# Banded symmetric Toeplitz matrices: where linear algebra borrows from difference equations

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A Toeplitz matrix, named after the German mathematician Otto Toeplitz (1881-1940), is of the form $T = [t_{r-s}]_{r,s=0}^{n-1}$. (It's ok, and convenient for Toeplitz matrices, to number rows and columns from 0 to $n-1$.) A symmetric Toeplitz matrix is of the form $T_n = [t_{|r-s|}]_{r,s=0}^{n-1}$. For example,

$$T_5 = \begin{bmatrix} t_0 & t_1 & t_2 & t_3 & t_4 \\ t_1 & t_0 & t_1 & t_2 & t_3 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ t_3 & t_2 & t_1 & t_0 & t_1 \\ t_4 & t_3 & t_2 & t_1 & t_0 \end{bmatrix}$$

is a $5 \times 5$ symmetric Toeplitz matrix. We will assume that $t_0, t_1, \ldots, t_{k-1}$ are all real numbers. From your linear algebra course you know that a symmetric matrix with real entries has real eigenvalues and is always diagonalizable; that is, $T_n$ has real eigenvalues and $n$ linearly independent eigenvectors.

A Toeplitz matrix is said to be <u>banded</u> if there is an integer $d < n-1$ such that $t_\ell = 0$ if $\ell > d$. In this case, we say that $T$ has bandwidth $d$. For example,

$$T_5 = \begin{bmatrix} t_0 & t_1 & t_2 & 0 & 0 \\ t_1 & t_0 & t_1 & t_2 & 0 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ 0 & t_2 & t_1 & t_0 & t_1 \\ 0 & 0 & t_2 & t_1 & t_0 \end{bmatrix}$$

is a $5 \times 5$ banded symmetric Toeplitz matrix with bandwidth 2.

The eigenvalue problem for very large ($n$ can be in the thousands!) symmetric banded Toeplitz matrices pops up in many statistical problems. In your linear algebra course you learned to solve the eigenvalue problem for a matrix $A$ by factoring its characteristic polynomial

$$p(\lambda) = \det(A - \lambda I).$$

Sorry, that's impossible for big matrices. In general there is no computationally useful way to obtain the characteristic polynomial of a large symmetric matrix (or any other large matrix). All methods for finding a single eigenvalue of an arbitrary $n \times n$ symmetric matrix carry a computational cost (it's called <u>complexity</u>) proportional to $n^3$. So, if you double the size of the matrix you make the problem of obtaining a single eigenvalue eight times more difficult. However, the situation is different for banded symmetric Toeplitz matrices.

Let's start with the simplest case: $d = 1$.

$$T_n = \begin{bmatrix} t_0 & t_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ t_1 & t_0 & t_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & t_1 & t_0 & t_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t_1 & t_0 & t_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & t_1 & t_0 \end{bmatrix}_{n \times n}$$

with $t_1 \neq 0$. This is a symmetric tridiagonal Toeplitz matrix. A vector

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

is a $\lambda$-eigenvector of $T_n$ if and only if

$$\begin{aligned} t_0 x_0 + t_1 x_1 &= \lambda x_0 \\ t_1 x_{j-1} + t_1 x_0 + t_1 x_{j+1} &= \lambda x_j, \quad 1 \leq j \leq n-1, \\ t_1 x_{n-1} + t_0 x_n &= \lambda x_n \end{aligned}$$

which we can rewrite as

$$t_1 x_{j-1} + (t_0 - \lambda) x_j + t_1 x_{j+1} = 0, \quad 0 \leq j \leq n,$$

(a homogeneous difference equation) if we define $x_0 = x_{n+1} = 0$ (boundary conditions).

The characteristic polynomial of the difference equation

$$t_1 x_{j-1} + (t_0 - \lambda)x_j + t_1 x_{j+1} = 0 \qquad \text{(DE)}$$

is

$$p(z;\lambda) = t_1 + (t_0 - \lambda)z + t_1 z^2 = t_1(z - z_1(\lambda))(z - z_2(\lambda));$$

thus, $p(z_1(\lambda)) = p(z_2(\lambda)) = 0$. (We don't know $z_1(\lambda)$ and $z_2(\lambda)$ yet; be patient.) If we let

$$x_j = c_1 z_1^j(\lambda) + c_2 z_2^j(\lambda)$$

where $c_1$ and $c_2$ are arbitrary constants, then the left side of (DE) equals

$$c_1 z_1^{j-1} p(z_1(\lambda)) + c_2 z_2^{j-1} p(z_2(\lambda)) = 0$$

for any choice of $c_1$ and $c_2$. Now let's work on the boundary conditions. Since $x_0 = 0$ if and only if $c_2 = -c_1$,

$$x_j = c(z_1^j(\lambda) - z_2^j(\lambda)).$$

Now $x_{n+1} = 0$ if and only if $(z_1(\lambda)/z_2(\lambda))^{n+1} = 1$, which is true if and only if

$$z_1(\lambda) = \gamma_q \exp\left(\frac{q\pi i}{n+1}\right) \quad \text{and} \quad z_2(\lambda) = \gamma_q \exp\left(\frac{-q\pi i}{n+1}\right),$$

where $\exp(i\theta) = e^{i\theta} = cos\theta + i\sin\theta$, $q = 1, \ldots, n$ and $\gamma_q$ is to be determined. (Letting $q = 0$ does not produce an eigenvector because if $z_1(\lambda) = z_2(\lambda)$) then $x_j = 0$ for all $j$).

Taking note that are $q$ possibilities, the eigenvectors have the form

$$x_q = \begin{bmatrix} x_{0q} \\ x_{1q} \\ \vdots \\ x_{n-1,q} \end{bmatrix}$$

where

$$x_{jq} = c(z_1^j(\lambda_q) - z_2^j(\lambda_q)),$$

$$z_{1q}(\lambda) = \gamma_q \exp\left(\frac{q\pi i}{n+1}\right), \quad \text{and} \quad z_{2q}(\lambda) = \gamma_q \exp\left(\frac{-q\pi i}{n+1}\right),$$

so

$$x_{jq} = c\left(\exp\frac{jq\pi i}{n+1} - \exp\frac{jq\pi i}{n+1}\right) = 2ci \sin\frac{jq\pi}{n+1}.$$

Since $c$ is arbitrary, it makes sense to let $c = 1/2\gamma_q i$. (Don't worry that maybe $\gamma_q = 0$; we'll see that it isn't.) Then

$$x_{jq} = \sin\frac{jq\pi}{n+1}, \quad 0 \le j \le n-1.$$

ALL SYMMETRIC TRIDIAGONAL TOEPLITZ MATRICES HAVE THE SAME EIGENVECTORS!

Now let's find $\lambda_q$, the eigenvalue associated with $q$.

$$t_1 + (t_0 - \lambda_q) + t_1 z^2 = t_1(z - z_1(\lambda))(z_2 - z_2(\lambda))$$

which equals

$$t_1 \left( z^2 - (z_1(\lambda) + z_2(\lambda))z + z_1(\lambda) z_2(\lambda) \right)$$

Since

$$z_1(\lambda) = \gamma_q \exp\left(\frac{q\pi i}{n+1}\right) \quad \text{and} \quad z_2(\lambda) = \gamma_q \exp\left(\frac{-q\pi i}{n+1}\right),$$

$$t_1 + (t_0 - \lambda_q)z + t_1 z^2 = t_1 \left( z^2 - 2\gamma_q z \cos\left(\frac{q\pi}{n+1}\right) + \gamma_q^2 \right).$$

Equating coefficients on the two sides yields $\gamma_q = 1$ and

$$\lambda_q = t_0 + 2t_1 \cos\left(\frac{q\pi}{n+1}\right), \quad 1 \le q \le n.$$

Now suppose $d > 1$. To see where we're going, a nonzero vector

$$x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

is a $\lambda$-eigenvector of

$$T_5 = \begin{bmatrix} t_0 & t_1 & t_2 & 0 & 0 \\ t_1 & t_0 & t_1 & t_2 & 0 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ 0 & t_2 & t_1 & t_0 & t_1 \\ 0 & 0 & t_2 & t_1 & t_0 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} t_0 & t_1 & t_2 & 0 & 0 \\ t_1 & t_0 & t_1 & t_2 & 0 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ 0 & t_2 & t_1 & t_0 & t_1 \\ 0 & 0 & t_2 & t_1 & t_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \lambda \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

or, equivalently,

$$\begin{aligned} t_0x_0 + t_1x_1 + t_2x_2 \quad &= \quad \lambda x_0 \\ t_1x_0 + t_0x_1 + t_1x_2 + t_2x_3 \quad &= \quad \lambda x_1 \\ t_2x_0 + t_1x_1 + t_2x_2 + t_3x_3 + t_4x_4 \quad &= \quad \lambda x_2 \\ t_2x_1 + t_1x_2 + t_0x_3 + t_1x_4 \quad &= \quad \lambda x_3 \\ t_2x_2 + t_1x_3 + t_0x_4 \quad &= \quad \lambda x_4, \end{aligned}$$

(repeated for clarity)

$$
\begin{aligned}
t_0x_0 + t_1x_1 + t_2x_2 &= \lambda x_0 \\
t_1x_0 + t_0x_1 + t_1x_2 + t_2x_3 &= \lambda x_1 \\
t_2x_0 + t_1x_1 + t_2x_2 + t_3x_3 + t_4x_4 &= \lambda x_2 \\
t_2x_1 + t_1x_2 + t_0x_3 + t_1x_4 &= \lambda x_3 \\
t_2x_2 + t_1x_3 + t_0x_4 &= \lambda x_4,
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
t_2x_{-2} + t_1x_{-1} + t_0x_0 + t_1x_1 + t_2x_2 &= \lambda x_0 \\
t_2x_{-1} + t_1x_0 + t_0x_1 + t_1x_2 + t_2x_3 &= \lambda x_1 \\
t_2x_0 + t_1x_1 + t_0x_2 + t_1x_3 + t_2x_4 &= \lambda x_2 \\
t_2x_1 + t_1x_2 + t_0x_3 + t_1x_4 + t_2x_5 &= \lambda x_3 \\
t_2x_2 + t_1x_3 + t_0x_4 + t_1x_5 + t_2x_6 &= \lambda x_4
\end{aligned}
$$

if we impose the boundary conditions

$$x_{-2} = x_{-1} = x_5 = x_6 = 0.$$

Better yet,

$$\sum_{\ell=-2}^{2} t_{|\ell|}x_{\ell+r} = \lambda x_r, \quad 0 \le r \le 4.$$

(repeated for clarity)

$$\sum_{\ell=-2}^{2} t_{|\ell|} x_{\ell+r} = \lambda x_r, \quad 0 \le r \le 4,$$

with boundary conditions

$$x_{-2} = x_{-1} = x_5 = x_6 = 0.$$

For the general case where $T_n = [t_{|r-s|}]_{r,s=0}^{n-1}$ with $t_\ell = 0$ if $\ell > d$, the eigenvalue problem can be written as

$$\sum_{\ell=-d}^{d} t_{|\ell|} x_{\ell+r} = \lambda x_r, \quad 0 \le r \le n-1, \tag{DE}$$

subject to

$$x_r = 0, \quad -d \le r \le -1, \quad n \le r \le n+d-1. \tag{BC}$$

Eqn. (DE) is a difference equation and the conditions in (BC) are called boundary conditions. Obviously, (DE) and (BC) both hold for any $\lambda$ if $x_r = 0$ for $-d \le r \le n+d-1$. However, that's not interesting, since an eigenvector must be nonzero. Finding the values of $\lambda$ for which (DE) has nonzero solutions that satisfy (BC) is a boundary value problem.

The characteristic polynomial of the difference equation

$$\sum_{\ell=-d}^{d} t_{|\ell|} x_{\ell+r} = \lambda x_r, \quad 0 \le r \le n-1, \tag{DE}$$

is

$$P(z,\lambda) = \sum_{\ell=-d}^{d} t_{|\ell|} z^{\ell} - \lambda.$$

The zeros of $P(z,\lambda)$ are continuous functions of $\lambda$ and, since $P(z,\lambda) = P(1/z,\lambda)$, they occur in reciprocal pairs

$$(z_1(\lambda), 1/z_1(\lambda)), \dots, (z_d(\lambda), 1/z_d(\lambda)).$$

It can be shown (don't you hate that?) that these zeros are distinct except for at most finitely many "bad values" of $\lambda$. We'll assume that none of these bad values are actually eigenvalues of $T_n$. (This is a pretty safe bet.) Then (DE) holds if

$$x_r = \sum_{s=1}^{d} \left( a_s z_s^r(\lambda) + b_s z_s^{-r}(\lambda) \right), \quad -d \le r \le n+d-1,$$

where $a_1, \dots, a_d$ and $b_1, \dots, b_d$ are arbitrary constants.

PROOF. Recall that $P(z,\lambda) = \sum_{\ell=-d}^{d} t_{|\ell|} z^{\ell} - \lambda$. If

$$x_r = \sum_{s=1}^{d} \left(a_s z_s^r(\lambda) + b_s z_s^{-r}(\lambda)\right), \ -d \le r \le n+d-1, \ \text{ then}$$

$$\begin{aligned}
\sum_{\ell=-d}^{d} t_{|\ell|} x_{\ell+r} - \lambda x_r &= \sum_{\ell=-d}^{d} t_{|\ell|} \sum_{s=1}^{d} \left(a_s z_s^{\ell+r} + b_s z_s^{-\ell-r}\right) \\
&\quad -\lambda \sum_{s=1}^{d} \left(a_s z_s^r(\lambda) + b_s z_s^{-r}(\lambda)\right) \\
&= \sum_{s=1}^{d} a_s z_s^r(\lambda) \left( \sum_{\ell=-d}^{d} t_{|\ell|} z_s^{\ell}(\lambda) - \lambda \right) \\
&+ \sum_{s=1}^{d} b_s z_s^{-r}(\lambda) \left( \sum_{\ell=-d}^{d} t_{|\ell|} z_s^{-\ell}(\lambda) - \lambda \right)
\end{aligned}$$

$$= \sum_{s=1}^{d} \left(a_s z_s^r(\lambda) P(z_s(\lambda),\lambda) + b_s z_s^{-r}(\lambda) P(1/z_s(\lambda),\lambda)\right) = 0$$

(look at the top of this page) for all $r$. Note that $a_1, \ldots, a_d$ and $b_1, \ldots, b_d$ are completely arbitrary up to this point.

Now we must choose them to satisfy the boundary conditions

$$x_r = 0, \quad -d \le r \le -1, \quad n \le r \le n+d-1; \quad \text{(BC)}$$

that is,

$$\sum_{s=1}^{d} \left( a_s z_s^r(\lambda) + b_s z_s^{-r}(\lambda) \right) = 0,$$

for $-d \le r \le -1$ and $n \le r \le n+d-1$. For example, if $d = 2$, we must have

$$\begin{bmatrix} z_1^{-1}(\lambda) & z_2^{-1}(\lambda) & z_1(\lambda) & z_2(\lambda) \\ z_1^{-2}(\lambda) & z_2^{-2}(\lambda) & z_1^2(\lambda) & z_2^2(\lambda) \\ z_1^{n}(\lambda) & z_2^{n}(\lambda) & z_1^{-n}(\lambda) & z_2^{-n}(\lambda) \\ z_1^{n+1}(\lambda) & z_2^{n+1}(\lambda) & z_1^{-n-1}(\lambda) & z_2^{-n-1}(\lambda) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

For clarity,

$$\begin{bmatrix} z_1^{-1}(\lambda) & z_2^{-1}(\lambda) & z_1(\lambda) & z_2(\lambda) \\ z_1^{-2}(\lambda) & z_2^{-2}(\lambda) & z_1^{2}(\lambda) & z_2^{2}(\lambda) \\ z_1^{n}(\lambda) & z_2^{n}(\lambda) & z_1^{-n}(\lambda) & z_2^{-n}(\lambda) \\ z_1^{n+1}(\lambda) & z_2^{n+1}(\lambda) & z_1^{-n-1}(\lambda) & z_2^{-n-1}(\lambda) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = 0.$$

Let

$$P(\lambda) = \begin{bmatrix} z_1^{-1}\lambda & z_2^{-1}(\lambda) \\ z_1^{-2}\lambda & z_2^{-2}(\lambda) \end{bmatrix}, \quad Q(\lambda) = \begin{bmatrix} z_1\lambda & z_2(\lambda) \\ z_1^{2}\lambda & z_2^{2}(\lambda) \end{bmatrix}$$

$$R_n(\lambda) = \begin{bmatrix} z_1^{n}\lambda & z_2^{n}(\lambda) \\ z_1^{n+1}\lambda & z_2^{n+1}(\lambda) \end{bmatrix},$$

$$S_n(\lambda) = \begin{bmatrix} z_1^{-n}\lambda & z_2^{-n}(\lambda) \\ z_2^{-n-1}\lambda & z_2^{-n-1}(\lambda) \end{bmatrix},$$

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Then the boundary conditions are satisfied if and only if

$$\begin{bmatrix} P(\lambda) & Q(\lambda) \\ R_n(\lambda) & S_n(\lambda) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

In general, let

$$P(\lambda) = [z_s^{-r}(\lambda)]_{r,s=1}^{d}, \quad Q(\lambda) = [z_s^{r}(\lambda)],$$

$$R_n(\lambda) = [z_s^{n+r-1}(\lambda)], \quad S_n(\lambda) = [z_s^{-n+r-1}(\lambda)]$$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{bmatrix}.$$

Then the boundary conditions are satisfied if and only if

$$\begin{bmatrix} P(\lambda) & Q(\lambda) \\ R_n(\lambda) & S_n(\lambda) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0. \qquad \text{(S)}$$

Let

$$D_n(\lambda) = \begin{vmatrix} P(\lambda) & Q(\lambda) \\ R_n(\lambda) & S_n(\lambda) \end{vmatrix} \quad \text{(determinant)}.$$

An eigenvector of $T_n$ must be a nonzero vector. Since (S) has only the trivial solution $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if $D_n(\lambda) \neq 0$, it follows that $\lambda$ is an eigenvalue of $T_n$ if and only if $D_n(\lambda) = 0$. For ways to find the zeros of $D_n(\lambda)$, see my papers RP-44, 61, 63, and 78. Since $D_n(\lambda)$ doesn't become more complicated as $n$ increases, the difficulty of finding individual eigenvalues of $T_n$ is independent of $n$.

We can take this a little further. The eigenvectors of a symmetric Toeplitz matrix have a special property that I haven't mentioned. To identify this property, let $J_n$ be the "flip matrix," which has 1's on its secondary diagonal and 0's elsewhere. For example,

$$J_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that

$$\begin{aligned} J_5^2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I_5. \end{aligned}$$

In general, $J_n^2 = I_n$; that is, $J_n$ is its own inverse.

Multiplying a vector by $J_n$ reverses ("flips") the components of the vector. For example, if

$$x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

then

$$J_5 x = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \\ x_0 \end{bmatrix}.$$

In general, if

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix} \quad \text{then} \quad J_n x = \begin{bmatrix} x_{n-1} \\ x_{n-2} \\ \vdots \\ x_1 \\ x_0 \end{bmatrix}$$

We say that a vector $x$ is symmetric if $J_n x = x$, or skew-symmetric if $J_n x = -x$. Adding symmetric vectors produces a symmetric vector, and multiplying a symmetric vector by a real number produces a symmetric vector; hence, the symmetric vectors in $\mathbb{R}^n$ form a subspace of $\mathbb{R}^n$. Similarly, the skew-symmetric vectors form a subspace of $\mathbb{R}^n$. If $n = 2m$ then each of these subspaces has dimension $m$. If $n = 2m + 1$ then the subspace of symmetric vectors has dimension $m + 1$ and the subspace of skew symmetric vectors has dimension $m$. The zero vector is the only vector that is both symmetric and skew-symmetric.

For example, if $n = 4$ then

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the subspace of symmetric vectors, while

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

form a basis for the subspace of skew-symmetric vectors.

If $n = 5$ then

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

form a basis for the subspace of symmetric vectors, while

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

form a basis for the subspace of skew-symmetric vectors. In general, if $n = 2m$ then the subspace of symmetric vectors and the subspace of skew-symmetric vectors are both $m$-dimensional. If $n = 2m + 1$ then the subspace of symmetric vectors is $(m + 1)$-dimensional and the subspace of skew-symmetric vectors is $m$-dimensional.

Multiplying a matrix on the left by $J_n$ reverses the rows of the matrix, so

$$
\begin{aligned}
J_5 T_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t_0 & t_1 & t_2 & t_3 & t_4 \\ t_1 & t_0 & t_1 & t_2 & t_3 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ t_3 & t_2 & t_1 & t_0 & t_1 \\ t_4 & t_3 & t_2 & t_1 & t_0 \end{bmatrix} \\
&= \begin{bmatrix} t_4 & t_3 & t_2 & t_1 & t_0 \\ t_3 & t_2 & t_1 & t_0 & t_1 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ t_1 & t_0 & t_1 & t_2 & t_3 \\ t_0 & t_1 & t_2 & t_3 & t_4 \end{bmatrix}
\end{aligned}
$$

Multiplying a matrix on the right by $J_n$ reverses the columns of the matrix, so

$$\begin{aligned}
J_5 T_5 J_5 &= (J_5 T_5) J_5 \\
&= \begin{bmatrix} t_4 & t_3 & t_2 & t_1 & t_0 \\ t_3 & t_2 & t_1 & t_0 & t_1 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ t_1 & t_0 & t_1 & t_2 & t_3 \\ t_0 & t_1 & t_2 & t_3 & t_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} t_0 & t_1 & t_2 & t_3 & t_4 \\ t_1 & t_0 & t_1 & t_2 & t_3 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ t_3 & t_2 & t_1 & t_0 & t_1 \\ t_4 & t_3 & t_2 & t_1 & t_0 \end{bmatrix} = T_5.
\end{aligned}$$

In general, $J_n T_n J_n = T_n$. What does this tell us about the eigenvectors of $T_n$?

First, suppose $\lambda$ is an eigenvalue of $T_n$ with multiplicity one, so the associated eigenspace is one-dimensional, and suppose that $T_n x = \lambda x$ with $x \neq 0$. Then $J_n T_n x = \lambda J_n x$. Since $J_n J_n = I_n$, it follows that $(J_n T_n J_n)(J_n x) = \lambda J_n x$. Therefore $T_n(J_n x) = \lambda(J_n x)$, because $J_n T_n J_n = T_n$. Since the $\lambda$-eigenspace of $T_n$ is one-dimensonal, it follows that $J_n x = cx$ for some constant $c$. Therefore, since $\|J_n x\| = \|x\|$ (that is, $x$ and $J_n x$ have the same length), it follows that $c = \pm 1$; that is, $x$ is either symmetric or skew-symmetric. The situation is more complicated if $\lambda$ is a repeated eigenvalue of $T_n$ with multiplicity $k$. However, it can be shown that if $k = 2\ell$ then the $\lambda$-eigenspace of $T_n$ has a basis consisting of $\ell$ symmetric and $\ell$ skew-symmetric vectors, while if $k = 2\ell + 1$ then the $\lambda$-eigenspace has a basis consistng of either $\ell$ symmetric and $\ell + 1$ skew-symmetric eigenvectors or $\ell + 1$ skew-symmetric and $\ell$-symmetric eigenvectors. In any case, if $n = 2k$ then $T_n$ has $k$ symmetric linearly independent eigenvectors and $k$ linearly independent skew-symmetric eigenvectors, while if $n = 2k + 1$ then $T_n$ has $k + 1$ linearly independent symmetric eigenvectors and $k$ linearly independent skew-symmetric vectors.

Recall that the components of the eigenvectors of $T_n$ satisfy

$$\sum_{\ell=-d}^{d} t_{|\ell|} x_{\ell+r} = \lambda x_r, \quad 0 \le r \le n-1, \qquad \text{(DE)}$$

subject to

$$x_r = 0, \quad -d \le r \le -1, \quad n \le r \le n+d-1, \quad \text{(BC)}$$

and are therefore of the form

$$x_r = \sum_{s=1}^{d} \left( a_s z_s^r(\lambda) + b_s z_s^{-r}(\lambda) \right), \quad -d \le r \le n+d-1. \qquad \text{(A)}$$

However, we now know that we can assume at the outset that the eigenvectors of $T_n$ are either symmetric, which means that $x_{n-r+1} = x_r$, or skew-symmetric, which means that $x_{n-r+1} = -x_r$. So let's build this into (A) at the start!

For clarity

$$x_r = \sum_{s=1}^{d} \left(a_s z_s^r(\lambda) + b_s z_s^{-r}(\lambda)\right), \quad -d \le r \le n+d-1.$$

To get a symmetric eigenvector, let

$$b_s = a_s z_s^{n+1}(\lambda) \quad \text{so} \quad x_r = \sum_{s=1}^{d} a_s \left(z_s^r(\lambda) + z_s^{n-r+1}(\lambda)\right).$$

Since $n-(n-r+1)-1 = r$, $x_{n-r+1} = x_r$. As for the boundary conditions

$$x_r = 0, \quad -d \le r \le -1, \quad n \le r \le n+d-1, \quad \text{(BC)}$$

it is enough to require that $x_r = 0$ for $-d \le r \le -1$, since this and the equality $x_{n-r+1} = x_r$ implies that $x_r = 0$ for $n \le r \le n+d-1$.

Therefore, $\lambda$ is an eigenvalue with an associated symmetric eigenvector if and only if

$$\det\left(\left[z_s^{-r}(\lambda) + z_s^{n+r+1}(\lambda)\right]_{r,s=1}^{d}\right) = 0.$$

For clarity

$$x_r = \sum_{s=1}^{d} \left(a_s z_s^r(\lambda) + b_s z_s^{-r}(\lambda)\right), \quad -d \le r \le n+d-1.$$

To get a skew-symmetric eigenvector, let

$$b_s = -a_s z_s^{n+1}(\lambda) \quad \text{so} \quad x_r = \sum_{s=1}^{d} a_s \left(z_s^r(\lambda) - z_s^{n-r+1}(\lambda)\right).$$

Therefore, $\lambda$ is an eigenvalue with an associated skew-symmetric eigenvector if and only if

$$\det\left(\left[z_s^{-r}(\lambda) - z_s^{n+r+1}(\lambda)\right]_{r,s=1}^{d}\right) = 0.$$