About q-Bernstein polynomials

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Due to the importance of Bernstein polynomials, many of their generalizations and related topics has been intensive research. In recent year, the q- Bernstein polynomials have attracted much interest and a great number of interesting results have been obtained [5], [6], [8], [13], [19], [20], [16], [17], [19] .

In 1912, S. N. Bernstein published his famous paper [2] containing a constructive proof of the Weierstrass Approximation Theorem. Using the Law of Large Numbers for a sequence of Bernoulli trials he defined polynomials called in nowadays *Bernstein polynomials*. Later it was found that Bernstein polynomials possess many remarkable properties, which made them an area of intensive research. A systematic treatment of the theory of Bernstein polynomials as it was until the 90's is presented [10],[16] . New papers are constantly coming out and new applications [3] and generalizations are being discovered [13]. The aim of these generalizations is to provide appropriate tools for studying various problems of analysis, geometry, statistical inference and computer science.

The rapid development of q- calculus has led to the discovery of new generalizations of Bernstein polynomials involving q- integers. The first person how make progress in this direction was Lupaş. In 1987 he introduced [7] a q- analogue of the Bernstein operator and investigated its approximating and shape- preserving properties [1].

In 1997, Phillips [14] introduced another generalization on Bernstein polynomials based on q- integers called *q- Bernstein polynomials*. The q- Bernstein polynomials attracted a lot of interest and were studied widely by a number of authors from different perspectives. A review of the results on the q- Bernstein polynomials, along with an extensive bibliography on this subject and a collection of open problems is given in [9]. The subject remains under ample study, and there have been new papers constantly coming out [11],[24], [25] .

It has been known ([14] and references therein) that some properties of the classical Bernstein polynomials are extended to the q- Bernstein polynomials. For example, the q- Bernstein polynomials posses the end- point interpolations property, the shape- preserving properties on the case $0 < q < 1$, and representation via divided differences. Like the Bernstein polynomials, the q- Bernstein polynomials reproduce linear functions, and are degree- reducing on the set of polynomials.

On the other hand, the approximation properties of the q- Bernstein polynomials are essentially different from those of the classical ones. What is more, the cases $0 < q < 1$ and $q > 1$ are not similar to each other. This absence of similarity is caused by the fact that, for $0 < q < 1$, the q- Bernstein polynomials are positive linear operators on $\mathbf{C}[0, 1]$, while for $q > 1$, the positivity does not hold any longer. It should be pointed out that in terms of the convergence properties, the similarity between the classical Bernstein and q- Bernstein polynomials ceases to be true even on the case $0 < q < 1$ [5], [17] . This is because, for $0 < q < 1$, the q- Bernstein polynomials, despite being positive linear operators, do not satisfy the conditions of Korovkin's Theorem. They do, however, satisfy the conditions of Wang' s Korovkin – type theorem [18], serving as a model example fot the theorem.

Due to the lack of positivity, the study of the convergence properties of the q - Bernstein polynomials in the case q < 1 turns out to be essentially more complicated than the one in the case 0 < q < 1. In spite of the intensive research conducted in this area recently, the class of functions in **C**[0, 1] uniformly approximated by their q - Bernstein polynomials when q > 1 is yet to be described. However, the results obtained for specific classes of functions have already revealed some new phenomena as well as interesting facts [10], [11], [20] . To some extent, the explanation for such an exotic behavior of the q- Bernstein polynomials is present in [19].

It has been proved there that basic q - Bernstein polynomials combine the fast increase magnitude with the sign oscillations on [0, 1]. This creates substantial hurdles in the numerical study of the q- Bernstein polynomials for q > 1. It is exactly this unexpected behavior of q - Bernstein polynomials with respect to convergence that makes the study of such properties interesting and challenging.

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