and all inequalities (71) do hold, since $1 = c_0 \geq 0 \geq c_2 = -3$, and $c_1 = 0 \leq 0 \leq 2 = c_3$. Accordingly,

$\rho = -1$ gives a pair of real orthogonal eigenvectors with real eigenvalues equal to plus and minus the bound (68), with $\lambda^2 = (c_0 - c_2)^2 + (c_1 - c_3)^2$ as in (74).

Indeed, with $\rho = -1$ we get from (59) & (60) that $d = 4,\ e = -2,\ \lambda = \pm\sqrt{20}$, giving real symmetric $\mathbf{H} = \left[\begin{smallmatrix} 4 & -2 \\ -2 & -4 \end{smallmatrix}\right]$, with orthogonal eigenvectors

$$\mathbf{v}^{(1)} = \begin{bmatrix} -2 \\ -4+\sqrt{20} \end{bmatrix}, \quad \mathbf{v}^{(2)} = \begin{bmatrix} -2 \\ -4-\sqrt{20} \end{bmatrix}. \tag{76}$$

Hence, $\mathbf{B}$ has the real orthogonal eigenvectors

$$\mathbf{w}^{(1)} = \begin{bmatrix} -2 \\ -4+\sqrt{20} \\ 2 \\ 4-\sqrt{20} \end{bmatrix}, \quad \mathbf{w}^{(2)} = \begin{bmatrix} -2 \\ -4-\sqrt{20} \\ 2 \\ 4+\sqrt{20} \end{bmatrix} \tag{77}$$

with eigenvalues $\lambda_1 = \sqrt{20},\ \lambda_2 = -\sqrt{20}$.

With $\rho = 1$ we get from (59) & (60) that $d = -2,\ e = 2,\ \lambda = 0$, giving $\mathbf{H} = \left[\begin{smallmatrix} -2 & 2 \\ -2 & 2 \end{smallmatrix}\right]$, with the single eigenvector

$$\mathbf{v}^{(3)} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \tag{78}$$

for double eigenvalue $\lambda_3 = \lambda_4 = 0$.

Consider the Jordan canonical form $\mathbf{HU} = \mathbf{UL}$ (cf. (37)):

$$\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & f \\ 2 & g \end{bmatrix} = \begin{bmatrix} 2 & f \\ 2 & g \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{79}$$

where f and g are the elements of the generalized eigenvector with double zero eigenvalue. This matrix equation is equivalent to $-2f + 2g = 2$. Hence, f is an arbitrary parameter and $g = 1 + f$.

Therefore, the Jordan canonical form $\mathbf{BW} = \mathbf{W\Lambda}$ (cf. (46)) has the matrix of 3 eigenvectors and 1 generalized eigenvector (with arbitrary parameter f)

$$\mathbf{W} = \begin{bmatrix} -2 & -2 & 2 & f \\ \sqrt{20}-4 & -\sqrt{20}-4 & 2 & 1+f \\ 2 & 2 & 2 & f \\ -\sqrt{20}+4 & \sqrt{20}+4 & 2 & 1+f \end{bmatrix}, \tag{80}$$

and the eigenvalues as the diagonal elements of

$$\mathbf{\Lambda} = \begin{bmatrix} \sqrt{20} & & & \\ & -\sqrt{20} & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}. \tag{81}$$

In this $\mathbf{W}$, each of columns 3 and 4 (with $\rho = 1$) is orthogonal (cf. (22)) to columns 1 and 2 (with $\rho = -1$). Columns 3 and 4 are linearly independent, and columns 1 and 2 are orthogonal; and so all 4 columns are linearly independent. If we take $f = -1/2$, so that

$$\mathbf{W} = \begin{bmatrix} -2 & -2 & 2 & -1/2 \\ \sqrt{20}-4 & -\sqrt{20}-4 & 2 & 1/2 \\ 2 & 2 & 2 & -1/2 \\ -\sqrt{20}+4 & \sqrt{20}+4 & 2 & 1/2 \end{bmatrix}, \qquad (82)$$

then column 4 is also orthogonal to column 3, and hence the entire matrix (82) is then orthogonal.

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