

## ONE GENERALIZATION OF THE CLASSICAL MOMENT PROBLEM

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ABSTRACT. Let  $*_P$  be a product on  $l_{\text{fin}}$  (a space of all finite sequences) associated with a fixed family  $(P_n)_{n=0}^{\infty}$  of real polynomials on  $\mathbb{R}$ . In this article, using methods from the theory of generalized eigenvector expansion, we investigate moment-type properties of  $*_P$ -positive functionals on  $l_{\text{fin}}$ .

If  $(P_n)_{n=0}^{\infty}$  is a family of the Newton polynomials  $P_n(x) = \prod_{i=0}^{n-1}(x - i)$  then the corresponding product  $\star = *_P$  is an analog of the so-called Kondratiev–Kuna convolution on a “Fock space”. We get an explicit expression for the product  $\star$  and establish a connection between  $\star$ -positive functionals on  $l_{\text{fin}}$  and a one-dimensional analog of the Bogoliubov generating functionals (the classical Bogoliubov functionals are defined correlation functions for statistical mechanics systems).

### 1. INTRODUCTION

It is well known that the classical moment problem can be viewed as a theory of spectral representations of positive functionals on some classical commutative algebra with involution. Namely, let  $l_{\text{fin}}$  be a space of all finite sequences  $f = (f_0, \dots, f_n, 0, 0, \dots)$  of complex numbers  $f_n$  and  $*$  denote the Cauchy product on  $l_{\text{fin}}$ , i.e.,

$$(1.1) \quad (f * g)_n := \sum_{i+j=n} f_i g_j = \sum_{k=0}^{\infty} f_k g_{n-k}$$

for all  $f = (f_n)_{n=0}^{\infty}, g = (g_n)_{n=0}^{\infty} \in l_{\text{fin}}$ . The space  $l_{\text{fin}}$  endowed with the product  $*$  is a commutative algebra with the involution  $f = (f_n)_{n=0}^{\infty} \mapsto \bar{f} := (\bar{f}_n)_{n=0}^{\infty}$ .

The classical moment problem is formulated as follows: *for a given sequence  $(\tau_n)_{n=0}^{\infty}$  of real numbers  $\tau_n$  when does there exist a non-negative finite Borel measure  $\mu$  on  $\mathbb{R}$  such that*

$$(1.2) \quad \tau_n = \int_{\mathbb{R}} x^n d\mu(x), \quad n \in \mathbb{N}_0 := \{0, 1, \dots\} ?$$

The answer is the following: *integral representation (1.2) holds if and only if  $\tau = (\tau_n)_{n=0}^{\infty}$  is a  $*$ -positive functional (more exactly, non-negative) on  $l_{\text{fin}}$ , i.e.,*

$$\tau(f * \bar{f}) = \sum_{j,k=0}^{\infty} \tau_{j+k} f_j \bar{f}_k \geq 0, \quad f = (f_n)_{n=0}^{\infty} \in l_{\text{fin}}.$$

In this article an essential role will be played by Yu. M. Berezansky’s method [3] of obtaining representation (1.2), which goes back to the works of M. G. Krein [19, 20]. This method is based on the theory of generalized eigenfunction expansion for selfadjoint operators and, in its modern version [4], can be formulated as follows. Let  $(l_{\text{fin}}, *)$  be

1991 *Mathematics Subject Classification*. Primary 44A60, 47A57.

*Key words and phrases*. Convolution, positive functional, moment problem, projection spectral theorem, Sheffer polynomials.

The work is partially supported by the DAAD, Project A/11/05321.

an algebra as above and  $\tau = (\tau_n)_{n=0}^\infty$  be a given  $*$ -positive functional. This pair can be associated, in a usual way, with a Hilbert space  $H_\tau$  generated by a quasiscalar product

$$(f, g)_{H_\tau} := \tau(f * \bar{g}), \quad f, g \in l_{\text{fin}}.$$

In this space  $H_\tau$  the translation operator

$$(1.3) \quad Jf := \delta_1 * f = (0, f_0, f_1, \dots), \quad f = (f_n)_{n=0}^\infty \in l_{\text{fin}}$$

(here  $\delta_1 := (0, 1, 0, 0, \dots) \in l_{\text{fin}}$ ) is Hermitian with equal defect numbers. Therefore it follows from the theory of generalized eigenfunction expansions that there exists a non-negative finite Borel measure  $\mu$  on  $\mathbb{R}$  (spectral measure) such that  $(x^n)_{n=0}^\infty$  is a generalized eigenvector of the operator  $J$  (more exactly, to its selfadjoint extension) with an eigenvalue  $x \in \mathbb{R}$  and the mapping (Fourier transform)

$$H_\tau \supset l_{\text{fin}} \ni f \mapsto (If)(x) := \sum_{n=0}^\infty f_n x^n \in L^2(\mathbb{R}, \mu)$$

is well-defined and isometric, i.e.,

$$(f, g)_{H_\tau} = \int_{\mathbb{R}} (If)(x) \overline{(Ig)(x)} d\mu(x), \quad f, g \in l_{\text{fin}}.$$

As a consequence of this Parseval equality, we immediately get representation (1.2):

$$\tau_n = (\delta_n, \delta_0)_{H_\tau} = \int_{\mathbb{R}} x^n d\mu(x),$$

where  $\delta_n = (\delta_{nj})_{j=0}^\infty$  denotes a  $\delta$ -sequence.

Among the advantages of Yu. M. Berezansky's method is that this method admits broad generalizations which give a possibility to investigate the following moment problems: strong Hamburger, trigonometric, complex, matrix and different many-dimensional analogs of them, including infinite-dimensional cases (in many-dimensional situation it is necessary to investigate the commuting families of Jacobi type operators), see [3, 9, 10, 4, 5, 6, 7, 11, 8] for more detailed presentation.

By analogy with the above described way of obtaining representation (1.2), we can get moment-type representations in the case when a family  $(x^n)_{n=0}^\infty$  of the monomials is replaced by a family  $(P_n)_{n=0}^\infty$  of polynomials  $P_n : \mathbb{R} \rightarrow \mathbb{R}$  (each  $P_n$  has a degree  $n$ ). In this situation, instead of the Cauchy product  $*$  (1.1), it is necessary to use the product

$$f *_P g := I_P^{-1}(I_P f \cdot I_P g), \quad (I_P f)(x) := \sum_{n=0}^\infty f_n P_n(x), \quad f, g \in l_{\text{fin}},$$

generated by the polynomials  $P_n(x)$  and, instead of (1.3), the operator

$$J_P f := \delta_1 *_P f, \quad f = (f_n)_{n=0}^\infty \in l_{\text{fin}}$$

(clearly, if  $P_n(x) = x^n$  then  $*$  =  $*_P$  and  $J_P = J$ ). Let  $H_\tau = H_{\tau, P}$  denotes a Hilbert space associated with the quasiscalar product  $(f, g)_{H_\tau} := \tau(f *_P \bar{g})$ . It can be shown that for a given  $*_P$ -positive functional  $\tau = (\tau_n)_{n=0}^\infty$  on  $l_{\text{fin}}$  there exists a non-negative finite Borel measure  $\mu$  on  $\mathbb{R}$  such that  $(P_n)_{n=0}^\infty$  is a generalized eigenvector of the operator  $J_P$  ( $J_P$  acts in  $H_\tau$ ) with an eigenvalue  $x \in \mathbb{R}$  and the mapping

$$H_\tau \supset l_{\text{fin}} \ni f \mapsto (I_P f)(x) := \sum_{n=0}^\infty f_n P_n(x) \in L^2(\mathbb{R}, \mu)$$

is a Fourier transform. The corresponding Parseval equality gives the moment-type representation

$$\tau_n = \int_{\mathbb{R}} P_n(x) d\mu(x), \quad n \in \mathbb{N}_0.$$

The described way of proving the latter representation is given below in Section 4. Let us mention that the idea of using the theory of generalized eigenvector expansion in a similar context is not new, see [4] for details.

If  $(P_n)_{n=0}^\infty$  is a family of the so-called Newton polynomials  $P_n(x) = (x)_n := \prod_{i=0}^{n-1} (x-i)$  then the corresponding product  $\star := \star_P$  on  $l_{\text{fin}}$  is an analog of the so-called Kondratiev–Kuna convolution on a “Fock space”. Formula (3.8) in Section 3 gives an explicit expression for the product  $\star$ . We refer to [15] for the definition and properties of the Kondratiev–Kuna convolution on a “Fock space”, see also Subsection 7.2.

In this article we also study the following problem: *for a given sequence  $(\tau_n)_{n=0}^\infty$  of real numbers  $\tau_n$  when does there exist a non-negative finite Borel measure  $\mu$  on  $\mathbb{R}$  such that a Laplace transform  $l_\mu(\lambda) := \int_{\mathbb{R}} e^{x\lambda} d\mu(x)$  is analytic in a neighborhood of zero in  $\mathbb{C}$  and  $\tau_n = \int_{\mathbb{R}} P_n(x) d\mu(x)$  for all  $n \in \mathbb{N}_0$ ?*

We give an answer on this problem in Section 6 for the case of the so-called Sheffer polynomials (i.e., polynomials with generating function of exponential type). The monomials and Newton polynomials are examples of the Sheffer polynomials. In the case of the monomials this problem is closely related to the problem of integral representation of exponentially convex functions (see Subsection 6.2), in the Newton polynomials’ case this problem is connected with a one-dimensional analog of the Bogoliubov generating functionals, i.e., with functions  $B : \mathcal{U} \rightarrow \mathbb{C}$  ( $\mathcal{U}$  is a neighborhood of  $0 \in \mathbb{C}$ ) of such type

$$B(\lambda) = \int_{\mathbb{R}} (1 + \lambda)^x d\mu(x) = \int_{\mathbb{R}} e^{x \log(1+\lambda)} d\mu(x), \quad \lambda \in \mathcal{U},$$

where  $\mu$  is a certain non-negative finite Borel measure on  $\mathbb{R}$  (see Subsection 6.3). Note that  $e^{x \log(1+\lambda)}$  is a generating function for the Newton polynomials  $(x)_n$ ,

$$e^{x \log(1+\lambda)} = \sum_{n=0}^\infty \frac{\lambda^n}{n!} (x)_n, \quad |\lambda| < 1.$$

We stress that the classical Bogoliubov functionals were introduced by N. N. Bogoliubov in [14] to define correlation functions for statistical mechanics systems.

The last part of this article (Section 7) is related, on the one hand, to the infinite-dimensional generalization of the classical moment problem (1.2) and, on the other hand, to some tasks of statistical physics. The main purpose of this section is to explain the motivation of this work and present a few known examples of the results and some open problems in the case of functions of infinite many variables.

## 2. PRELIMINARIES

**2.1. Projection spectral theorem.** In this subsection we recall some results concerning the projection spectral theorem and the quasianalytic criterion of selfadjointness of operators (for a detailed explanation see e.g. books [3, 9, 12]).

Let  $\mathcal{H}$  be a complex separable Hilbert space and

$$(2.1) \quad \mathcal{H}_- \supset \mathcal{H} \supset \mathcal{H}_+ \supset \mathcal{D}$$

be a fixed rigging of  $\mathcal{H}$ . We suppose that  $\mathcal{H}_+$  is a Hilbert space which is topologically (i.e., densely and continuously) and quasinuclearly (i.e., the inclusion operator is of Hilbert–Schmidt type) embedded into  $\mathcal{H}$ ,  $\mathcal{H}_-$  is the dual of  $\mathcal{H}_+$  with respect to the zero space  $\mathcal{H}$  (with the pairing  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ ), and  $\mathcal{D}$  is a linear, separable, topological space that is topologically embedded into  $\mathcal{H}_+$ .

The following projection spectral theorem holds (see [3], Ch. 5; [9], Ch. 3; [12], Ch. 15).

**Theorem 2.1.** *Let  $A$  be a self-adjoint operator defined on  $\text{Dom}(A)$  in  $\mathcal{H}$ . Assume that*

- *$A$  is standardly connected with chain (2.1), i.e.,  $\mathcal{D} \subset \text{Dom}(A)$  and the restriction  $A \upharpoonright \mathcal{D}$  of the operator  $A$  on  $\mathcal{D}$  acts from  $\mathcal{D}$  into  $\mathcal{H}_+$  continuously.*

- *A has a strong cyclic vector  $\Omega$ , that is there exists a vector  $\Omega \in \mathcal{H}$  such that  $\Omega \in \text{Dom}(A^n)$  for all  $n \in \mathbb{N}$  and a set  $\{A^n \Omega \mid n \in \mathbb{N}_0\}$  is total in  $\mathcal{H}_+$  (i.e., a set  $\text{span}\{A^n \Omega \mid n \in \mathbb{N}_0\}$  is dense in  $\mathcal{H}_+$ ).*

Then there exists a non-negative finite Borel measure  $\mu$  on  $\mathbb{R}$  (spectral measure, defined on the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R})$ ) such that

- *For  $\mu$ -almost every  $x \in \mathbb{R}$  there is a unique vector  $\xi(x) \in \mathcal{H}_-$  (the so-called generalized eigenvector of  $A$  with an eigenvalue  $x$ ) such that*

$$\langle \xi(x), Af \rangle_{\mathcal{H}} = x \langle \xi(x), f \rangle_{\mathcal{H}}, \quad f \in \mathcal{D}.$$

- *The mapping*

$$(2.2) \quad \mathcal{H} \supset \mathcal{D} \ni f \mapsto (I_A f)(\cdot) := \langle f, \xi(\cdot) \rangle_{\mathcal{H}} \in L^2(\mathbb{R}, \mu)$$

is well-defined and isometric, i.e.,

$$(2.3) \quad (f, g)_{\mathcal{H}} = \int_{\mathbb{R}} (I_A f)(x) \overline{(I_A g)(x)} d\mu(x), \quad f, g \in \mathcal{H}.$$

Extending the mapping  $I_A$  by continuity to the whole space  $\mathcal{H}$  we obtain an isometric operator  $I_A : \mathcal{H} \rightarrow L^2(\mathbb{R}, \mu)$ .

*Remark 2.2.* Let an operator  $A$  satisfies all assumptions of Theorem 2.1 and, moreover, the closure of  $A \upharpoonright \mathcal{D}$  in  $H$  coincides with  $A$ . Then by a well-known fact (see, e.g., [12], Ch. 15, § 3) the extension (by continuity) of mapping (2.2) is a unitary operator  $I_A : \mathcal{H} \rightarrow L^2(\mathbb{R}, \mu)$  acting from the whole space  $\mathcal{H}$  onto the whole space  $L^2(\mathbb{R}, \mu)$ . The image of  $A$  under  $I_A$  is the operator of multiplication by  $x$  in  $L^2(\mathbb{R}, \mu)$ .

Let us recall the quasianalytic criterion of self-adjointness. For a Hermitian operator  $A$  defined on  $\text{Dom}(A)$  in  $\mathcal{H}$ , a vector  $f \in \bigcap_{n=1}^{\infty} \text{Dom}(A^n)$  is called quasianalytic if

$$(2.4) \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{\|A^n f\|_{\mathcal{H}}}} = \infty.$$

**Theorem 2.3.** *A Hermitian operator  $A$  in  $\mathcal{H}$  is essentially self-adjoint if and only if the space  $\mathcal{H}$  contains a total set of quasianalytic vectors.*

Versions of this theorem are published in [22, 23], see also [3], Ch. 8, § 5. For the given form of it, see [12], Ch. 13, § 9.

**2.2. Spaces and riggings.** Denote by  $\mathbb{C}^{\infty}$  a linear space of all sequences  $f = (f_n)_{n=0}^{\infty}$  of complex numbers  $f_n \in \mathbb{C}$ , and by  $l_{\text{fin}}$  its linear subspace consisting of finite sequences  $f = (f_0, \dots, f_n, 0, 0, \dots)$ . Henceforth, we will denote by  $\delta_n$  the  $\delta$ -sequence,

$$(2.5) \quad \delta_n = (\delta_{nj})_{j=0}^{\infty} = (\underbrace{0, \dots, 0}_{n \text{ times}}, 1, 0, 0, \dots).$$

Then each vector  $f$  from  $l_{\text{fin}}$  can be interpreted as a finite sum  $\sum_{n=0}^{\infty} f_n \delta_n$ .

For a fixed weight  $p = (p_n)_{n=0}^{\infty}$ ,  $p_n > 0$ , we denote by

$$l^2(p) := \left\{ f = (f_n)_{n=0}^{\infty} \in \mathbb{C}^{\infty} \mid \|f\|_{l^2(p)}^2 := \sum_{n=0}^{\infty} |f_n|^2 p_n < \infty \right\}$$

the  $l^2$ -type space with a corresponding scalar product  $(\cdot, \cdot)_{l^2(p)}$ . In the case of the weight  $1 = (1, 1, \dots)$  we will use a standard notation  $l^2 := l^2(1)$ .

Let  $p = (p_n)_{n=0}^{\infty}$ ,  $p_n \geq 1$ . Then the space  $l^2(p)$  is densely and continuously embedded into the space  $l^2$  and therefore one can construct the chain (the rigging of  $l^2$ )

$$(2.6) \quad l^2(p^{-1}) \supset l^2 \supset l^2(p) \supset l_{\text{fin}},$$

where  $p^{-1} := (p_n^{-1})_{n=0}^\infty$  and  $l^2(p^{-1}) = (l^2(p))'$  is the dual space of  $l^2(p)$  with respect to the zero space  $l^2$ . Denote by  $\langle \cdot, \cdot \rangle_{l^2}$  the dual pairing between elements of  $l^2(p^{-1})$  and  $l^2(p)$  inducted by the scalar product in  $l^2$ , i.e.,

$$\langle \xi, g \rangle_{l^2} := \sum_{n=0}^\infty \xi_n \bar{g}_n, \quad \xi \in l^2(p^{-1}), \quad g \in l^2(p).$$

Together with (2.6), we consider a rigging of  $l^2$  connected with a special weight  $p$ . Namely, for each  $q \in \mathbb{N}$ , we set

$$\gamma(q) = ((n!)^2 2^{qn})_{n=0}^\infty$$

and introduce the so-called Kondratiev-type  $l^2$ -spaces

$$l^2(\gamma(q)) \quad \text{and} \quad l^2_+ := \text{pr} \lim_{q \in \mathbb{N}} l^2(\gamma(q)).$$

Then the dual spaces of  $l^2(\gamma(q))$  and  $l^2_+$  with respect to the zero space  $l^2$  are

$$l^2(\gamma^{-1}(q)) \quad \text{and} \quad l^2_- := (l^2_+)' = \text{ind} \lim_{q \in \mathbb{N}} l^2(\gamma^{-1}(q))$$

respectively (here  $\gamma_n^{-1}(q) = (n!)^{-2} 2^{-qn}$ ). Thus, we get a rigging

$$\mathbb{C}^\infty = l'_{\text{fin}} \supset l^2_- \supset l^2(\gamma^{-1}(q)) \supset l^2 \supset l^2(\gamma(q)) \supset l^2_+ \supset l_{\text{fin}}.$$

Here we identify, in the usual way, the space  $\mathbb{C}^\infty$  with the space  $l'_{\text{fin}}$  of all linear functionals on  $l_{\text{fin}}$ . In the sequel we won't distinguish  $\mathbb{C}^\infty$  and  $l'_{\text{fin}}$ .

Now we recall one important property of the space  $l^2_-$ . Denote by  $\text{Hol}_0(\mathbb{C})$  a set of all (germs of) functions  $\phi : \mathbb{C} \rightarrow \mathbb{C}$  that are holomorphic at  $0 \in \mathbb{C}$  and, for each  $\xi = (\xi_n)_{n=0}^\infty \in l^2_-$ , define the so-called  $S$ -transform by the formula

$$(S\xi)(\lambda) := \sum_{n=0}^\infty \frac{\lambda^n}{n!} \xi_n, \quad \lambda \in \mathcal{U},$$

where  $\mathcal{U}$  is a (depending on  $\xi$ ) neighborhood of  $0 \in \mathbb{C}$ .

The following result shows that each vector  $\xi$  from  $l^2_-$  is uniquely determined by its  $S$ -transform (see [17] for the infinite dimensional analogue of this fact).

**Theorem 2.4.** *The  $S$ -transform*

$$S : l^2_- \rightarrow \text{Hol}_0(\mathbb{C}), \quad \xi = (\xi_n)_{n=0}^\infty \mapsto (S\xi)(\lambda) := \sum_{n=0}^\infty \frac{\lambda^n}{n!} \xi_n,$$

is a one-to-one map between  $l^2_-$  and  $\text{Hol}_0(\mathbb{C})$ .

*Proof.* Let  $\xi = (\xi_n)_{n=0}^\infty \in l^2_-$ , i.e., there exists  $q \in \mathbb{N}$  such that

$$\xi \in l^2(\gamma^{-1}(q)) \quad \text{or, equivalently,} \quad \sum_{n=0}^\infty |\xi_n|^2 2^{-qn} (n!)^{-2} < \infty.$$

Using the Cauchy-Bunyakovsky-Schwarz inequality, for  $|\lambda| < 2^{-\frac{q}{2}}$ , we get

$$\begin{aligned} |(S\xi)(\lambda)| &= \left| \sum_{n=0}^\infty \frac{\lambda^n}{n!} \xi_n \right| \leq \sum_{n=0}^\infty \frac{|\lambda|^n}{n!} |\xi_n| \\ &\leq \left( \sum_{n=0}^\infty |\lambda|^{2n} 2^{qn} \right)^{\frac{1}{2}} \left( \sum_{n=0}^\infty |\xi_n|^2 2^{-qn} (n!)^{-2} \right)^{\frac{1}{2}} < \infty. \end{aligned}$$

Thus,  $S\xi \in \text{Hol}_0(\mathbb{C})$ .

For the converse, suppose that  $\phi \in \text{Hol}_0(\mathbb{C})$ , that is there exists  $r > 0$  such that the function  $\phi$  admits the representation

$$\phi(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \xi_n < \infty, \quad |\lambda| < r,$$

with

$$\xi_n = \frac{d^n \phi}{d\lambda^n}(\lambda) \Big|_{\lambda=0} = \frac{n!}{2\pi i} \oint_{|\zeta|=r_0} \frac{\phi(\zeta)}{\zeta^{n+1}} d\zeta, \quad 0 < r_0 < r.$$

As a consequence of the latter integral representation, for some  $C > 0$ , we get

$$|\xi_n| \leq n! C^{n+1}, \quad n \in \mathbb{N}_0.$$

Choosing  $q \in \mathbb{N}$  in such way that  $C^2 2^{-q} < 1$ , we obtain

$$\sum_{n=0}^{\infty} |\xi_n|^2 2^{-qn} (n!)^{-2} \leq C^2 \sum_{n=0}^{\infty} \frac{C^{2n}}{2^{qn}} < \infty.$$

So,  $\xi := (\xi_n)_{n=0}^{\infty} \in l^2_-$  and  $S\xi = \phi$ .

The fact that  $\text{Ker}(S) := \{\xi \in l^2_- \mid S\xi = 0\} = \{0\}$  is obvious.  $\square$

**Corollary 2.5.** *A sequences  $\xi = (\xi_n)_{n=0}^{\infty} \in \mathbb{C}^{\infty}$  belongs to the space  $l^2_-$  if and only if there exists a constant  $C > 0$  such that*

$$|\xi_n| \leq n! C^{n+1}, \quad n \in \mathbb{N}_0.$$

### 3. CONVOLUTIONS ON THE SPACE OF FINITE SEQUENCES

**3.1. Definition and properties of convolutions.** Let  $(P_n)_{n=0}^{\infty}$  be a fixed family of real-valued polynomials  $P_n : \mathbb{R} \rightarrow \mathbb{R}$  such that each  $P_n$  has a degree  $n$ . Thus,  $(P_n)_{n=0}^{\infty}$  is a linear basis in the space  $\mathcal{P} := \mathbb{C}[x]$  of all complex-valued polynomials  $F : \mathbb{R} \rightarrow \mathbb{C}$ .

Define a convolution (product)  $*_{\mathcal{P}}$  on the space  $l_{\text{fin}}$  by setting

$$(3.1) \quad f *_{\mathcal{P}} g := I_{\mathcal{P}}^{-1}(I_{\mathcal{P}} f \cdot I_{\mathcal{P}} g), \quad f, g \in l_{\text{fin}},$$

where

$$(3.2) \quad I_{\mathcal{P}} : l_{\text{fin}} \rightarrow \mathbb{C}[x], \quad f = (f_n)_{n=0}^{\infty} \mapsto (I_{\mathcal{P}} f)(x) := \sum_{n=0}^{\infty} f_n P_n(x),$$

is a natural bijection between  $l_{\text{fin}}$  and  $\mathbb{C}[x]$ . The space  $l_{\text{fin}}$  with product  $*_{\mathcal{P}}$  becomes a commutative algebra  $\mathcal{A}$  with the unity  $\delta_0 = \{1, 0, 0, \dots\}$  and the involution

$$(3.3) \quad l_{\text{fin}} \ni f = (f_n)_{n=0}^{\infty} \mapsto \bar{f} := (\bar{f}_n)_{n=0}^{\infty} \in l_{\text{fin}},$$

where  $\bar{f}_n$  denotes the complex conjugation. Clearly, choosing different bases in the space  $\mathbb{C}[x]$  we obtain different products in the space  $l_{\text{fin}}$ .

In the space  $\mathbb{C}[x]$  we introduce a scalar product by setting

$$(F, G)_{\mathcal{P}} := (I_{\mathcal{P}}^{-1} F, I_{\mathcal{P}}^{-1} G)_{l^2} = \sum_{n=0}^{\infty} f_n \bar{g}_n,$$

$$F(\cdot) = \sum_{n=0}^{\infty} f_n P_n(\cdot), \quad G(\cdot) = \sum_{n=0}^{\infty} g_n P_n(\cdot) \in \mathbb{C}[x].$$

The sequence  $(P_n)_{n=0}^{\infty}$  makes an orthonormal basis in  $\mathbb{C}[x]$  and therefore each polynomial  $F \in \mathbb{C}[x]$  admits the representation

$$(3.4) \quad F(x) = \sum_{n=0}^{\infty} (F, P_n)_{\mathcal{P}} P_n(x), \quad x \in \mathbb{R}$$

(note that  $(F, P_n)_{\mathcal{P}} = 0$  for  $n$  greater than the degree of  $F$ ).

The following result holds, see also [4].

**Lemma 3.1.** *For all  $f, g \in l_{\text{fin}}$  and  $n \in \mathbb{N}_0 := \{0, 1, \dots\}$  we have*

$$(3.5) \quad (f *_P g)_n = \sum_{j,k=0}^{\infty} f_j g_k (P_j P_k, P_n)_{\mathcal{P}}.$$

*Proof.* Using formulas (3.1), (3.2) and (3.4), for all  $f, g \in l_{\text{fin}}$  and  $x \in \mathbb{R}$ , we get

$$\begin{aligned} (I_P(f *_P g))(x) &= (I_P f)(x) \cdot (I_P g)(x) = \sum_{n=0}^{\infty} (I_P f \cdot I_P g, P_n)_{\mathcal{P}} P_n(x) \\ &= \sum_{n=0}^{\infty} \left( \sum_{j,k=0}^{\infty} f_j g_k (P_j P_k, P_n)_{\mathcal{P}} \right) P_n(x) = \sum_{n=0}^{\infty} (f *_P g)_n P_n(x). \end{aligned}$$

So, formula (3.5) takes place. □

**3.2. Examples.** In some special cases we can count a more explicit expression of product (3.5). Let us consider two examples.

1) Let  $P_n(x) = x^n$  be a monomial. Then  $*_P = *$  is an ordinary convolution (*Cauchy product*)  $*$  of two sequences  $f = (f_n)_{n=0}^{\infty}, g = (g_n)_{n=0}^{\infty} \in l_{\text{fin}}$

$$(3.6) \quad (f *_P g)_n = (f * g)_n = \sum_{i+j=n} f_i g_j = \sum_{k=0}^{\infty} f_k g_{n-k}.$$

This fact is a direct consequence of (3.5) and the following formula:

$$(P_j P_k, P_n)_{\mathcal{P}} = (x^j x^k, x^n)_{\mathcal{P}} = (x^{j+k}, x^n)_{\mathcal{P}} = \delta_{j+k,n}.$$

2) Let  $P_n(x) = (x)_n$  (here  $(x)_n$  denotes the Pochhammer symbol) be the so-called *Newton (or binomial) polynomial*. By definition

$$P_n(x) = (x)_n := \begin{cases} 1, & \text{if } n = 0, \\ x(x-1) \cdots (x-n+1), & \text{if } n \in \mathbb{N}. \end{cases}$$

In terms of Gamma function, we have

$$(x)_n = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}, \quad n \in \mathbb{N}.$$

Note that  $(x)_n, n \in \mathbb{N}_0$ , is an example of *Sheffer polynomials*, see Section 5 for details. The corresponding generating function of  $(x)_n$  has the form

$$(3.7) \quad P(x, \lambda) := (1 + \lambda)^x = e^{x \log(1+\lambda)} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (x)_n, \quad |\lambda| < 1.$$

In the case  $P_n(x) = (x)_n$  we will denote convolution (3.1) by  $\star := *_P$ .

**Theorem 3.2.** *For all  $f = (f_n)_{n=0}^{\infty}, g = (g_n)_{n=0}^{\infty} \in l_{\text{fin}}$  and  $n \in \mathbb{N}_0$  we have*

$$(3.8) \quad (f *_P g)_n = (f \star g)_n = \sum_{i+j+k=n} \frac{(i+j)!(i+k)!}{i!k!j!} f_{i+j} g_{j+k}.$$

*Proof.* Due to (3.5) it suffices to show that

$$(3.9) \quad ((x)_j (x)_k, (x)_n)_{\mathcal{P}} = \begin{cases} \frac{j!k!}{(n-j)!(n-k)!(j+k-n)!}, & j, k \in \{0, \dots, n\}, j+k \geq n, \\ 0, & \text{otherwise.} \end{cases}$$

Using (3.7) and the multinomial formula

$$(a_1 + a_2 + \dots + a_m)^n = \sum_{k_1+k_2+\dots+k_m=n} \frac{n!}{k_1!k_2!\dots k_m!} \prod_{1 \leq i \leq m} a_i^{k_i},$$

for all  $|\lambda|, |\mu| < \varepsilon$  ( $\varepsilon > 0$  is small enough), we get

$$\begin{aligned} e^{x \log(1+\lambda)} e^{x \log(1+\mu)} &= e^{x \log(1+\lambda+\mu+\lambda\mu)} = \sum_{n=0}^{\infty} \frac{(\lambda + \mu + \lambda\mu)^n}{n!} (x)_n \\ (3.10) \quad &= \sum_{n=0}^{\infty} \sum_{i+j+k=n} \frac{\lambda^{i+j} \mu^{k+j}}{i!j!k!} (x)_n. \end{aligned}$$

On the other hand, taking into account the formula

$$(x)_j(x)_k = \sum_{n=0}^{\infty} ((x)_j(x)_k, (x)_n)_{\mathcal{P}}(x)_n, \quad x \in \mathbb{R},$$

for all  $|\lambda|, |\mu| < \varepsilon$ , we obtain

$$\begin{aligned} e^{x \log(1+\lambda)} e^{x \log(1+\mu)} &= \sum_{j,k=0}^{\infty} \frac{\lambda^j \mu^k}{j! k!} (x)_j(x)_k \\ (3.11) \quad &= \sum_{n=0}^{\infty} \sum_{j,k=0}^{\infty} \frac{\lambda^j \mu^k}{j!k!} ((x)_j(x)_k, (x)_n)_{\mathcal{P}}(x)_n. \end{aligned}$$

Comparing the coefficients at  $(x)_n$  and then at  $\lambda^j \mu^k$  in formulas (3.10) and (3.11) we get equality (3.9). □

*Remark 3.3.* It should be noticed that the product  $\star$  is a one-dimensional analog of the so-called *Kondratiev–Kuna convolution* on a “Fock space”, see e.g. [15] and Subsection 7.2 below for the definition and properties of the Kondratiev–Kuna convolution.

#### 4. THE MOMENT PROBLEM

As above let  $(P_n)_{n=0}^{\infty}$  be a fixed family of real-valued polynomials  $P_n \in \mathbb{C}[x]$  such that each  $P_n$  has a degree  $n$  and  $\mathcal{A} = l_{\text{fin}}$  be a commutative algebra with the product  $*_{\mathcal{P}}$  (3.1).

**Definition 4.1.** A functional  $\tau = (\tau_n)_{n=0}^{\infty} \in \mathbb{C}^{\infty}$  is said to be a *moment functional* (or, a *moment sequences*) on  $(\mathcal{A}, *_{\mathcal{P}})$  if there exists a non-negative Borel measure  $\mu$  on  $\mathbb{R}$  such that

$$(4.1) \quad \tau_n = \int_{\mathbb{R}} P_n(x) d\mu(x), \quad n \in \mathbb{N}_0.$$

Obviously, if  $\tau = (\tau_n)_{n=0}^{\infty} \in \mathbb{C}^{\infty}$  is a moment functional on  $(\mathcal{A}, *_{\mathcal{P}})$  then actually  $\tau_n \in \mathbb{R}$  for all  $n \in \mathbb{N}_0$  and the measure  $\mu$  from representation (4.1) is finite. The *moment problem* on  $\mathcal{A}$  is to characterize those linear functionals  $\tau \in \mathbb{C}^{\infty}$  which are moment functionals. A solution of this problem is given in the next theorem.

**Theorem 4.2.**  $\tau = (\tau_n)_{n=0}^{\infty} \in \mathbb{C}^{\infty}$  is a moment functional on  $\mathcal{A} = l_{\text{fin}}$  if and only if  $\tau$  is  $*_{\mathcal{P}}$ -positive (more exactly, non-negative) on  $\mathcal{A}$ , that is

$$(4.2) \quad \tau(f *_{\mathcal{P}} \bar{f}) = \sum_{n=0}^{\infty} \tau_n(f *_{\mathcal{P}} \bar{f})_n = \sum_{n=0}^{\infty} \tau_n \left( \sum_{j,k=0}^{\infty} f_j \bar{f}_k (P_j P_k, P_n)_{\mathcal{P}} \right) \geq 0$$

for all  $f = (f_n)_{n=0}^{\infty} \in \mathcal{A}$ .



A method of proving this result is similar to considerations of [4] and is based on the theory of generalized eigenfunction expansion. In the case of the classical moment problem this method was first proposed by Yu. M. Berezansky in [3], Ch. 8.

*Proof.* The necessity of condition (4.2) is trivial. Indeed,

$$\begin{aligned} \tau(f *_P \bar{f}) &= \sum_{n=0}^{\infty} \tau_n(f *_P \bar{f})_n = \sum_{n=0}^{\infty} \tau_n \left( \sum_{j,k=0}^{\infty} f_j \bar{f}_k (P_j P_k, P_n)_P \right) \\ &= \int_{\mathbb{R}} \sum_{j,k=0}^{\infty} f_j \bar{f}_k \left( \sum_{n=0}^{\infty} (P_j P_k, P_n)_P P_n(x) \right) d\mu(x) \\ &= \int_{\mathbb{R}} \left| \sum_{j=0}^{\infty} f_j P_j(x) \right|^2 d\mu(x) \geq 0, \quad f = (f_n)_{n=0}^{\infty} \in \mathcal{A}. \end{aligned}$$

For the proof of the sufficiency of condition (4.2), we will apply Theorem 2.1 to a certain self-adjoint operator connected with our moment problem.

Let  $\tau \in \mathbb{C}^{\infty}$  be a positive functional on  $\mathcal{A}$ , that is (4.2) holds. Using this functional and convolution  $*_P$  we construct in a standard way a Hilbert space  $H_{\tau}$ . Namely, we define  $H_{\tau}$  as a Hilbert space associated with the quasiscalar product

$$(4.3) \quad (f, g)_{H_{\tau}} := \tau(f *_P \bar{g}), \quad f, g \in \mathcal{A}.$$

For the construction of  $H_{\tau}$ , at first it is necessary to pass from  $\mathcal{A}$  to the factor space  $\dot{\mathcal{A}} := \mathcal{A}/\{f \in \mathcal{A} \mid (f, f)_{H_{\tau}} = 0\}$  and then to take the completion of  $\dot{\mathcal{A}}$ . For simplicity we will suppose that  $\dot{\mathcal{A}} \equiv \mathcal{A}$ , i.e.,  $(f, f)_{H_{\tau}} = 0$  if and only if  $f = 0$ . Note that an investigation of the general case is possible but technically it is more complicated (for the corresponding constructions in the case of the classical moment problem, see [3], Ch. 8, § 1, Subsect. 4 or in [9], Ch. 5, § 5, Subsect. 1–3).

For the sake of simplicity we will assume that  $P_0(x) = 1$  and  $P_1(x) = x$ . Using (3.1) and (3.2) we define an operator

$$(4.4) \quad J_P : l_{\text{fin}} \rightarrow l_{\text{fin}}, \quad J_P f := I_P^{-1} \mathbb{J} I_P = \delta_1 *_P f, \quad f \in l_{\text{fin}},$$

where  $\delta_1 = (0, 1, 0, 0, \dots)$ ,  $I_P$  is defined by formula (3.2) and  $\mathbb{J}$  is the operator of multiplication by  $x$  in the space  $\mathbb{C}[x]$ , i.e.,

$$(\mathbb{J}F)(x) := P_1(x)F(x) = xF(x), \quad F \in \mathbb{C}[x].$$

The operator  $J : l_{\text{fin}} \rightarrow l_{\text{fin}}$  is Hermitian in the Hilbert space  $H_{\tau}$ ,

$$(J_P f, g)_{H_{\tau}} = \tau(\delta_1 *_P f *_P \bar{g}) = \tau(f *_P \overline{\delta_1 *_P g}) = (f, J_P g)_{H_{\tau}}, \quad f, g \in l_{\text{fin}},$$

and, moreover, it is real with respect to involution (3.3), i.e.,  $\overline{J_P f} = J_P \bar{f}$ ,  $f \in l_{\text{fin}}$ . Therefore, by a theorem of von Neumann  $J_P$  has self-adjoint extensions.

Denote by  $A$  a certain self-adjoint extension of  $J_P$  on  $H_{\tau}$ . We will apply Theorem 2.1 to this operator. Now the role of chain (2.1) will play the rigging

$$(4.5) \quad (l^2(p))'_{H_{\tau}} \supset H_{\tau} \supset l^2(p) \supset l_{\text{fin}},$$

where  $(l^2(p))'_{H_{\tau}} = \mathcal{H}_-$  is the negative space with respect to the positive space  $l^2(p)$  and the zero space  $H_{\tau} = \mathcal{H}$ . The space  $l_{\text{fin}} = \mathcal{D}$  is provided with uniformly finite coordinate-wise convergence, i.e., the sequence  $\{f^{(j)}, j \in \mathbb{N}\} \subset l_{\text{fin}}$  converge to  $f \in l_{\text{fin}}$  if and only if there exists  $N \in \mathbb{N}$  such that  $f_n^{(j)} = 0$  for all  $n > N$ ,  $j \in \mathbb{N}$  and  $f_n^{(j)} \rightarrow f_n$  as  $j \rightarrow \infty$  for all  $n \in \mathbb{N}_0$ .

**Lemma 4.3.** *There exists a weight  $p = (p_n)_{n=0}^{\infty}$ ,  $p_n \geq 1$ , such that the embedding  $l^2(p) \hookrightarrow H_{\tau}$  is well-defined and quasinuclear.*

*Proof.* Let us set

$$(4.6) \quad K_{jk} := \sum_{n=0}^{\infty} \tau_n(P_j P_k, P_n)_{\mathcal{P}}, \quad j, k \in \mathbb{N}_0$$

(note that for any  $j, k \in \mathbb{N}_0$  the sum in (4.6) is finite). Due to (4.2) the matrix  $K = (K_{jk})_{j,k=0}^{\infty}$  is nonnegative definite, i.e.,

$$\sum_{j,k=0}^{\infty} K_{jk} f_j \bar{f}_k = \sum_{j,k=0}^{\infty} \left( \sum_{n=0}^{\infty} \tau_n(P_j P_k, P_n)_{\mathcal{P}} \right) f_j \bar{f}_k = \tau(f *_{\mathcal{P}} \bar{f}) \geq 0, \quad f \in l_{\text{fin}}.$$

Hence,

$$(4.7) \quad |K_{jk}|^2 \leq K_{jj} K_{kk}, \quad j, k \in \mathbb{N}_0.$$

Let  $q = (q_n)_{n=0}^{\infty}$ ,  $q_n \geq 1$ , be such that  $\sum_{n=0}^{\infty} K_{nn} q_n^{-1} < \infty$ . Then from (4.3), (4.2) and (4.7) it follows that, for all  $f \in l_{\text{fin}}$ ,

$$\|f\|_{H_{\tau}}^2 = \tau(f *_{\mathcal{P}} \bar{f}) = \sum_{j,k=0}^{\infty} K_{jk} f_j \bar{f}_k \leq \left( \sum_{j=0}^{\infty} \frac{K_{jj}}{q_j} \right) \|f\|_{l_2(q)}^2.$$

Therefore,  $l^2(q) \hookrightarrow H_{\tau}$  topologically. But if  $\sum_{n=0}^{\infty} q_n p_n^{-1} < \infty$ , then  $l^2(p) \hookrightarrow l^2(q)$  quasinuclearly. The composition of these two embeddings gives that  $l^2(p) \hookrightarrow H_{\tau}$  is quasinuclear.  $\square$

In what follows we fix a weight  $p = (p_n)_{n=0}^{\infty}$ ,  $p_n \geq 1$ , such that the embedding  $l^2(p) \hookrightarrow H_{\tau}$  is quasinuclear. It is clear that the operator  $A$  is standardly connected with chain (4.5). Let us show that the vector  $\Omega = \delta_0 = (1, 0, 0, \dots) \in l_{\text{fin}}$  is a strong cyclic vector for  $A$ .

To this end, it suffices to show that  $\text{span}\{A^n \Omega \mid n \in \mathbb{N}_0\} = l_{\text{fin}}$ . But this is evidently true, since  $I_P : l_{\text{fin}} \rightarrow \mathbb{C}[x]$  is bijection,  $\text{span}\{x^n \mid n \in \mathbb{N}_0\} = \mathbb{C}[x]$  and by (4.4)

$$A^n \Omega = J^n \delta_0 = I_P^{-1}(x^n), \quad n \in \mathbb{N}_0.$$

So, the operator  $A$  satisfies all assumptions of Theorem 2.1. Let  $\mu$  be the corresponding spectral measure of  $A$  and  $\xi(x) \in (l^2(p))'_{H_{\tau}}$  be the generalized eigenvector of  $A$  with an eigenvalue  $x \in \mathbb{R}$ . According to Theorem 2.1 we have

$$(4.8) \quad \langle \xi(x), Af \rangle_{H_{\tau}} = x \langle \xi(x), f \rangle_{H_{\tau}}, \quad f \in l_{\text{fin}},$$

and the mapping

$$(4.9) \quad H_{\tau} \supset l_{\text{fin}} \ni f \mapsto (I_A f)(\cdot) := \langle f, \xi(\cdot) \rangle_{H_{\tau}} \in L^2(\mathbb{R}, \mu)$$

is isometric.

To prove (4.1), it suffices to check that

$$(4.10) \quad (I_A f)(x) = (I_P f)(x) = \sum_{n=0}^{\infty} f_n P_n(x), \quad f \in l_{\text{fin}},$$

for  $\mu$ -almost all  $x \in \mathbb{R}$ .

Indeed, suppose that (4.10) takes place. Then by (2.3) we have

$$(4.11) \quad (f, g)_{H_{\tau}} = \int_{\mathbb{R}} (I_P f)(x) \overline{(I_P g)(x)} d\mu(x), \quad f, g \in l_{\text{fin}}.$$

Therefore, taking into account the equalities  $\tau_n = \tau(\delta_n) = \tau(\delta_n *_{\mathcal{P}} \delta_0) = (\delta_n, \delta_0)_{H_{\tau}}$  and  $(I_P \delta_n)(x) = P_n(x)$ , we get

$$\tau_n = (\delta_n, \delta_0)_{H_{\tau}} = \int_{\mathbb{R}} P_n(x) d\mu(x), \quad n \in \mathbb{N}_0.$$

Let us check (4.10). According to [4], Lemma 2.2, there exists a unique determined unitary operator  $U : (l^2(p))'_{H_\tau} \rightarrow l^2(p^{-1})$  such that

$$\langle U\eta, g \rangle_{l^2} = \langle \eta, g \rangle_{H_\tau}, \quad \eta \in (l^2(p))'_{H_\tau}, \quad g \in l^2(p).$$

Therefore, it suffices to show that

$$(U\xi)(x) = P(x) := (P_n(x))_{n=0}^\infty, \quad x \in \mathbb{R},$$

or, equivalently,

$$\langle P(x), Af \rangle_{l^2} = x \langle P(x), f \rangle_{l^2}, \quad x \in \mathbb{R}, \quad f \in l_{\text{fin}}.$$

But the latter equality takes place, since on the one hand

$$x \langle P(x), f \rangle_{l^2} = x \sum_{n=0}^\infty f_n P_n(x) = x \cdot (I_P f)(x).$$

On the other hand, taking into account that  $P_1(x) = x$ ,  $Af = \delta_1 *_{\mathcal{P}} f$ ,  $f \in l_{\text{fin}}$ , and

$$(\delta_1 *_{\mathcal{P}} f)_n = \sum_{k=0}^\infty f_k (P_1 P_k, P_n)_{\mathcal{P}}, \quad n \in \mathbb{N}_0,$$

we get

$$\begin{aligned} \langle P(x), Af \rangle_{l^2} &= \langle P(x), \delta_1 *_{\mathcal{P}} f \rangle_{l^2} = \sum_{n=0}^\infty P_n(x) \left( \sum_{k=0}^\infty f_k (P_1 P_k, P_n)_{\mathcal{P}} \right) \\ &= \sum_{n=0}^\infty \left( \sum_{k=0}^\infty f_k (P_1 P_k, P_n)_{\mathcal{P}} P_n(x) \right) = \sum_{n=0}^\infty (P_1 I_P f, P_n)_{\mathcal{P}} P_n(x) \\ &= P_1(x) \cdot (I_P f)(x) = x \cdot (I_P f)(x). \end{aligned}$$

Thus, Theorem 4.2 is proved.  $\square$

*Remark 4.4.* Let  $P_n(x) = x^n$ ,  $n \in \mathbb{N}_0$ . Then  $*_{\mathcal{P}} = *$  is the Cauchy product (3.6) and the corresponding moment problem is called the *Hamburger moment problem*.

From Theorem 4.2 and formula (3.6) we immediately get the following classical result:  $\tau = (\tau_n)_{n=0}^\infty \in \mathbb{C}^\infty$  is a moment functional on  $(\mathcal{A}, *)$  (moment sequences) if and only if

$$(4.12) \quad \tau(f * \bar{f}) = \sum_{j,k=0}^\infty \tau_{j+k} f_j \bar{f}_k \geq 0, \quad f \in l_{\text{fin}}.$$

Note that now the operator  $J_{\mathcal{P}} : l_{\text{fin}} \rightarrow l_{\text{fin}}$ ,  $J_{\mathcal{P}} f := \delta_1 * f$ , is an ordinary right shift (or, in another terminology, a creation operator), that is

$$J_{\mathcal{P}} f = J(f_0, f_1, \dots) = (0, f_0, f_1, \dots), \quad f = (f_n)_{n=0}^\infty \in l_{\text{fin}},$$

or in a matrix form

$$J_{\mathcal{P}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{pmatrix}.$$

*Remark 4.5.* Let  $P_n(x) = (x)_n = x(x-1)\cdots(x-n+1)$ . Then  $*_{\mathcal{P}} = \star$  has form (3.8) and as a direct consequence of Theorem 4.2 we get:  $\tau = (\tau_n)_{n=0}^\infty \in \mathbb{C}^\infty$  is a moment functional on  $(\mathcal{A}, \star)$  if and only if

$$(4.13) \quad \tau(f \star \bar{f}) = \sum_{i,j,k=0}^\infty \frac{(i+j)!(i+k)!}{i!k!j!} \tau_{i+j+k} f_{i+j} \bar{f}_{j+k} \geq 0, \quad f \in l_{\text{fin}}.$$

It easy to see that the Newton polynomials  $(x)_n$  obey the recurrence relation

$$(4.14) \quad x(x)_n = (x)_{n+1} + n(x)_n.$$

Indeed, let  $a_- : l_{\text{fin}} \rightarrow l_{\text{fin}}$  be an annihilation operator, i.e.,

$$a_-((f_n)_{n=0}^\infty) = (f_1, 2f_2, \dots, nf_n, \dots).$$

Then on the one hand, the operator  $\partial := I_P a_- I_P^{-1} : \mathbb{C}[x] \rightarrow \mathbb{C}[x]$  acts by the formula

$$\partial(x)_n = n(x)_{n-1}, \quad n \in \mathbb{N}_0.$$

On the other hand, it can be proved that, for any polynomial  $F \in \mathbb{C}[x]$ ,

$$(\partial F)(x) = F(x + 1) - F(x) \quad \text{and, therefore,} \quad \partial(x)_n = (x + 1)_n - (x)_n.$$

Thus,  $n(x)_{n-1} = (x + 1)_n - (x)_n$  and therefore (4.14) holds.

It follows from (4.14) that the operator  $J_P : l_{\text{fin}} \rightarrow l_{\text{fin}}$ ,  $J_P f := \delta_1 \star f$ , has the following matrix representation

$$J_P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 2 & 0 & 0 & \dots \\ 0 & 0 & 1 & 3 & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{pmatrix}.$$

5. SHEFFER POLYNOMIALS AND ANALYTIC MEASURES

The *Sheffer polynomials*  $(P_n)_{n=0}^\infty$  are defined via their exponential generating function

$$(5.1) \quad P(x, \lambda) := \gamma(\lambda)e^{\alpha(\lambda)x} = \sum_{n=0}^\infty \frac{\lambda^n}{n!} P_n(x), \quad x \in \mathbb{R}, \quad \lambda \in \mathcal{U},$$

where  $\mathcal{U}$  is a some neighborhood of zero in  $\mathbb{C}$ ,  $\gamma$  and  $\alpha$  are analytic functions in  $\mathcal{U}$  such that  $\alpha(0) = 0$ ,  $\alpha'(0) \neq 0$  and  $\gamma(0) = 1$ . Using the classical Faa di Bruno formula it can be showed that each  $P_n(x)$  is a polynomial of exact degree  $n \in \mathbb{N}_0$ .

We observe that many classical polynomial families are Sheffer — the monomials, Newton, Bernoulli, Hermite, Poisson-Charlier polynomials and many others. Note also that the Sheffer polynomials have remarkable applications in various fields, such as probability, numerical analysis, Rota’s umbral calculus and so on. We refer, e.g., to [28, 25, 24] and [1] for more details.

For every  $x \in \mathbb{R}$  the function  $P(x, \cdot)$  is an analytic in a neighborhood of  $0 \in \mathbb{C}$ . Therefore,

$$P_n(x) = \frac{d^n}{d\lambda^n} P(x, \lambda) \Big|_{\lambda=0} = \frac{n!}{2\pi i} \oint_{|\zeta|=r} \frac{P(x, \zeta)}{\zeta^{n+1}} d\zeta,$$

where  $r > 0, r \in \mathcal{U}$ . As a consequence, for all  $\varepsilon > 0$  there exists  $r_\varepsilon > 0$  such that

$$(5.2) \quad |P_n(x)| \leq \frac{n!}{r_\varepsilon^n} \sup_{|\lambda|=r_\varepsilon} |\gamma(\lambda)e^{\alpha(\lambda)x}| \leq \frac{2n!}{r_\varepsilon^n} e^{\varepsilon|x|}, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}_0,$$

where  $r_\varepsilon \in \mathcal{U}$  is chosen in such a way that  $|\alpha(\lambda)| \leq \varepsilon$  and  $\gamma(\lambda) \leq 2$  for  $|\lambda| = r_\varepsilon$ .

Let  $\mu$  be a non-negative finite Borel measure on  $\mathbb{R}$  such that a Laplace transform

$$l_\mu(\lambda) := \int_{\mathbb{R}} e^{x\lambda} d\mu(x)$$

is well-defined in a neighborhood of zero in  $\mathbb{C}$ . It is easy to check the following result.

**Proposition 5.1.** *Let a measure  $\mu$  on  $\mathcal{B}(\mathbb{R})$  be such that  $e^{x\lambda}$  belongs to  $L^1(\mathbb{R}, \mu)$  for  $|\lambda| < \varepsilon$  (for some  $\varepsilon > 0$ ). Then the Laplace transform  $l_\mu$  of  $\mu$  admits the representation*

$$(5.3) \quad l_\mu(\lambda) := \int_{\mathbb{R}} e^{x\lambda} d\mu(x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{\mathbb{R}} x^n d\mu(x), \quad |\lambda| < \varepsilon,$$

and, as a consequences,  $l_\mu$  is analytic in a neighborhood of zero in  $\mathbb{C}$ , i.e.,  $l_\mu \in \text{Hol}_0(\mathbb{C})$ .

*Proof.* Let us fix  $\lambda \in \mathbb{C}$  such that  $|\lambda| < \varepsilon$ . Since  $e^{x\lambda} \in L^1(\mathbb{R}, \mu)$  then  $\cosh(x|\lambda|) \in L^1(\mathbb{R}, \mu)$  and by the monotone convergence theorem we have

$$\int_{\mathbb{R}} \cosh(x|\lambda|) d\mu(x) = \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{(2n)!} \int_{\mathbb{R}} x^{2n} d\mu(x) < \infty.$$

Therefore,  $x^{2n} \in L^1(\mathbb{R}, \mu)$ . Using the Shwarz inequality we get

$$\int_{\mathbb{R}} |x|^n d\mu(x) \leq \sqrt{\mu(\mathbb{R})} \left( \int_{\mathbb{R}} |x|^{2n} d\mu(x) \right)^{\frac{1}{2}} < \infty,$$

i.e.,  $x^n \in L^1(\mathbb{R}, \mu)$  for all  $n \in \mathbb{N}$ . Since  $|\sum_{n=0}^N \frac{\lambda^n}{n!} x^n| \leq 2 \cosh(x|\lambda|)$ , by the dominated convergence theorem we obtain

$$l_\mu(\lambda) = \int_{\mathbb{R}} e^{x\lambda} d\mu(x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{\mathbb{R}} x^n d\mu(x) < \infty, \quad |\lambda| < \varepsilon.$$

□

Denote by  $\mathcal{M}_a(\mathbb{R})$  the set of all non-negative finite analytic measures  $\mu$  on  $\mathcal{B}(\mathbb{R})$ , i.e.,

$$\mathcal{M}_a(\mathbb{R}) := \{ \mu : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty) \mid \mu - \text{measure}, l_\mu \in \text{Hol}_0(\mathbb{C}) \}.$$

Equivalent descriptions of analytic measures are given by the following lemma (see [18] for the infinite dimensional analogue of this result).

**Theorem 5.2.** *The following statement are equivalent*

- (1)  $\mu \in \mathcal{M}_a(\mathbb{R})$ .
- (2) There exists a constant  $C > 0$  such that

$$\left| \int_{\mathbb{R}} x^n d\mu(x) \right| < n! C^{n+1}, \quad n \in \mathbb{N}_0.$$

- (3) There exists a constant  $r > 0$  such that  $e^{|x|r} \in L^1(\mathbb{R}, \mu)$ .
- (4) There exists a constant  $\varepsilon > 0$  such that  $P(x, \lambda) = \gamma(\lambda) e^{\alpha(\lambda)x} \in L^1(\mathbb{R}, \mu)$  for  $|\lambda| < \varepsilon$ , where  $P(x, \lambda)$  is a generating function of the Sheffer polynomials  $P_n(x)$ .

*Proof.* Let us check the following chain 1)  $\Rightarrow$  2)  $\Rightarrow$  3)  $\Rightarrow$  4)  $\Rightarrow$  1).

1)  $\Rightarrow$  2). This fact immediately follows from representation (5.3).

2)  $\Rightarrow$  3). For the moments of even order we have

$$\int_{\mathbb{R}} |x|^{2n} d\mu(x) = \int_{\mathbb{R}} x^{2n} d\mu(x) \leq (2n)! C^{2n+1}, \quad n \in \mathbb{N}_0.$$

The moments of arbitrary order can be estimated by the Cauchy-Bunyakovsky-Schwarz inequality

$$\int_{\mathbb{R}} |x|^n d\mu(x) \leq \sqrt{\mu(\mathbb{R})} \left( \int_{\mathbb{R}} |x|^{2n} d\mu(x) \right)^{\frac{1}{2}} \leq \sqrt{\mu(\mathbb{R})} C C^n \sqrt{(2n)!} \leq \sqrt{\mu(\mathbb{R})} C (2C)^n n!,$$

since  $(2n)! \leq 4^n (n!)^2$ . Chose  $r < (2C)^{-1}$  then

$$\int_{\mathbb{R}} e^{|x|r} d\mu(x) = \sum_{n=0}^{\infty} \frac{r^n}{n!} \int_{\mathbb{R}} |x|^n d\mu(x) = \sqrt{\mu(\mathbb{R})} C \sum_{n=0}^{\infty} (r2C)^n < \infty.$$

3)  $\Rightarrow$  4). Let  $r > 0$  be such as in statement (3) and  $\varepsilon > 0$  be chosen in such way that  $\varepsilon \in B_0$  and  $|\alpha(\lambda)| \leq r$  for  $|\lambda| < \varepsilon$ . Then for all  $x \in \mathbb{R}$  and all  $\lambda \in \mathbb{C}$  such that  $|\lambda| < \varepsilon$  we have

$$|P(x, \lambda)| = |\gamma(\lambda)e^{\alpha(\lambda)x}| \leq Ce^{r|x|}, \quad C := \sup_{|\lambda| \leq \varepsilon} |\gamma(\lambda)|.$$

So,  $P(x, \lambda) \in L^1(\mathbb{R}, \mu)$  for  $|\lambda| < \varepsilon$ .

4)  $\Rightarrow$  1). Since  $\alpha$  is the analytic function in  $\mathcal{U}$ ,  $\alpha(0) = 0$  and  $\alpha'(0) \neq 0$  then there exists  $\tilde{\varepsilon} > 0$  such that  $\{\lambda \in \mathbb{C} \mid |\lambda| < \tilde{\varepsilon}\} \subset \text{Ran}(\alpha)$ . Therefore,  $e^{x\lambda} \in L^1(\mathbb{R}, \mu)$  for  $|\lambda| < \tilde{\varepsilon}$ , i.e.,  $\mu \in \mathcal{M}_a(\mathbb{R})$ .  $\square$

**Corollary 5.3.** *Let  $\mu \in \mathcal{M}_a(\mathbb{R})$  and  $P(x, \lambda) := \gamma(\lambda)e^{\alpha(\lambda)x}$  be a generating function of the Sheffer polynomials  $P_n(x)$ . Then the following formula holds*

$$\int_{\mathbb{R}} P(x, \lambda) d\mu(x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{\mathbb{R}} P_n(x) d\mu(x) < \infty, \quad |\lambda| < \varepsilon,$$

for some  $\varepsilon > 0$ .

*Proof.* Clearly,  $P^2(x, \lambda) = \gamma^2(\lambda)e^{2\alpha(\lambda)x}$  is a generating function of the Sheffer polynomials. Therefore, by Theorem 5.2 we have  $P^2(\cdot, \lambda) \in L^1(\mathbb{R}, \mu)$  for  $|\lambda| < \varepsilon$ , i.e.,  $P(\cdot, \lambda) \in L^2(\mathbb{R}, \mu)$  for  $|\lambda| < \varepsilon$ . Using the latter, (5.1) and the continuity property of the inner product, we get

$$\begin{aligned} \int_{\mathbb{R}} P(x, \lambda) d\mu(x) &= (P(\cdot, \lambda), 1)_{L^2(\mathbb{R}, \mu)} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (P_n(\cdot), 1)_{L^2(\mathbb{R}, \mu)} \\ &= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_{\mathbb{R}} P_n(x) d\mu(x) < \infty, \quad |\lambda| < \varepsilon. \end{aligned}$$

$\square$

*Remark 5.4.*  $\mu \in \mathcal{M}_a(\mathbb{R})$  if and only if there exists a constant  $C > 0$  such that

$$(5.4) \quad \int_{\mathbb{R}} |P_n(x)|^2 d\mu(x) \leq (n!)^2 C^{n+1}, \quad n \in \mathbb{N}_0,$$

where  $(P_n)_{n=0}^{\infty}$  is a family of Sheffer polynomials on  $\mathbb{R}$ .

*Proof.* Let  $\mu \in \mathcal{M}_a(\mathbb{R})$ . Using (5.2) for  $2\varepsilon < r$  ( $r > 0$  from Theorem 5.2) and Theorem 5.2 we get

$$\int_{\mathbb{R}} |P_n(x)|^2 d\mu(x) \leq \frac{4(n!)^2}{r_{\varepsilon}^{2n}} \int_{\mathbb{R}} e^{2\varepsilon|x|} d\mu < \infty, \quad n \in \mathbb{N}_0.$$

Hence, (5.4) takes place.

Conversely, let (5.4) holds. Then

$$\left\| \sum_{n=M}^N \frac{|\lambda|^n}{n!} P_n(\cdot) \right\|_{L^2(\mathbb{R}, \mu)} \leq \sum_{n=M}^N \frac{|\lambda|^n}{n!} \|P_n(\cdot)\|_{L^2(\mathbb{R}, \mu)} \leq \sqrt{C} \sum_{n=M}^N (\lambda\sqrt{C})^n.$$

So,  $P(\cdot, \lambda) \in L^2(\mathbb{R}, \mu)$  for  $|\lambda| < (\sqrt{C})^{-1}$  and, therefore, from Theorem 5.2 follows that  $\mu \in \mathcal{M}_a(\mathbb{R})$ .  $\square$

## 6. ANALYTIC MOMENT FUNCTIONALS

**6.1. Definition and properties.** As above let  $P(x, \lambda) := \gamma(\lambda)e^{\alpha(\lambda)x}$  be a generating function of the Sheffer polynomials  $P_n(x)$  and  $\mathcal{A} = l_{\text{fin}}$  be an algebra with the product  $*_P$  (3.1). In the sequel, we will fix such family  $(P_n(x))_{n=0}^{\infty}$  of Sheffer polynomials and we assume, in addition, that  $P_n(x)$ ,  $n \in \mathbb{N}_0$ , are real-valued polynomials.

**Definition 6.1.** A functional  $\tau \in \mathbb{C}^{\infty}$  is said to be an *analytic moment functional* on  $(\mathcal{A}, *_P)$  if there exists an analytic measure  $\mu \in \mathcal{M}_a(\mathbb{R})$  such that

$$(6.1) \quad \tau_n = \int_{\mathbb{R}} P_n(x) d\mu(x), \quad n \in \mathbb{N}_0.$$

Clearly,  $\tau \in \mathbb{C}^{\infty}$  is an analytic moment functional on  $\mathcal{A}$  if and only if  $\tau$  is a moment functional on  $\mathcal{A}$  and the measure  $\mu$  in representation (6.1) belongs to  $\mathcal{M}_a(\mathbb{R})$ .

*Remark 6.2.* Let  $\tau = (\tau_n)_{n=0}^{\infty} \in \mathbb{C}^{\infty}$  be an analytic moment functional on  $\mathcal{A}$  and  $\mu$  be the corresponding measure on  $\mathcal{B}(\mathbb{R})$  such that (6.1) holds. Then Corollary 5.3 shows that the moments  $\tau_n$  are the Taylor coefficients of the function generalized Laplace transform

$$l(\cdot) := \int_{\mathbb{R}} P(x, \cdot) d\mu(x) \in \text{Hol}_0(\mathbb{C}).$$

That is,  $l$  is the generating function for the moments  $\tau_n$  and

$$\tau_n = l^{(n)}(0) = \frac{d^n}{d\lambda^n} l(\lambda) \Big|_{\lambda=0}, \quad n \in \mathbb{N}_0.$$

By using the  $S$ -transform (see Theorem 2.4), this means that  $\tau = S^{-1}l$ .

The aim of this section is to find conditions on  $\tau \in \mathbb{C}^{\infty}$  that would guarantee existence of a measure  $\mu$  on  $\mathcal{B}(\mathbb{R})$  such that  $\mu \in \mathcal{M}_a(\mathbb{R})$  and (6.1) takes place.

**Theorem 6.3.** *Necessary conditions that  $\tau = (\tau_n)_{n=0}^{\infty} \in \mathbb{C}^{\infty}$  is an analytic moment functional on  $\mathcal{A}$  are the following:  $\tau$  is  $*_P$ -positive on  $\mathcal{A}$  (i.e., (4.2) holds) and  $\tau \in l_-^2$ .*

*Sufficient conditions that  $\tau = (\tau_n)_{n=0}^{\infty} \in \mathbb{C}^{\infty}$  is an analytic moment functional on  $\mathcal{A}$  are the following:  $\tau$  is  $*_P$ -positive on  $\mathcal{A}$  and there is a constant  $C > 0$  such that*

$$(6.2) \quad \tau(\delta_n *_P \delta_n) = \sum_{k=0}^{2n} \tau_k (\delta_n *_P \delta_n)_k \leq (n!)^2 C^{n+1}, \quad n \in \mathbb{N}_0,$$

where the vector  $\delta_n \in l_{\text{fin}}$  is defined by (2.5).

*Proof. Necessity.* Let  $\tau \in \mathbb{C}^{\infty}$  be an analytic moment functional on  $\mathcal{A}$ . Then from Theorem 4.2 it immediately follows that  $\tau$  is  $*_P$ -positive on  $\mathcal{A}$ . Since by Remark 6.2 the function  $l(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \tau_n$  belongs to the space  $\text{Hol}_0(\mathbb{C})$ , the fact that  $\tau \in l_-^2$  is a direct consequence of Theorem 2.4.

*Sufficiency.* Suppose that  $\tau$  is  $*_P$ -positive on  $\mathcal{A}$  and (6.2) holds. Then according to Theorem 4.2 the  $*_P$ -positiveness of  $\tau$  insures that (6.1) holds. Next, using (4.11), (4.10) and (6.2) we get

$$\begin{aligned} \tau(\delta_n *_P \delta_n) &= (\delta_n, \delta_n)_{H_{\tau}} = \int_{\mathbb{R}} (I_P \delta_n)^2(x) d\mu(x) \\ &= \int_{\mathbb{R}} P_n^2(x) d\mu(x) \leq (n!)^2 C^{n+1}, \quad n \in \mathbb{N}_0. \end{aligned}$$

So, from Remark 5.4 we conclude that  $\mu \in \mathcal{M}_a(\mathbb{R})$ . □

**Theorem 6.4.** *If  $\tau = (\tau_n)_{n=0}^{\infty} \in \mathbb{C}^{\infty}$  is an analytic moment functional on  $(\mathcal{A}, *_P)$  then the measure  $\mu$  in representation (6.1) is uniquely defined.*

*Proof.* At first, we prove the statement for the case  $P_n(x) = x^n$ . So, we need to show that for an analytic moment functional  $\tau = (\tau_n)_{n=0}^\infty \in \mathbb{C}^\infty$  on  $(\mathcal{A}, *)$  ( $*$  is defined by (4.12)) the measure  $\mu$  in the representation

$$(6.3) \quad \tau_n = \int_{\mathbb{R}} x^n d\mu(x), \quad n \in \mathbb{N}_0,$$

is unique.

It is well known (see, e.g., [3], Ch. 8, Theorem 1.1) that the measure  $\mu$  in representation (6.3) is unique if and only if the operator  $l_{\text{fin}} \ni f \mapsto Jf = \delta_1 * f \in l_{\text{fin}}$  is essentially self-adjoint (i.e., has a unique self-adjoint extension) in the space  $\mathcal{H}_\tau$  (see the proof of Theorem 4.2 for the definition of  $\mathcal{H}_\tau$ ). Since  $\text{span}\{\delta_n \mid n \in \mathbb{N}_0\} = l_{\text{fin}}$  and  $l_{\text{fin}}$  is dense in  $\mathcal{H}_\tau$ , then according to the quasianalytic criterion of self-adjointness (see Theorem 2.3) it is sufficient to check that every vector  $\delta_k, k \in \mathbb{N}_0$ , is quasianalytic, i.e., equality (2.4) holds for every  $\delta_k$ .

It is easy to see that  $J^n \delta_k = \delta_{k+n}, \|J^n \delta_k\|_{\mathcal{H}_\tau}^2 = \|\delta_{k+n}\|_{\mathcal{H}_\tau}^2 = \tau_{2k+2n}$ . Since  $\mu \in \mathcal{M}_a(\mathbb{R})$ , then there exists  $C > 0$  such that  $|\tau_n| \leq n!C^{n+1}$  for all  $n \in \mathbb{N}_0$  and therefore

$$\sum_{n=1}^\infty \frac{1}{\sqrt[n]{\|J^n \delta_k\|_{\mathcal{H}_\tau}}} = \sum_{n=1}^\infty \frac{1}{\sqrt[n]{\tau_{2k+2n}}} = \infty, \quad k \in \mathbb{N}_0,$$

i.e., the measure  $\mu$  in representation (6.3) is unique.

Let us prove the general case. Suppose that measures  $\mu_1, \mu_2 \in \mathcal{M}_a(\mathbb{R})$  are such that  $\mu_1 \neq \mu_2$  and

$$\int_{\mathbb{R}} P_n(x) d\mu_1(x) = \int_{\mathbb{R}} P_n(x) d\mu_2(x), \quad n \in \mathbb{N}_0.$$

Then it is easy to check by induction that

$$\int_{\mathbb{R}} x^n d\mu_1(x) = \int_{\mathbb{R}} x^n d\mu_2(x), \quad n \in \mathbb{N}_0.$$

So,  $\mu_1 = \mu_2$ , due to the above proven, which leads to a contradiction. □

*Remark 6.5.* From the proof of Theorem 6.4 and Remark 2.2 it easily follows the next well known result: *If  $\mu \in \mathcal{M}_a(\mathbb{R})$  then the set of all polynomials  $\mathbb{C}[x]$  is dense in the space  $L^2(\mathbb{R}, \mu)$ .*

**6.2. Analytic moment functionals connected with the monomials.** Let  $P(x, \lambda) = e^{x\lambda}$  and  $\mathcal{A} = l_{\text{fin}}$  be an algebra with the Cauchy product  $*$  (3.6).

**Theorem 6.6.** *A functional  $\tau \in \mathbb{C}^\infty$  is an analytic moment functional on  $(\mathcal{A}, *)$ , i.e., there exists a measure  $\mu \in \mathcal{M}_a(\mathbb{R})$  such that*

$$(6.4) \quad \tau(\delta_n) = \tau_n = \int_{\mathbb{R}} x^n d\mu(x), \quad n \in \mathbb{N}_0,$$

*if and only if  $\tau$  is  $*$ -positive on  $\mathcal{A}$  (i.e., (4.12) holds) and  $\tau \in l_-^2$ .*

*For an analytic moment functional  $\tau \in \mathbb{C}^\infty$  on  $(\mathcal{A}, *)$  the measure  $\mu$  in representation (6.4) is unique defined.*

*Proof.* The necessity immediately follows from Theorem 6.3.

Let us prove the sufficiency. Assume that  $\tau$  is  $*$ -positive and  $\tau \in l_-^2$ . Then from Remark 4.4 we conclude that  $\tau$  is a moment functional on  $(\mathcal{A}, *)$ , i.e., there exists a Borel measure  $\mu$  on  $\mathbb{R}$  such that (6.4) holds.

Let us check that  $\mu \in \mathcal{M}_a(\mathbb{R})$ . Since  $\tau = (\tau_n)_{n=0}^\infty \in l_-^2$ , from Corollary 2.5 it follows that there exists  $C > 0$  such that

$$|\tau_n| = \left| \int_{\mathbb{R}} x^n d\mu(x) \right| \leq n!C^{n+1}, \quad n \in \mathbb{N}_0.$$



Hence, from Theorem 5.2 we conclude that  $\mu \in \mathcal{M}_a(\mathbb{R})$ .

The last assertion of the theorem directly follows from Theorem 6.4.  $\square$

Let us show that the class of analytic moment functionals on  $(\mathcal{A}, \star)$  is closely related to the class of exponentially convex functions. Recall that a function  $k : (-2a, 2a) \rightarrow \mathbb{C}$ , where  $0 < a \leq \infty$ , is called exponentially convex if

$$(6.5) \quad \sum_{i,j=0}^{\infty} k(x_i + x_j) f_i \bar{f}_j \geq 0$$

for all  $f = (f_n)_{n=0}^{\infty} \in l_{\text{fin}}$  and  $x_i, x_j \in (-a, a)$ .

The classical Bernstein's theorem asserts (see, e.g., [2], Ch. 5, § 5; [3], Ch. 8, § 3): A continuous function  $k : (-2a, 2a) \rightarrow \mathbb{C}$  is exponentially convex if and only if there exists a non-negative finite Borel measure  $\mu$  on  $\mathbb{R}$  such that

$$k(\lambda) = l_{\mu}(\lambda) = \int_{\mathbb{R}} e^{x\lambda} d\mu(x), \quad \lambda \in (-2a, 2a).$$

The measure  $\mu$  in the latter representation is unique. It follows from Theorem 5.2 that in fact  $\mu \in \mathcal{M}_a(\mathbb{R})$  and therefore  $k : (-2a, 2a) \rightarrow \mathbb{C}$  is an analytic in a neighborhood of zero in  $\mathbb{R}$ .

From Bernstein's theorem, Theorem 6.6 and Remark 6.2 (for  $P(x, \lambda) = e^{x\lambda}$ ) we get the following result.

**Theorem 6.7.** A functional  $\tau = (\tau_n)_{n=0}^{\infty} \in \mathbb{C}^{\infty}$  is an analytic moment functional on  $(\mathcal{A}, \star)$  if and only if the function

$$k(\lambda) := \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \tau_n$$

is well defined and exponentially convex in some neighborhood of zero in  $\mathbb{R}$ .

Vice versa, an analytic in a neighborhood  $\mathcal{U}$  of zero in  $\mathbb{R}$  function  $k : \mathcal{U} \rightarrow \mathbb{C}$  is exponentially convex if and only if the functional

$$\tau = (\tau_n)_{n=0}^{\infty} = (k^{(n)}(0))_{n=0}^{\infty}, \quad \tau_n := k^{(n)}(0) = \left. \frac{d^n}{d\lambda^n} k(\lambda) \right|_{\lambda=0},$$

is an analytic moment functional on  $(\mathcal{A}, \star)$ .

**Corollary 6.8.** An analytic in a neighborhood  $\mathcal{U}$  of zero in  $\mathbb{R}$  function  $k : \mathcal{U} \rightarrow \mathbb{C}$  is exponentially convex if and only if

$$\sum_{i,j=0}^{\infty} k^{(i+j)}(0) f_i \bar{f}_j \geq 0, \quad f = (f_n)_{n=0}^{\infty} \in l_{\text{fin}}.$$

### 6.3. Analytic moment functionals connected with the Newton polynomials.

Let  $P(x, \lambda)$  be a generating function of the Newton polynomials  $(x)_n = \prod_{i=0}^{n-1} (x - i)$ , that is

$$P(x, \lambda) := (1 + \lambda)^x = e^{x \log(1+\lambda)} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (x)_n, \quad |\lambda| < 1,$$

and  $\mathcal{A} = l_{\text{fin}}$  be an algebra with the product  $\star$  (3.8).

Now an analogue of Theorem 6.6 holds.

**Theorem 6.9.** A functional  $\tau \in \mathbb{C}^{\infty}$  is an analytic moment functional on  $(\mathcal{A}, \star)$ , i.e., there exists a measure  $\mu \in \mathcal{M}_a(\mathbb{R})$  such that

$$(6.6) \quad \tau(\delta_n) = \tau_n = \int_{\mathbb{R}} (x)_n d\mu(x), \quad n \in \mathbb{N}_0,$$

if and only if  $\tau$  is  $\star$ -positive on  $\mathcal{A}$  (i.e., (4.13) holds) and  $\tau \in l_-^2$ .

For an analytic moment functional  $\tau \in \mathbb{C}^\infty$  on  $(\mathcal{A}, \star)$  the measure  $\mu$  in (6.6) is unique.

*Proof.* The necessity immediately follows from Theorem 6.3.

Let us prove the converse. Suppose that  $\tau$  is  $\star$ -positive on  $\mathcal{A}$  and  $\tau \in l^2_-$ . Then due to Theorem 6.3 it is sufficient to show that there exists  $C > 0$  such that

$$\tau(\delta_n \star \delta_n) = \int_{\mathbb{R}} (x)_n^2 d\mu(x) \leq (n!)^2 C^{n+1}, \quad n \in \mathbb{N}_0.$$

Since  $\tau = (\tau_n)_{n=0}^\infty \in l^2_-$ , there exists  $\tilde{C} > 0$  such that  $|\tau_n| \leq n! \tilde{C}^{n+1}$  for all  $n \in \mathbb{N}_0$ . Hence, taking into account that (see (3.8))

$$(\delta_n \star \delta_n)_m = \begin{cases} \frac{(n!)^2}{((m-n)!)^2 (2n-m)!}, & \text{if } m \in \{n, \dots, 2n\}, \\ 0, & \text{otherwise,} \end{cases}$$

we get

$$(6.7) \quad \tau(\delta_n \star \delta_n) = \sum_{m=n}^{2n} \tau_m \frac{(n!)^2}{((m-n)!)^2 (2n-m)!} \leq \sum_{m=n}^{2n} \tilde{C}^{m+1} \frac{m!(n!)^2}{((m-n)!)^2 (2n-m)!}.$$

Let us estimate the expression

$$\frac{m!(n!)^2}{((m-n)!)^2 (2n-m)!}.$$

Using the bound for the binomial coefficients

$$\frac{m!}{n!(m-n)!} \leq 2^m, \quad m \in \mathbb{N}_0,$$

we get

$$(6.8) \quad \begin{aligned} \frac{m!(n!)^2}{((m-n)!)^2 (2n-m)!} &= \frac{(m!)^2 (n!)^4}{((m-n)!)^2 (n!)^2 m! (2n-m)!} \\ &\leq 4^m \frac{(n!)^4}{m! (2n-m)!} \leq 4^m (n!)^2 \end{aligned}$$

for all  $m \in \{n, \dots, 2n\}$ .

From (6.7) and (6.8) we conclude that

$$\tau(\delta_n \star \delta_n) = \int_{\mathbb{R}} (x)_n^2 d\mu(x) \leq \tilde{C}^{2n+1} 4^{2n+1} (n!)^2 \leq (n!)^2 C^{n+1},$$

where  $C := \max\{8\tilde{C}^2, 4\tilde{C}\}$ . So, the sufficiency is proved.

The last assertion of the theorem directly follows from Theorem 6.4. □

Let us establish a relation between the analytic moment functional on  $(\mathcal{A}, \star)$  and a one-dimensional analog of the Bogoliubov generating functionals. We say that a function  $B : \mathcal{U} \rightarrow \mathbb{C}$  ( $\mathcal{U}$  is a neighborhood of zero in  $\mathbb{C}$ ) is a Bogoliubov functional in  $\mathcal{U}$  if  $B$  admits the following integral representation

$$(6.9) \quad B(\lambda) = \int_{\mathbb{R}} (1 + \lambda)^x d\mu(x) = \int_{\mathbb{R}} e^{x \log(1+\lambda)} d\mu(x), \quad \lambda \in \mathcal{U},$$

with some non-negative finite Borel measure  $\mu$  on  $\mathbb{R}$ . It follows from Theorem 5.2 that the measure  $\mu$  in representation (6.9) is actually analytical, i.e.,  $\mu \in \mathcal{M}_a(\mathbb{R})$ .

It should be noticed that the classical Bogoliubov or generating functionals were introduced by N. N. Bogoliubov in [14] to define correlation functions for statistical mechanics systems (this functional is defined by analogue with (6.9) but for measures on the space of finite configuration). We refer to, e.g., [21, 16] for details, historical remarks and references therein.

An analogue of Theorem 6.7 holds.

**Theorem 6.10.** *A functional  $\tau = (\tau_n)_{n=0}^\infty \in \mathbb{C}^\infty$  is an analytic moment functional on  $(\mathcal{A}, \star)$  if and only if the function*

$$B(\lambda) := \sum_{n=0}^\infty \frac{\lambda^n}{n!} \tau_n$$

is the Bogoliubov functional in some neighborhood of zero in  $\mathbb{C}$ .

Vice versa, an analytic in a neighborhood  $\mathcal{U}$  of zero in  $\mathbb{C}$  function  $B : \mathcal{U} \rightarrow \mathbb{C}$  is the Bogoliubov functional in  $\mathcal{U}$  if and only if the functional

$$\tau = (\tau_n)_{n=0}^\infty = (B^{(n)}(0))_{n=0}^\infty, \quad \tau_n := B^{(n)}(0) = \frac{d^n}{d\lambda^n} B(\lambda) \Big|_{\lambda=0},$$

is an analytic moment functional on  $(\mathcal{A}, \star)$ .

**Corollary 6.11.** *An analytic in a neighborhood  $\mathcal{U}$  of zero in  $\mathbb{C}$  function  $B : \mathcal{U} \rightarrow \mathbb{C}$  is the Bogoliubov functional in  $\mathcal{U}$  if and only if*

$$\sum_{i,j,k=0}^\infty \frac{(i+j)!(i+k)!}{i!k!j!} B^{(i+j+k)}(0) f_{i+j} \bar{f}_{j+k} \geq 0, \quad f = (f_n)_{n=0}^\infty \in l_{\text{fin}}.$$

### 7. INFINITE DIMENSIONAL CASE

The theory outlined in previous sections has an essential development to the case of functions of infinite many variables, see e.g. [9, 10, 4, 5, 11] for details. Without going into details we present here a few examples of the results and open problems.

**7.1. Infinite dimensional power moment problem.** Let  $\mathcal{F}(H)$  be a symmetric Fock space over a real separable Hilbert space  $H$ , that is

$$\mathcal{F}(H) := \mathbb{C} \oplus \bigoplus_{n=1}^\infty H_{\mathbb{C}}^{\odot n},$$

where  $\odot$  stands for the symmetric tensor product ( $\otimes$  is the ordinary tensor product), the subindex  $\mathbb{C}$  denotes the complexification of a real space. Thus,  $\mathcal{F}(H)$  is a complex Hilbert space of sequences  $f = (f_n)_{n=0}^\infty$  such that  $f_n \in H_{\mathbb{C}}^{\odot n}$  ( $H_{\mathbb{C}}^{\odot 0} := \mathbb{C}$ ) and

$$\|f\|_{\mathcal{F}(H)}^2 = |f_0|^2 + \sum_{n=1}^\infty \|f_n\|_{H_{\mathbb{C}}^{\odot n}}^2 < \infty.$$

For simplicity, in the sequel we will suppose that  $H = L^2(\mathbb{R}) := L^2(\mathbb{R}, dt)$  and one will always identify, in the usual way, the space  $L_{\mathbb{C}}^2(\mathbb{R})^{\odot n}$  with the space  $L_{\mathbb{C}, \text{sym}}^2(\mathbb{R}^n)$  of all symmetric functions from  $L_{\mathbb{C}}^2(\mathbb{R}^n)$ .

Let us construct a convenient for us rigging of the Fock space  $\mathcal{F}(L^2(\mathbb{R}))$ . To this end, we start with the classical rigging

$$(7.1) \quad \mathcal{D}' \supset L^2(\mathbb{R}) \supset \mathcal{D},$$

where  $\mathcal{D} = \mathcal{D}(\mathbb{R})$  is the Schwartz space of infinite differentiable functions on  $\mathbb{R}$  with compact supports,  $\mathcal{D}' = \mathcal{D}'(\mathbb{R})$  is the Schwartz space of distributions dual of  $\mathcal{D}$  with respect to the zero space  $L^2(\mathbb{R})$ . We denote by  $\langle \cdot, \cdot \rangle$  the dual pairing between elements of  $\mathcal{D}'$  and  $\mathcal{D}$ . We preserve the notation  $\langle \cdot, \cdot \rangle$  for the dual pairings in tensor powers and complexifications of chain (7.1).

Using (7.1) we construct the rigging

$$(7.2) \quad \mathcal{F}'_{\text{fin}}(\mathcal{D}) \supset \mathcal{F}(L^2(\mathbb{R})) \supset \mathcal{F}_{\text{fin}}(\mathcal{D}),$$

where  $\mathcal{F}_{\text{fin}}(\mathcal{D})$  is a space of all finite sequences  $f = (f_n)_{n=0}^\infty$ ,  $f_n \in \mathcal{D}'_{\mathbb{C}} \circledast n$  (i.e.,  $f_n = 0$  for all  $n \geq \text{some } N \in \mathbb{N}_0$ ),  $\mathcal{F}'_{\text{fin}}(\mathcal{D}) = \times_{n=0}^\infty (\mathcal{D}'_{\mathbb{C}}) \circledast n$  is the dual of  $\mathcal{F}_{\text{fin}}(\mathcal{D})$  with respect to  $\mathcal{F}(L^2(\mathbb{R}))$  (it consists of all sequences of the form  $(\xi_n)_{n=0}^\infty$ ,  $\xi_n \in (\mathcal{D}'_{\mathbb{C}}) \circledast n$ ). Note that in our case the role of the spaces  $\mathcal{F}'_{\text{fin}}(\mathcal{D})$ ,  $\mathcal{F}(L^2(\mathbb{R}))$  and  $\mathcal{F}_{\text{fin}}(\mathcal{D})$  are the same as the role of the spaces  $\mathbb{C}^\infty$ ,  $l^2$  and  $l_{\text{fin}}$  in the one-dimensional case.

Denote by  $\mathcal{P}(\mathcal{D}')$  the space of all continuous polynomials on  $\mathcal{D}'$ ,

$$\mathcal{P}(\mathcal{D}') := \left\{ F : \mathcal{D}' \rightarrow \mathbb{C} \mid \exists (f_n)_{n=0}^\infty \in \mathcal{F}_{\text{fin}}(\mathcal{D}) : F(x) = \sum_{n=0}^\infty \langle x^{\otimes n}, f_n \rangle, x \in \mathcal{D}' \right\}.$$

By analogy with the one-dimensional case (see (3.1)), using the bijection

$$I : \mathcal{F}_{\text{fin}}(\mathcal{D}) \rightarrow \mathcal{P}(\mathcal{D}'), \quad f = (f_n)_{n=0}^\infty \mapsto (If)(x) := \sum_{n=0}^\infty \langle x^{\otimes n}, f_n \rangle,$$

we introduce a product  $*$  on  $\mathcal{F}_{\text{fin}}(\mathcal{D})$  by setting

$$(7.3) \quad f * g := I^{-1}(If \cdot Ig), \quad f, g \in \mathcal{F}_{\text{fin}}(\mathcal{D}).$$

It is easy to check that (cf. (3.6))

$$(f * g)_n = \sum_{i+j=n} f_i \odot g_j = \sum_{k=0}^\infty f_k \odot g_{n-k}, \quad f, g \in \mathcal{F}_{\text{fin}}(\mathcal{D}).$$

So,  $\mathcal{F}_{\text{fin}}(\mathcal{D})$  becomes a commutative algebra  $\mathcal{A}(\mathcal{D})$  with the product  $*$ , unity  $\delta_0 = \{1, 0, 0, \dots\}$  and the natural involution  $f \mapsto \bar{f}$  inducted by usual complex conjugation.

Let us pass to the infinite dimensional power moment problem.

**Definition 7.1.** We say that  $\tau = (\tau_n)_{n=0}^\infty \in \mathcal{F}'_{\text{fin}}(\mathcal{D}) = \times_{n=0}^\infty (\mathcal{D}'_{\mathbb{C}}) \circledast n$  is a *moment functional on  $(\mathcal{A}(\mathcal{D}), *)$*  if there exists a finite Borel measure  $\mu$  on  $\mathcal{D}'$  such that

$$(7.4) \quad \tau_n = \int_{\mathcal{D}'} x^{\otimes n} d\mu(x), \quad \text{i.e.,} \quad \langle \tau_n, \cdot \rangle = \int_{\mathcal{D}'} \langle x^{\otimes n}, \cdot \rangle d\mu(x), \quad n \in \mathbb{N}_0.$$

Before stating the result note that the Schwartz space  $\mathcal{D}$  can be interpreted as a projective limit of some Sobolev spaces  $D_\sigma$ ,  $\sigma \in \Sigma$ , i.e.,  $\mathcal{D} = \text{pr } \lim_{\sigma \in \Sigma} D_\sigma$ , where  $\Sigma$  denotes some set of indexes, see e.g. [9, 12] for more details.

The following statement follows from [9] (see also [4]).

**Theorem 7.2.** Let  $\tau = (\tau_n)_{n=0}^\infty \in \mathcal{F}'_{\text{fin}}(\mathcal{D})$  and the following two conditions are fulfilled:

- (1)  $\tau$  is  $*$ -positive (more exactly, non-negative) on  $\mathcal{A}(\mathcal{D}) = \mathcal{F}_{\text{fin}}(\mathcal{D})$ , that is

$$(7.5) \quad \tau(f * \bar{f}) = \sum_{j,k=0}^\infty \langle \tau_{j+k}, f_j \odot \bar{f}_k \rangle \geq 0, \quad f \in \mathcal{A}(\mathcal{D}).$$

- (2) there exists an index  $\sigma = \sigma(\tau) \in \Sigma$  such that  $\tau_n \in D_{-\sigma, \mathbb{C}}^{\circledast n} = (D_{\sigma, \mathbb{C}}^{\circledast n})'$  for all  $n \in \mathbb{N}$  and the class

$$(7.6) \quad C\{s_n\}, \quad s_n = \sqrt{\|\tau_{2n}\|_{D_{-\sigma, \mathbb{C}}^{\circledast 2n}}}$$

is quasianalytic (for example,  $s_n = n!$ ).

Then  $\tau$  is a moment functional on  $(\mathcal{A}(\mathcal{D}), *)$  and the measure  $\mu$  in representation (7.4) is uniquely defined.

Conversely, for every moment functional  $\tau$   $(\mathcal{A}(\mathcal{D}), *)$  conditions (7.5) is fulfilled.

The proof of this result is analogous to that of Theorem 4.2. Namely, just as in the case of the one-dimensional moment problem, Theorem 7.2 is a result of the application of the projection spectral theorem to the family  $(J(\varphi))_{\varphi \in \mathcal{D}}$  of “creation” operators

$$(7.7) \quad J(\varphi) : \mathcal{A}(\mathcal{D}) = \mathcal{F}_{\text{fin}}(\mathcal{D}) \rightarrow \mathcal{A}(\mathcal{D}), \quad J(\varphi)f := (0, \varphi, 0, 0, \dots) * f,$$

acting in a Hilbert space  $H_\tau$  associated with the quasiscalar product

$$(7.8) \quad (f, g)_{H_\tau} = \tau(f * \bar{g}), \quad f, g \in \mathcal{F}_{\text{fin}}(\mathcal{D}).$$

Note also that, unlike the one-dimensional case, only conditions (7.5) it is not sufficient for existence of representation (7.4). This is connected with impossibility, in general, to extend a family of commuting Hermitian operators to some family of strongly commuting selfadjoint operators. Condition (7.6) implies that the corresponding Hermitian operators are essential selfadjoint and strongly commuting (therefore, the measure  $\mu$  in Theorem 7.2 is unique). It is possible to give a condition weaker than (7.6) which makes it possible to extend these Hermitian operators to selfadjoint commuting operators, in this case the measure  $\mu$  is not unique. For the corresponding result, see [9], Ch. 5, § 2.

*Remark 7.3.* Using results from [18], it can be shown that the infinite dimensional analog of Theorem 6.6 holds. Namely, a functional  $\tau \in \mathcal{F}'_{\text{fin}}(\mathcal{D})$  admits representation (7.4) with the analytic measure  $\mu \in \mathcal{M}_a(\mathcal{D}')$  (i.e.,  $\int_{\mathcal{D}'} \exp\langle x, \lambda \rangle d\mu(x) < \infty$  for all  $\lambda$  from some neighborhood of  $0 \in \mathcal{D}'_{\mathbb{C}}$ ) if and only if  $\tau$  is  $*$ -positive on  $\mathcal{A}(\mathcal{D}')$  (i.e., (7.5) holds) and  $\tau$  belongs to the space  $\mathcal{F}_-$ . Here  $\mathcal{F}_-$  is defined (similar to  $l^2_-$ ) by the formula

$$\mathcal{F}_- := \text{ind} \lim_{\sigma \in \Sigma, q \in \mathbb{N}} \mathcal{F}(-\sigma, -q) \subset \mathcal{F}'_{\text{fin}}(\mathcal{D}),$$

where  $\mathcal{F}(-\sigma, -q)$  is the so-called Kondratiev-type Fock space,

$$\mathcal{F}(-\sigma, -q) := \left\{ f = (f_n)_{n=0}^\infty \in \mathcal{F}'_{\text{fin}}(\mathcal{D}) \mid \|f\|_{\mathcal{F}(-\sigma, -q)}^2 = \sum_{n=0}^\infty \|f_n\|_{D_{-\sigma, \mathbb{C}}^{\otimes n}}^2 (n!)^{-2} 2^{-qn} < \infty \right\}.$$

**7.2. Moment problem associated with correlation functions.** In this subsection it is convenient for us to interpret the Fock space  $\mathcal{F}(L^2(\mathbb{R}))$  as the space of functions on the space of finite configurations on  $\mathbb{R}$ . Namely, denote by  $\Gamma^{(n)}$  the space of  $n$ -point configuration, i.e.,

$$\Gamma^{(n)} := \{ \eta \subset \mathbb{R} \mid |\eta| = n \},$$

where  $|\cdot|$  means cardinality of a set. As a set,  $\Gamma^{(n)}$  coincides with the symmetrization of

$$\widehat{\mathbb{R}}^n := \{ (t_1, \dots, t_n) \in \mathbb{R}^n \mid t_n \neq t_j \text{ if } k \neq j \}.$$

Hence,  $\Gamma^{(n)}$  inherits the topology of  $\mathbb{R}^n$ . Denote by  $\mathcal{B}(\Gamma^{(n)})$  the corresponding Borel  $\sigma$ -algebra on  $\Gamma^{(n)}$  and introduce a measure  $m^{(n)}$  on  $\mathcal{B}(\Gamma^{(n)})$  as the image of product  $m^{\otimes n}$  of Lebesgue measures  $dm(t) = dt$  on  $\mathcal{B}(\mathbb{R})$ . It is clear that

$$L^2(\Gamma^{(n)}, m^{(n)}) = L^2_{\mathbb{C}, \text{sym}}(\mathbb{R}^n, m^{\otimes n}).$$

The space  $\Gamma_0$  of (all) finite configuration is defined as the topological disjoint union

$$\Gamma_0 = \bigsqcup_{n=0}^\infty \Gamma^{(n)}.$$

Denote by  $\nu$  the *Lebesgue-Poisson measure* on the Borel  $\sigma$ -algebra  $\mathcal{B}(\Gamma_0)$ ,

$$\nu := \sum_{n=0}^\infty \frac{1}{n!} m^{(n)}, \quad m^{(0)}(\emptyset) := 1,$$

and by  $L^2(\Gamma_0, \nu)$  the corresponding  $L^2$ -space. Clearly, the Fock space  $\mathcal{F}(L^2(\mathbb{R}))$  can be identified with the space  $L^2(\Gamma_0, \nu)$  via

$$\mathcal{F}(L^2(\mathbb{R})) \ni (f_n)_{n=0}^\infty \sim \sum_{n=0}^\infty F_n(\cdot) \in L^2(\Gamma_0, \nu),$$

where  $F_0(\emptyset) := f_0$  and

$$F_n(\eta) := \begin{cases} f_n(t_1, \dots, t_n), & \text{if } \eta = (t_1, \dots, t_n) \in \Gamma^{(n)} \\ 0, & \text{otherwise} \end{cases},$$

for all  $n \in \mathbb{N}$ . So,

$$\mathcal{F}(L^2(\mathbb{R})) \cong L^2(\Gamma_0, \nu) = \bigoplus_{n=0}^{\infty} L^2(\Gamma^{(n)}, m^{(n)}).$$

In what follows we won't distinguish between a vector  $f = (f_n)_{n=0}^{\infty}$  from the Fock space  $\mathcal{F}(L^2(\mathbb{R}))$  (and from  $\mathcal{F}_{\text{fin}}(\mathcal{D})$ ) and the corresponding function  $f(\eta)$ ,  $\eta \in \Gamma_0$ , i.e.,

$$\mathcal{F}(L^2(\mathbb{R})) \ni f = (f_n)_{n=0}^{\infty} = f(\eta), \quad \eta \in \Gamma_0, \quad f_n = f \upharpoonright \Gamma^{(n)}.$$

We will need also the space  $\Gamma$  of infinite configurations on  $\mathbb{R}$ , i.e., the space of all locally finite subsets in  $\mathbb{R}$

$$\Gamma := \{\gamma \subset \mathbb{R} \mid |\gamma \cap \Lambda| < \infty \text{ for all compact } \Lambda \subset \mathbb{R}\}.$$

We consider the  $\sigma$ -algebra  $\mathcal{B}(\Gamma)$  as the smallest  $\sigma$ -algebra for which all the mappings  $N_{\Lambda} : \Gamma \rightarrow \mathbb{N}_0$ ,  $N_{\Lambda}(\gamma) := |\gamma \cap \Lambda|$ , are measurable for all bounded Borel set  $\Lambda \subset \mathbb{R}$ . Note that each element  $\gamma \in \Gamma$  can be identified with a generalized function:

$$\Gamma \ni \gamma \mapsto \sum_{t \in \gamma} \delta_t \in \mathcal{D}',$$

where  $\delta_t$  denotes the delta function (Dirac measure) at  $t$ . In this way, the space  $\Gamma$  is embedded in the Schwartz space of distributions  $\mathcal{D}'$ .

Let us pass to a definition of the so-called *Kondratiev–Kuna convolution*  $\star$ . This convolution acts in  $\mathcal{F}_{\text{fin}}(\mathcal{D})$  and we define it by analogy with (7.3) but using instead of the monomials  $\langle x^{\otimes n}, f_n \rangle$  an infinite dimensional analog of the Newton polynomials.

Recall that infinite dimensional Newton polynomials  $\chi_n(x) \in (\mathcal{D}')^{\otimes n}$  are defined as coefficients of the following expansion (cf. (3.7))

$$e^{\langle x, \log(1+\lambda) \rangle} = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \chi_n(x), \lambda^{\otimes n} \rangle, \quad x \in \mathcal{D}', \quad \lambda \in \mathcal{D}_{\mathbb{C}}.$$

It is well known that (see e.g. [13])

$$\langle \chi_n(x), \lambda^{\otimes n} \rangle = \sum_{m=0}^{n-1} (-1)^{n-m-1} \frac{(n-1)!}{m!} \langle \lambda^{n-m}, x \rangle \langle \chi_m(x), \lambda^{\otimes m} \rangle, \quad n \in \mathbb{N}_0,$$

and the mapping

$$(7.9) \quad I_{\chi} : \mathcal{F}_{\text{fin}}(\mathcal{D}) \rightarrow \mathcal{P}(\mathcal{D}'), \quad f = (f_n)_{n=0}^{\infty} \mapsto (I_{\chi} f)(x) := \sum_{n=0}^{\infty} \langle \chi_n(x), f_n \rangle$$

is bijection. Therefore, we can introduce the convolution  $\star$  on  $\mathcal{F}_{\text{fin}}(\mathcal{D})$  by setting

$$f \star g := I_{\chi}^{-1}(I_{\chi} f \cdot I_{\chi} g), \quad f, g \in \mathcal{F}_{\text{fin}}(\mathcal{D}).$$

It follows from e.g. [15, 11] that

$$(f \star g)(\eta) = \sum_{\eta_1 \sqcup \eta_2 \sqcup \eta_3 = \eta} f(\eta_1 \cup \eta_2) g(\eta_2 \cup \eta_3), \quad \eta \in \Gamma_0, \quad f, g \in \mathcal{F}_{\text{fin}}(\mathcal{D}),$$

where the summation is taken over all partitions of  $\eta$  in three parts (parts may be empty). Note that (3.8) is a one-dimensional analog of the latter formula.

So, the space  $\mathcal{F}_{\text{fin}}(\mathcal{D})$  endowed with the product  $\star$  becomes a commutative algebra  $\mathcal{A}(\mathcal{D})$  with unity  $\delta_0 = \{1, 0, 0, \dots\}$  and the natural involution  $f \mapsto \bar{f}$  induced by usual complex conjugation.

Let us pass to the corresponding infinite dimensional moment problem.

**Definition 7.4.** We say that  $\tau = (\tau_n)_{n=0}^\infty \in \mathcal{F}'_{\text{fin}}(\mathcal{D}) = \times_{n=0}^\infty (\mathcal{D}'_{\mathbb{C}})^{\odot n}$  is a *moment functional* on  $(\mathcal{A}(\mathcal{D}), \star)$  if there exists a finite Borel measure  $\mu$  on  $\mathcal{D}'$  such that

$$(7.10) \quad \tau_n = \int_{\mathcal{D}'} \chi_n(x) d\mu(x), \quad \text{i.e.,} \quad \langle \tau_n, \cdot \rangle = \int_{\mathcal{D}'} \langle \chi_n(x), \cdot \rangle d\mu(x), \quad n \in \mathbb{N}_0.$$

From [11] (see also [10, 5]) follows the following result.

**Theorem 7.5.** Let  $\tau = (\tau_n)_{n=0}^\infty \in \mathcal{F}'_{\text{fin}}(\mathcal{D})$  and the following three conditions are fulfilled:

(1) there exists a  $\sigma$ -finite measure  $\rho$  on  $\mathcal{B}(\Gamma_0)$  such that

$$(7.11) \quad \tau(f) = \int_{\Gamma_0} f(\eta) d\rho(\eta), \quad f \in \mathcal{A}(\mathcal{D}) = \mathcal{F}_{\text{fin}}(\mathcal{D}).$$

(2)  $\tau$  is  $\star$ -positive on  $\mathcal{A}(\mathcal{D})$ , that is

$$(7.12) \quad \tau(f \star \bar{f}) = \int_{\Gamma_0} (f \star \bar{f})(\eta) d\rho(\eta) \geq 0, \quad f \in \mathcal{A}(\mathcal{D}).$$

(3) for every compact  $\Lambda \subset \mathbb{R}$  there exists a constant  $C_\Lambda > 0$  such that

$$(7.13) \quad \rho(\Gamma_\Lambda^{(n)}) \leq C_\Lambda^n, \quad n \in \mathbb{N},$$

where  $\Gamma_\Lambda^{(n)} := \{\eta \subset \Lambda \mid |\eta| = n\}$ .

Then  $\tau$  is a moment functional on  $(\mathcal{A}(\mathcal{D}), \star)$  and the measure  $\mu$  in representation (7.10) is uniquely defined.

Conversely, every moment functional  $\tau \in \mathcal{F}'_{\text{fin}}(\mathcal{D})$  is  $\star$ -positive on  $\mathcal{A}(\mathcal{D})$ .

A way of proving this result is similar to that of Theorem 7.2 but in this case instead of (7.7) and (7.8) it is necessary to use the operators

$$J(\varphi) : \mathcal{A}(\mathcal{D}) = \mathcal{F}_{\text{fin}}(\mathcal{D}) \rightarrow \mathcal{A}(\mathcal{D}), \quad J(\varphi)f := (0, \varphi, 0, 0, \dots) \star f,$$

and the quasiscalar product

$$(f, g)_{H_\tau} = \tau(f \star \bar{g}), \quad f, g \in \mathcal{F}_{\text{fin}}(\mathcal{D}).$$

*Remark 7.6.* It is possible to give a condition on  $\rho$  weaker than (7.13) which guarantee the existence of the measure  $\mu$  on  $\mathcal{D}'$  such that (7.10) holds, see [11] for details.

*Remark 7.7.* Let us explain the connection of the moment problem on  $(\mathcal{A}(\mathcal{D}), \star)$  with some essential objects of statistical mechanics.

At first we recall the definitions of these objects, see e.g. [15, 16] for a detailed explanation. The so-called  $K$ -transform maps the functions defined on  $\Gamma_0$  into functions defined on  $\Gamma$  by the formula

$$K : \mathcal{F}_{\text{fin}}(\mathcal{D}) \rightarrow \mathcal{P}(\Gamma), \quad f \mapsto (Kf)(\gamma) := \sum_{\eta \in \gamma} f(\eta),$$

where the summation is taken over all finite subconfigurations of  $\gamma$  (for short  $\eta \in \gamma$ ) and  $\mathcal{P}(\Gamma)$  denotes the space of all continuous polynomials on  $\Gamma \subset \mathcal{D}'$ . Note here that the  $K$ -transform coincides with mapping  $I_\chi$  from (7.9). More exactly,

$$(7.14) \quad (I_\chi f)(\gamma) = (Kf)(\gamma), \quad f \in \mathcal{F}_{\text{fin}}(\mathcal{D}), \quad \gamma \in \Gamma.$$

For a probability measure  $\mu$  on  $\Gamma$ , the so-called correlation measure  $\rho_\mu$  corresponding to  $\mu$  is a  $\sigma$ -finite measure on  $\Gamma_0$  defined by

$$\int_{\Gamma_0} f(\eta) d\rho_\mu(\eta) = \int_{\Gamma} (Kf)(\gamma) d\mu(\gamma), \quad f \in \mathcal{F}_{\text{fin}}(\mathcal{D}).$$

If the measure  $\rho_\mu$  is absolutely continuous with respect to the Lebesgue–Poisson measure  $\nu$  then the corresponding Radon–Nikodym derivative  $k_\mu(\eta) = \frac{d\rho_\mu}{d\nu}(\eta)$ ,  $\eta \in \Gamma_0$ , is called a correlation functional of a measure  $\mu$ . Note that in this case the functions

$$k_{\mu,n}(t_1, \dots, t_n) := \begin{cases} k_\mu(\{t_1, \dots, t_n\}), & \text{if } (t_1, \dots, t_n) \in \widehat{\mathbb{R}}^n \\ 0, & \text{otherwise} \end{cases},$$

are well-known correlation functions of statistical physics, see e.g. [26, 27].

In several applications, a  $\sigma$ -finite measure  $\rho$  on  $\Gamma_0$  appears as a given object and the problem is to show that this  $\rho$  can be seen as a correlation measure for a probability measure on  $\Gamma$ . Due to (7.14) it is easy to see that this problem is a particular case of the moment problem on  $(\mathcal{A}(\mathcal{D}), \star)$ . Namely, a given measure  $\rho$  on  $\Gamma_0$  is a correlation measure for a probability measure  $\mu$  on  $\Gamma$  if and only if the corresponding functional  $\tau$  defined by (7.11) admits representation (7.10) with this  $\mu$  (i.e.,  $\tau$  is a moment functional on the algebra  $(\mathcal{A}, \star)$ ).

It should be noticed that Theorem 7.5 gives the sufficient conditions that  $\rho$  is a correlation measure. More exactly, let  $\rho$  be a given measure on  $\Gamma_0$ . Suppose that the corresponding functional  $\tau$  (defined by (7.11)) satisfies all conditions of Theorem 7.5 and, moreover,

$$\sum_{n=0}^{\infty} 2^n \rho(\Gamma_\Lambda^{(n)}) < \infty$$

for every compact  $|\Lambda| \subset \mathbb{R}$ . Then  $\Gamma$  is the set of full measure  $\mu$  and due to equality (7.14) and representation (7.10) we have

$$\tau(f) = \int_{\Gamma_0} f(\eta) d\rho(\eta) = \int_{\Gamma} (Kf)(\gamma) d\mu(\gamma), \quad f \in \mathcal{F}_{\text{fin}}(\mathcal{D}).$$

So,  $\rho$  is the correlation measure of  $\mu$ . If, moreover,  $\rho$  is absolutely continuous with respect to the Lebesgue–Poisson measure  $\nu$  then the corresponding correlation functional  $k_\mu$  of  $\mu$  coincides with  $\tau$ , i.e.,  $\tau = k_\mu = \frac{d\rho}{d\nu}$ .

*Remark 7.8.* We formulate some problems, the investigation of which are essential for the above described theory:

- (1) To give sufficient conditions on a functional  $\tau = (\tau_n)_{n=0}^{\infty} \in \mathcal{F}'_{\text{fin}}(\mathcal{D})$  which would guarantee the existence of representation (7.10), i.e., to give conditions for validity of Theorem 7.5 different from (7.11) or (7.13).
- (2) To prove an infinite dimensional analog of Theorem 6.9 and as a consequence to get an analog of Theorem 6.10 for the classical Bogoliubov functional.
- (3) To investigate the situation when a measure  $\rho$  on  $\Gamma_0$  is a correlation measure for a probability measure on  $\Gamma$ . More exactly, to give sufficient conditions on  $\rho$  which would guarantee the existence of representation (7.10) for the functional  $\tau \in \mathcal{F}'_{\text{fin}}(\mathcal{D})$  determined by  $\rho$  and, moreover, these conditions should assure that the measure  $\mu$  from (7.10) is concentrated on  $\Gamma$ .

*Acknowledgments.* I would like to thank Prof. Yu. M. Berezansky and Dr. K. Yusenko for valuable comments.

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Received 11/07/2011