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On the Faber Polynomials of the Univalent Functions of Class $\sum$

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With the help of the ordinary Bell polynomials we find the simplest combinatorial form for the coefficients of the Faber polynomials $\phi_n(t)$ expanded in powers of $t-\alpha_0$. We also find a remarkable inequality between the moduli $|\phi_n'(t)|$ for $|t|\leqslant 1$ and the Fibonacci numbers u_{2n} with even subscripts.

1. ORDINARY BELL POLYNOMIALS

For arbitrary $x_1, x_2, \ldots$, the ordinary Bell polynomials D_{nk} are generated by the formal expansions (see Comtet [1, p. 136, Remark])

$$\left(\sum_{m=1}^{\infty} x_m z^m\right)^k \equiv \sum_{n=k}^{\infty} D_{nk} z^n, \qquad k=1,2,\ldots. \tag{1}$$

If we apply the Faà di Bruno "precise formula" for the nth derivative of composite functions, developed in our paper [2, pp. 82–84, Section 2], to the composite function

$$\left(\sum_{m=1}^{\infty} x_m z^m\right)^k \equiv t^k \circ \left(\sum_{m=1}^{\infty} x_m z^m\right),$$

then we find that D_{nk}, $1\leqslant k\leqslant n$, $n\geqslant 1$, in (1) are homogeneous and isobaric polynomials of degree k and weight n with respect to $x_1, \ldots, x_{n-k+1}$ with integral coefficients, and they have the explicit form

$$D_{nk} \equiv D_{nk}(x_1, \ldots, x_{n-k+1}) \equiv \sum \frac{k!(x_1)^{\nu_1}\cdots(x_{n-k+1})^{\nu_{n-k+1}}}{\nu_1!\cdots\nu_{n-k+1}!}, \tag{2}$$

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where the sum is taken over all nonnegative integers $v_1, ..., v_{n-k+1}$ satisfying

$$v_1 + v_2 + \cdots + v_{n-k+1} = k, \qquad v_1 + 2v_2 + \cdots + (n-k+1)\, v_{n-k+1} = n. \tag{3}$$

For $k=0$ $(n \geqslant 0)$ and $0 \leqslant n < k$ $(k \geqslant 1)$, we set

$$D_{n0} \equiv D_{n0}(x_1, ..., x_{n+1}) \equiv 0, \qquad n = 1, 2, ...,$$
$$D_{00} \equiv D_{00}(x_1) \equiv 1, \tag{4}$$
$$D_{nk} \equiv 0, \qquad 0 \leqslant n < k, \qquad k \geqslant 1.$$

In our papers [2–8] we have used the polynomials

$$C_{nk}(x_1, ..., x_{n-k+1}) \equiv \frac{1}{k!} D_{nk}(x_1, ..., x_{n-k+1})$$

in the theory of univalent functions (see also Harmelin [9]). The polynomials D_{nk} satisfy the recurrence relations (see [2, 3, 6, 8, 9])

$$D_{nk} = \sum_{\mu=1}^{n-k+1} x_\mu D_{n-\mu, k-1}, \qquad 1 \leqslant k \leqslant n,\ n \geqslant 1,\ D_{n0} = 0,\ D_{00} = 1, \tag{5}$$

and

$$nD_{nk} = k \sum_{\mu=1}^{n-k+1} \mu x_\mu D_{n-\mu, k-1}, \qquad 1 \leqslant k \leqslant n,\ n \geqslant 1,\ D_{n0} = 0,\ D_{00} = 1. \tag{6}$$

The first and the last polynomials are

$$D_{n1} = x_n, \qquad D_{nn} = x_1^n, \qquad n \geqslant 1. \tag{7}$$

For $1 \leqslant n \leqslant 5$ from (5) and (7) we obtain the following short table (see in [1, p. 309], a longer table for $1 \leqslant n \leqslant 10$)

$$\begin{aligned}
&D_{11} = x_1; && D_{21} = x_2, && D_{22} = x_1^2;\\
&D_{31} = x_3, && D_{32} = 2x_1 x_2, && D_{33} = x_1^3;\\
&D_{41} = x_4, && D_{42} = 2x_1 x_3 + x_2^2, && D_{43} = 3x_1^2 x_2,\\
&D_{44} = x_1^4; && D_{51} = x_5, && D_{52} = 2x_1 x_4 + 2x_2 x_3,\\
&D_{53} = 3x_1^2 x_3 + 3x_1 x_2^2, && D_{54} = 4x_1^3 x_2, && D_{55} = x_1^5.
\end{aligned} \tag{8}$$

Another application of the Faà di Bruno precise formula, this time to the composite function

$$\left(1+\sum_{n=1}^{\infty} x_n z^n\right)^{\lambda} \equiv t^{\lambda} \circ \left(1+\sum_{n=1}^{\infty} x_n z^n\right), \qquad 1^{\lambda}=1,$$

for an arbitrary complex number λ, yields the formal expansion (see [2, p. 84, Formulas (25)–(26)])

$$\left(1+\sum_{n=1}^{\infty} x_n z^n\right)^{\lambda} = 1+\sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} \binom{\lambda}{k} D_{nk}(x_1, ..., x_{n-k+1}). \tag{9}$$

From (2) and (3) it is clear that (compare with [1, p. 136, Relations [3*l*] and [3*l*′]], and [9, Relation (1.11)])

$$\begin{aligned} D_{nk}&(x_1, ..., x_{n-k+1}) \\ &= \sum_{\nu=0}^{k} \binom{k}{\nu} x_1^{\nu} D_{n-\nu,k-\nu}(x_1=0, x_2, ..., x_{n-k+1}) \\ &= \sum_{\nu=\max(0,2k-n)}^{k} \binom{k}{\nu} x_1^{\nu} D_{n-k,k-\nu}(x_2, ..., x_{n-2k+\nu+2}) \end{aligned} \tag{10}$$

for $1 \leqslant k \leqslant n$, $n \geqslant 1$. (According to (4) relation (10) is true for $k=0$ and $n \geqslant 0$ as well.)

2. Faber Polynomials

Let $\sum$ denote the class of functions

$$F(z)=z+\sum_{n=0}^{\infty} \alpha_n z^{-n}, \tag{11}$$

which are meromorphic and univalent for $|z|>1$, and let S denote the class of functions

$$f(z)=\sum_{n=1}^{\infty} a_n z^n, \qquad a_1=1, \tag{12}$$

which are analytic and univalent for $|z|<1$, i.e., the functions

$$F(z)=\frac{1}{f(1/z)} \in \sum, \qquad |z|>1. \tag{13}$$

The Faber polynomials $\phi_n(t)$ of degrees $n=1, 2, ...$ with respect to $F(z)$ are generated by the Taylor expansion (see Pommerenke [10, p. 57] or [3])

$$\log \frac{F(z)-t}{z} = -\sum_{n=1}^{\infty} \frac{1}{n} \phi_n(t) z^{-n} \tag{14}$$

for an arbitrary complex number t and sufficiently large $|z|>1$. Differentiation of (14) with respect to z and (11) give the recurrence relation (compare with [10, p. 57])

$$\phi_{n+1}(t) = (t-\alpha_0)\,\phi_n(t) - \sum_{s=0}^{n-1} \alpha_{n-s}\phi_s(t) - n\alpha_n,$$
$$n=1, 2, ..., \qquad \phi_0(t)=1, \qquad \phi_1(t)=t-\alpha_0. \tag{15}$$

By aid of (15) the following polynomials $\phi_2(t)$, $\phi_3(t)$, $\phi_4(t)$, ... can be found successively (see this table in [10, p. 57]). In our paper [3], with the help of the ordinary Bell polynomials in α_0, α_1, ..., we found simple explicit formulas for the coefficients of the Faber polynomials $\phi_n(t)$ expanded in powers of t. Further, Johnston [11, p. 1236, Theorem 1], found explicit cumbersome formulas for the coefficients of the Faber polynomials expanded in powers of $t-\alpha_0$. In this section, with the help of formulas (5)–(6) and (9)–(10), we obtain the simplest combinatorial form for the coefficients of the Faber polynomials $\phi_n(t)$ expanded in powers of $t-\alpha_0$.

THEOREM 1. *The Faber polynomials $\phi_n(t)$ expanded in powers of $t-\alpha_0$ have the explicit form*

$$\phi_n(t) = c_0^{(n)} + \sum_{v=1}^{n} c_v^{(n)}(t-\alpha_0)^v, \qquad n=1, 2, ..., \tag{16}$$

where

$$c_v^{(n)} = n \sum_{k=0}^{[(n-v)/2]} \frac{(-1)^k}{k+v} \binom{k+v}{k} D_{n-k-v,k}(\alpha_1, ..., \alpha_{n-2k-v+1}) \tag{17}$$

for $v=1, ..., n$, and $c_0^{(1)}=0$ and

$$c_0^{(n)} = n \sum_{k=1}^{[n/2]} \frac{(-1)^k}{k} D_{n-k,k}(\alpha_1, ..., \alpha_{n-2k+1}) \tag{18}$$

for $n=2, 3, ...$.

Remark. For an arbitrary real number x, the symbol $[x]$ denotes the greatest integer less than or equal to x.

Proof. Differentiation of (14) with respect to t gives

$$\frac{z}{F(z)-t}=1+\sum_{n=1}^{\infty}\frac{1}{n+1}\phi'_{n+1}(t)\,z^{-n}. \tag{19}$$

On the other hand, from (11) and (9) we have

$$\begin{aligned}\frac{z}{F(z)-t}&=\left[1+(\alpha_0-t)\,z^{-1}+\sum_{n=2}^{\infty}\alpha_{n-1}z^{-n}\right]^{-1}\\&=1+\sum_{n=1}^{\infty}z^{-n}\left[\sum_{k=0}^{n-1}(-1)^k D_{nk}(\alpha_0-t,\alpha_1,...,\alpha_{n-k})+(t-\alpha_0)^n\right],\end{aligned} \tag{20}$$

having in mind (7) and (4). Equating the coefficients of z^{-n} in (19) and (20), we obtain

$$\frac{1}{n+1}\phi'_{n+1}(t)=\sum_{k=0}^{n-1}(-1)^k D_{nk}(\alpha_0-t,\alpha_1,...,\alpha_{n-k})+(t-\alpha_0)^n \tag{21}$$

for $n=1,2,...$. From (10) we have

$$\begin{aligned}&D_{nk}(\alpha_0-t,\alpha_1,...,\alpha_{n-k})\\&\quad=\sum_{v=0}^{k}\binom{k}{v}(\alpha_0-t)^v D_{n-k,k-v}(\alpha_1,...,\alpha_{n-2k+v+1})\end{aligned} \tag{22}$$

for $0\leqslant k\leqslant n-1$, $n\geqslant 1$, where if $2k-n>0$, the terms in the sum are replaced by zeroes for $0\leqslant v<2k-n$. Thus (21) and (22) yield

$$\begin{aligned}\frac{1}{n+1}\phi'_{n+1}(t)=&\sum_{v=0}^{n}(t-\alpha_0)^v\\&\cdot\sum_{k=0}^{[(n-v)/2]}(-1)^k\binom{k+v}{v}D_{n-k-v,v}(\alpha_1,...,\alpha_{n-2k-v+1})\end{aligned} \tag{23}$$

for $n=0,1,2,...$, keeping in mind (4). If we integrate (23) with respect to t from α_0 to t, we obtain the formulas (16) and (17), where $c_0^{(n)}\equiv\phi_n(\alpha_0)$, $n=1,2,...$, must be found. For our purpose, from (15) for $t=\alpha_0$ we obtain the recurrence relation

$$\begin{aligned}c_0^{(n+1)}&=-\sum_{s=1}^{n-1}\alpha_{n-s}c_0^{(s)}-(n+1)\,\alpha_n,\\&n=2,3,...,\qquad c_0^{(1)}=0,\qquad c_0^{(2)}=-2\alpha_1.\end{aligned} \tag{24}$$

For $n=2$ from (24) we obtain $c_0^{(3)}=-3\alpha_2$; i.e., the formula (18) is true for $n=2$ and $n=3$. If we assume that the formula (18) is true for any integer $n\geqslant 3$, then from (24), (18), (4), (5), and (6) it follows that

$$\begin{aligned}
c_0^{(n+1)} &= \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} \sum_{s=k}^{n-1} s\alpha_{n-s} D_{s-k,k}(\alpha_1, \dots) - (n+1)\,\alpha_n \\
&= \sum_{k=1}^{[(n-1)/2]} \frac{(-1)^{k+1}}{k} \sum_{s=2k}^{n-1} s\alpha_{n-s} D_{s-k,k}(\alpha_1, \dots) - (n+1)\,\alpha_n \\
&= \sum_{k=1}^{[(n-1)/2]} \frac{(-1)^{k+1}}{k} \sum_{\mu=1}^{n-2k} (n-\mu)\,\alpha_\mu D_{n-k-\mu,k} - (n+1)\,\alpha_n \\
&= \sum_{k=1}^{[(n-1)/2]} \frac{(-1)^{k+1}}{k} \left(n - \frac{n-k}{k+1}\right) D_{n-k,k+1} - (n+1)\,\alpha_n \\
&= (n+1) \sum_{k=1}^{[(n+1)/2]} \frac{(-1)^k}{k} D_{n+1-k,k}(\alpha_1, \dots, \alpha_{n-2k+2}).
\end{aligned} \tag{25}$$

From the comparison of (25) and (18) we conclude that the formula (18) is true for any integer $n\geqslant 2$.

This completes the proof of Theorem 1.

For $\nu=n, n-1, n-2, n-3, n-4, n-5, n-6, n-7, \dots$ from (17), (18), (8), and (7), we obtain the table

$$c_n^{(n)}=1,\ n\geqslant 1; \qquad c_{n-1}^{(n)}=0,\ n\geqslant 1;$$

$$c_{n-2}^{(n)}=-n\alpha_1,\ n\geqslant 2; \qquad c_{n-3}^{(n)}=-n\alpha_2,\ n\geqslant 3;$$

$$c_{n-4}^{(n)}=\frac{n(n-3)}{2}\alpha_1^2-n\alpha_3, \qquad n\geqslant 4;$$

$$c_{n-5}^{(n)}=n[(n-4)\,\alpha_1\alpha_2-\alpha_4], \qquad n\geqslant 5;$$

$$c_{n-6}^{(n)}=-\frac{n(n-4)(n-5)}{6}\alpha_1^3 + \frac{n(n-5)}{2}(2\alpha_1\alpha_3+\alpha_2^2)-n\alpha_5, \qquad n\geqslant 6;$$

$$c_{n-7}^{(n)}=-n\left[\frac{(n-5)(n-6)}{2}\alpha_1^2\alpha_2-(n-6)(\alpha_1\alpha_4+\alpha_2\alpha_3)+\alpha_6\right], \qquad n\geqslant 7;$$

. . . .

For $0\leqslant\nu\leqslant n-2$, $n\geqslant 3$, the formulas (17) and (18) can be united into one formula:

$$c_\nu^{(n)} = n \sum_{k=1}^{[(n-\nu)/2]} \frac{(-1)^k}{k+\nu} \binom{k+\nu}{k} D_{n-k-\nu,k}(\alpha_1, \dots, \alpha_{n-2k-\nu+1}). \tag{26}$$

Evidently, in comparison with the Johnston results [11, p. 1236, Formulas (7) and (8)], our formula (26) is simpler.

3. An Inequality for the First Derivatives of the Faber Polynomials

The aim of this section is the following

Theorem 2. *Let the functions* $F(z) \in \sum$ *be determined by* (12) *and* (13). *Then in the disc* $|t| \leqslant 1$ *the derivatives* $\phi_n'(t)$ *of the Faber polynomials* $\phi_n(t)$ *of* $F(z)$, *determined by* (14), *satisfy the sharp inequalities*

$$|\phi_n'(t)| \leqslant n u_{2n}, \qquad n = 1, 2, ..., \tag{27}$$

where u_{2n} *is the* $2n$th *Fibonacci number, i.e.,*

$$u_{2n} = \frac{(3+\sqrt{5})^n - (3-\sqrt{5})^n}{2^n \sqrt{5}}, \qquad n = 1, 2, ... \tag{28}$$

For $n \geqslant 2$, *the equality in* (27) *holds only for the Koebe function*

$$f(z) = \frac{z}{(1-\varepsilon z)^2} = \sum_{n=1}^{\infty} n\varepsilon^{n-1} z^n \in S, \qquad |\varepsilon| = 1, \tag{29}$$

at the point $t = \varepsilon$.

Proof. In [3, p. 434, Theorem 3], we found that, in terms of the coefficients a_n in (12), the Faber polynomials $\phi_n(t)$ have the form

$$\begin{aligned} \phi_n(t) &= nb_n + n \sum_{k=1}^{n} \frac{t^k}{k} D_{nk}(a_1, ..., a_{n-k+1}), \qquad n \geqslant 1, \\ b_n &= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} D_{nk}(a_2, ..., a_{n-k+2}), \qquad n \geqslant 1. \end{aligned} \tag{30}$$

In addition, Louis de Branges [12] proved the Bieberbach conjecture for the functions (12) of the class S that

$$|a_n| \leqslant n, \qquad n = 2, 3, ..., \tag{31}$$

where for some n the equality holds only for the Koebe function (29). Thus, from (30) and (31) with the help of (2) and (3), we obtain the inequalities

$$|\phi_n'(t)| \leqslant n \sum_{k=1}^{n} D_{nk}(1, 2, ..., n-k+1) \tag{32}$$

for $n=1, 2, \ldots$ and $|t| \leqslant 1$, where for $n=2, 3, \ldots$ the equality holds only for the Koebe function (29) at the point $t=\varepsilon$ for which

$$\phi_n'(\varepsilon) = n\varepsilon^{n-1} \sum_{k=1}^{n} D_{nk}(1, 2, \ldots, n-k+1), \qquad |\varepsilon| = 1. \tag{33}$$

Further, in our papers [5–7] we found the equations

$$D_{nk}(1, 2, \ldots, n-k+1) = \binom{n+k-1}{n-k}, \qquad 1 \leqslant k \leqslant n, \qquad n \geqslant 1, \tag{34}$$

and

$$\sum_{k=1}^{n} \binom{n+k-1}{n-k} = \frac{(3+\sqrt{5})^n - (3-\sqrt{5})^n}{2^n \sqrt{5}}, \qquad n \geqslant 1, \tag{35}$$

respectively. On the other hand, for the Fibonacci numbers $u_1 = u_2 = 1$, $u_n = u_{n-2} + u_{n-1}$, $n \geqslant 3$, the well-known formula

$$u_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \sqrt{5}}, \qquad n = 1, 2, \ldots, \tag{36}$$

holds. Therefore, from (32)–(36) we obtain the relations (27)–(28). (Hence the expansion

$$\frac{1}{1-\varepsilon f(z)} = 1 + \sum_{n=1}^{\infty} u_{2n} \varepsilon^n z^n, \qquad |z| < \frac{3-\sqrt{5}}{2},$$

where $f(z)$ is the Koebe function (29), generates the Fibonacci numbers u_{2n} with even subscripts.)

This completes the proof of Theorem 2.

COROLLARY. *Under the conditions and notations of Theorem 2, for arbitrary complex numbers t_1 and t_2 with $t_1 \neq t_2$ and $|t_{1,2}| \leqslant 1$, we have the precise inequalities*

$$\left| \frac{\phi_n(t_1) - \phi_n(t_2)}{t_1 - t_2} \right| < n u_{2n}, \qquad n = 2, 3, \ldots. \tag{37}$$

Proof. The inequalities (37) follow from the relations

$$\left| \frac{\phi_n(t_2) - \phi_n(t_1)}{t_2 - t_1} \right| = \left| \int_0^1 \phi_n'[t_1 + (t_2 - t_1)\tau] \, d\tau \right| < n u_{2n}$$

for $n = 2, 3, \ldots$, having in mind (27).

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