#A2 INTEGERS 14 (2014)

AN APPROACH TO q-SERIES

Jorge Luis Cimadevilla Villacorta
Gijón, Spain
villacorta1968@hotmail.com

Received: 10/17/10, Revised: 11/19/13, Accepted: 12/7/13, Published: 1/6/14

Abstract

In this paper we prove a new q-series transformation using a very simple method.

–Dedicado al Profesor Chan Heng Huat.

1. Introduction

From the theorems of L. Euler [6] and C. F. Gauss [8] to the works by E. Heine [11], S. Ramanujan [2,3], L. J. Rogers [13], W. N. Bailey [4], G. E. Andrews [1], G. Gasper and M. Rahman [7], and many others, there is a plethora of literature on q-series.

However, there are few tools available to prove these theorems. Functional equations, partial fractions, combinatorial reasoning and the powerful "Bailey's lemma" introduced by W. N. Bailey [4] are among the techniques used by mathematicians.

"What is a q-series," an eloquent paper written by B. C. Berndt [5], describes some of the main results and characters in the history of this subject.

No special knowledge about q-series is necessary to understand the development of this paper. Using a potentially new method, we derive a strange q-series transformation, from which several results follow. We employ the traditional notation in this field

$$(a;q)_n = (1-a)(1-aq)...(1-aq^{n-1}), \quad n \in \mathbf{Z}^+,$$

$$(a;q)_0 = 1,$$

$$(a;q)_\infty = \prod_{m=1}^{\infty} (1-aq^m).$$

2. The Main Result

This paper is devoted to proving the general transformation which we state in the following proposition.

Proposition 1. *Let* $a_m(q)$ *and* $b_m(q)$, $m \in \boldsymbol{Z}^+$, *be rational functions of* q *where* $|q| < 1$ *and* $|z| < 1$. *If*

$$\sum_{m=0}^{\infty} a_m(q) y^m = (yq^2; q^2)_\infty \sum_{m=0}^{\infty} b_m(q) y^m, \tag{1}$$

then

$$\frac{(q;q)_\infty}{(-z,q)_\infty} \sum_{m=0}^{\infty} b_m(q)(-z;q)_m + 2z\frac{(q^2;q^2)_\infty}{(z^2,q^2)_\infty} \sum_{m=0}^{\infty} a_m(q) q^m (z^2;q^2)_m = \frac{(q;q)_\infty}{(z,q)_\infty} \sum_{m=0}^{\infty} b_m(q)(z;q)_m. \tag{2}$$

3. A Pair of Lemmas

Before proving Proposition 1 we establish several lemmas.

Lemma 2. *Let* $|q| < 1$ *and* $|z| < 1$. *Then*

$$\sum_{m=0}^{\infty} \frac{(-1)^m q^{\frac{m^2+m}{2}}}{(q;q)_m (1 - zq^m)} = \frac{(q;q)_\infty}{(z;q)_\infty}. \tag{3}$$

Proof. We need only to rewrite the last expression as

$$\sum_{m=0}^{\infty} \frac{(-1)^m q^{\frac{m^2+m}{2}}}{(q;q)_m (1 - zq^m)} = \sum_{m=0}^{\infty} (q^{m+1};q)_\infty z^m = (q;q)_\infty \sum_{m=0}^{\infty} \frac{z^m}{(q;q)_m} = \frac{(q;q)_\infty}{(z;q)_\infty}.$$

To do so, we use two results by Euler [6]

$$\sum_{m=0}^{\infty} \frac{z^m}{(q;q)_m} = \frac{1}{(z;q)_\infty}$$

and

$$\sum_{m=0}^{\infty} \frac{(-z)^m q^{\frac{m^2-m}{2}}}{(q;q)_m} = (z;q)_\infty.$$

□

Lemma 3. *Let* $|q| < 1$ *and* $|z| <1$. *Then*

$$\sum_{n=0}^{\infty} \frac{1}{1-zq^n} (\sum_{i=0}^{n} \frac{(-1)^{n-i} q^{\frac{(n-i)^2+n-i}{2}}}{(q;q)_{n-i}} b_i(q)) = \frac{(q;q)_\infty}{(z;q)_\infty} \sum_{n=0}^{\infty} b_n(q)(z;q)_n, \tag{4}$$

Proof. The infinite double series

$$\sum_{n=0}^{\infty} b_n(q) \sum_{m=0}^{\infty} \frac{(-1)^m q^{\frac{m^2+m}{2}}}{(q;q)_m (1-zq^{m+n})} = \sum_{n=0}^{\infty} b_n(q) \frac{(q;q)_\infty}{(zq^n;q)_\infty}$$

can be rewritten in the more appropriate form

$$\sum_{n=0}^{\infty} \frac{1}{1-zq^n} (\sum_{i=0}^{n} b_i(q) \frac{(-1)^{n-i} q^{\frac{(n-i)^2+(n-i)}{2}}}{(q;q)_{n-i}}) = \frac{(q;q)_\infty}{(z;q)_\infty} \sum_{n=0}^{\infty} b_n(q)(z;q)_n.$$

□

4. An Additional Lemma and an Important Result

In the following section we establish an additional lemma and highlight an important result used in its proof.

Lemma 4. *Let* $|q| < 1$ *and* $|z| < 1$. *Then*

$$\sum_{n=0}^{\infty} \frac{q^n}{1-z^2 q^{2n}} (\sum_{i=0}^{n} \frac{(-1)^{n-i} a_i(q) q^{(n-i)^2}}{(q^2;q^2)_{n-i}}) = \frac{(q^2;q^2)_\infty}{(z^2;q^2)_\infty} \sum_{n=0}^{\infty} a_n(q) q^n (z^2;q^2)_n. \tag{5}$$

Proof. In this case, we begin with the double series

$$\sum_{n=0}^{\infty} a_n(q) q^n (\sum_{m=0}^{\infty} \frac{(-1)^m q^{m^2+m}}{(q^2;q^2)_n (1-z^2 q^{2m+2n})}) = \sum_{n=0}^{\infty} a_n(q) q^n \frac{(q^2;q^2)_\infty}{(z^2 q^{2n};q^2)_\infty},$$

We can then rewrite the last identity as we did in (4). □

Now we can multiply (8) by $(yq;q^2)_\infty$ to find

$$(yq;q^2)_\infty \sum_{m=0}^{\infty} a_m(q) y^m = (yq;q)_\infty \sum_{m=0}^{\infty} b_m(q) y^m.$$

Equating coefficients of y^n on both sides, we have

$$\sum_{i=0}^{n} \frac{(-1)^{n-i} a_i(q) q^{(n-i)^2}}{(q^2;q^2)_{n-i}} = \sum_{i=0}^{n} \frac{(-1)^{n-i} b_i(q) q^{\frac{(n-i)^2+(n-i)}{2}}}{(q;q)_{n-i}}. \tag{6}$$

Therefore we can substitute the right side of (6) into the left-hand side of (4) to obtain the important result mentioned earlier

$$\sum_{n=0}^{\infty} \frac{1}{1-zq^n} \left(\sum_{i=0}^{n} \frac{(-1)^{n-i} q^{\frac{(n-i)^2+n-i}{2}}}{(q;q)_{n-i}} b_i(q)\right) = \sum_{n=0}^{\infty} \frac{1}{1-zq^n} \left(\sum_{i=0}^{n} \frac{(-1)^{n-i} a_i(q) q^{(n-i)^2}}{(q^2;q^2)_{n-i}}\right)$$
$$= \frac{(q;q)_\infty}{(z;q)_\infty} \sum_{n=0}^{\infty} b_n(q)(z;q)_n. \tag{7}$$

5. Proof of the Main Identity

We first restate the main identity of the paper before providing a proof.

Proposition 1. *Let $a_m(q)$ and $b_m(q)$, $m \in \boldsymbol{Z}^+$, be rational functions of q where $|q| < 1$ and $|z| < 1$. If*

$$\sum_{m=0}^{\infty} a_m(q) y^m = (yq^2;q^2)_\infty \sum_{m=0}^{\infty} b_m(q) y^m, \tag{8}$$

then

$$\frac{(q;q)_\infty}{(-z,q)_\infty} \sum_{m=0}^{\infty} b_m(q)(-z;q)_m + 2z \frac{(q^2;q^2)_\infty}{(z^2,q^2)_\infty} \sum_{m=0}^{\infty} a_m(q) q^m (z^2;q^2)_m$$
$$= \frac{(q;q)_\infty}{(z,q)_\infty} \sum_{m=0}^{\infty} b_m(q)(z;q)_m. \tag{9}$$

Proof. We only need to replace z with $-z$ in (7), multipy (5) by $2z$, and then add both identities:

$$\frac{1}{2} \frac{(q;q)_\infty}{(z;q)_\infty} \sum_{n=0}^{\infty} b_n(q)(z;q)_n + \frac{1}{2} \frac{(q;q)_\infty}{(-z;q)_\infty} \sum_{n=0}^{\infty} b_n(q)(-z;q)_n + z \frac{(q^2;q^2)_\infty}{(z^2;q^2)} \sum_{n=0}^{\infty} a_n(q) q^n (z^2;q^2)_n$$
$$= \sum_{n=0}^{\infty} \frac{1}{1-zq^n} \left(\sum_{i=0}^{n} \frac{(-1)^{n-i} a_i(q) q^{(n-i)^2}}{(q^2;q^2)_{n-i}}\right) = \frac{(q;q)_\infty}{(z;q)_\infty} \sum_{n=0}^{\infty} b_n(q)(z;q)_n.$$

□

6. Applications and Examples

Application 5: Mock theta functions. We take the product

$$(-yq^2;q^2)_\infty(yq^2;q^2)_\infty = (y^2q^4;q^4)_\infty,$$

where

$$(-yq^2;q^2)_\infty = \sum_{n=0}^{\infty} \frac{q^{n^2+n}y^n}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} b_n(q)y^n$$

and

$$(y^2q^4;q^4)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{2(n^2+n)}y^{2n}}{(q^4;q^4)_n} = \sum_{n=0}^{\infty} a_n(q)y^n.$$

If we then insert $a_n(q)$ and $b_n(q)$ from above into (9), we find

$$\frac{(q;q)_\infty}{(-z;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+n}(-z;q)_n}{(q^2;q^2)_n} + 2z\frac{(q^2;q^2)_\infty}{(z^2;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(-1)^n q^{2(n^2+n)}q^{2n}(z^2;q^2)_{2n}}{(q^4;q^4)_n} = \frac{(q;q)_\infty}{(z;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2+n}(z;q)_n}{(q^2;q^2)_n}. \quad (10)$$

We now need the classical result of Rogers [13]

$$\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}.$$

Thus, in (10) by replacing z with q, we finally obtain

$$\frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty}{(-q;q)_\infty} + \frac{2}{q}\sum_{n=0}^{\infty}(-1)^n q^{2(n+1)^2}(q^2;q^4)_n = \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(-q;q)_n}. \quad (11)$$

This is a famous identity first proved by G. N. Watson [14] for the fifth order mock theta functions.

Application 6: A New Result from a Classic Identity. In this case we choose

$$\sum_{m=0}^{\infty} a_m(q)y^m = (-yq;q^2)_\infty = \sum_{m=0}^{\infty}\frac{q^{m^2}}{(q^2;q^2)_m}$$

and

$$\sum_{m=0}^{\infty} b_m(q)y^m = \frac{(-yq;q^2)_\infty}{(yq^2;q^2)_\infty} = \sum_{m=0}^{\infty} y^m q^{2m}\frac{(-q^{-1};q^2)_m}{(q^2;q^2)_m}.$$

Inserting these results into (9) we find

$$\frac{(q,q)_\infty}{(-z;q)_\infty}\sum_{m=0}^{\infty} q^{2m}\frac{(-q^{-1};q^2)_m(-z;q)_m}{(q^2;q^2)_m}+2z\frac{(q^2;q^2)_\infty}{(z^2;q^2)_\infty}\sum_{m=0}^{\infty}\frac{q^{m^2+m}(z^2;q^2)_n}{(q^2;q^2)_m}$$
$$=\frac{(q,q)_\infty}{(z;q)_\infty}\sum_{m=0}^{\infty} q^{2m}\frac{(-q^{-1};q^2)_m(z;q)_m}{(q^2;q^2)_m}. \quad (12)$$

In the last identity V. A. Lebesgue's [12] classic result appears:

$$\sum_{m=0}^{\infty}\frac{q^{m^2+m}(z^2;q^2)_m}{(q^2;q^2)_m}=(z^2q^2;q^4)_\infty(-q^2;q^2)_\infty.$$

Therefore we can rewrite (12) as

$$\frac{(q,q)_\infty}{(-z;q)_\infty}\sum_{m=0}^{\infty} q^{2m}\frac{(-q^{-1};q^2)_m(-z;q)_m}{(q^2;q^2)_m}+2z\frac{(q^4;q^4)_\infty}{(z^2;q^4)_\infty}$$
$$=\frac{(q,q)_\infty}{(z;q)_\infty}\sum_{m=0}^{\infty} q^{2m}\frac{(-q^{-1};q^2)_m(z;q)_m}{(q^2;q^2)_m}, \quad (13)$$

which appears to be a new result.

Application 7: An Interesting Identity. In this example, we take

$$\sum_{m=0}^{\infty} a_m(q)y^m=(yq^2;q^2)_\infty=\sum_{m=0}^{\infty}(-1)^m\frac{q^{m^2+m}}{(q^2;q^2)_m},$$

where

$$b_0=1,$$
$$b_1,b_2,b_3...=0.$$

If we use these results in (9), we find

$$\frac{(q;q)_\infty}{(-z;q)_\infty}+2z\frac{(q^2;q^2)_\infty}{(z^2;q^2)_\infty}\sum_{m=0}^{\infty}(-1)^m\frac{q^{m^2+2m}(z^2;q^2)_m}{(q^2;q^2)_m}=\frac{(q;q)_\infty}{(z;q)_\infty}. \quad (14)$$

Using the previous identity, we can obtain the following interesting result. If

$$\frac{1}{(q;q^2)_\infty}=\sum_{m=0}^{\infty} c_m q^m$$

then

$$\sum_{m=1}^{\infty} c_{2m-1}q^m=(q^2,q^2)_\infty\sum_{m=0}^{\infty}(-1)^m\frac{q^{(m+1)^2}(q;q^2)_m}{(q^2;q^2)_m}. \quad (15)$$

Another possibility is using the second Göllnitz-Gordon [9,10] identity

$$\sum_{m=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^2,q^2)_n} = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}$$

in (14) where $q \to -q$ and $z = i\sqrt{q}$. From this we obtain

$$\frac{(iq^{\frac{1}{2}};q^2)_\infty(-iq^{\frac{3}{2}};q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty} + \frac{2i\sqrt{q}}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty} = \frac{(-iq^{\frac{1}{2}};q^2)_\infty(iq^{\frac{3}{2}};q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}.$$

The following identity is equivalent. If

$$(-q;q^2)_\infty = \sum_{m=0}^{\infty} a_m q^m,$$

then

$$\sum_{m=0}^{\infty}(-1)^m a_{2m+1} q^m = \frac{(q^1;q^8)_\infty(q^7;q^8)_\infty(-q^4;q^4)_\infty}{(q^2;q^4)_\infty}.$$

7. An Alternative Proof of the Main Identity

G. E. Andrews graciously provided the author with the following alternative proof of identity (9):

$$-\frac{(q;q)_\infty}{(-z;q)_\infty}\sum_{m=0}^{\infty} b_m(q)(-z;q)_m + \frac{(q;q)_\infty}{(z;q)_\infty}\sum_{m=0}^{\infty} b_m(q)(z;q)_m =$$

$$= (q;q)_\infty \sum_{m=0}^{\infty} b_m(q)\left(\frac{-1}{(-zq^m;q)_\infty} + \frac{1}{(zq^m;q)_\infty}\right)$$

$$= (q;q)_\infty \sum_{m=0}^{\infty} b_m(q) \sum_{j=0}^{\infty} \frac{z^j q^{mj}((-1)^{j+1}+1)}{(q;q)_j}$$

$$= 2(q;q)_\infty \sum_{j=0}^{\infty} \frac{z^{2j+1}}{(q;q)_{2j+1}} \sum_{m=0}^{\infty} b_m(q) q^{m(2j+1)}$$

$$= 2(q;q)_\infty \sum_{j=0}^{\infty} \frac{z^{2j+1}}{(q;q)_{2j+1}} \frac{1}{(q^{2j+3};q^2)_\infty} \sum_{m=0}^{\infty} a_m(q) q^{m(2j+1)}$$

$$= 2\frac{(q;q)_\infty}{(q;q^2)_\infty} \sum_{j=0}^{\infty} \frac{z^{2j+1}}{(q^2;q^2)_\infty} \sum_{m=0}^{\infty} a_m(q) q^{m(2j+1)}$$

$$= 2(q^2;q^2)_\infty \sum_{m=0}^{\infty} zq^m a_m(q) \sum_{j=0}^{\infty} \frac{z^{2j}q^{2jm}}{(q^2;q^2)_j}$$

$$= 2(q^2;q^2)_\infty \sum_{m=0}^{\infty} \frac{zq^m a_m(q)}{(z^2q^{2m};q^2)_\infty} = 2\frac{(q^2;q^2)_\infty}{(z^2;q^2)_\infty} \sum_{m=0}^{\infty} zq^m a_m(q)(z^2;q^2)_m.$$

Acknowledgement. The author would like to thank George E. Andrews, Timothy Kehoe and Jack Rossbach for their contributions as well as their patience.

References

[1] Personal website of G. E. Andrews, available at www.math.psu.edu/andrews, Pennsylvania State University.

[2] G. E. Andrews and B. C. Berndt, *Ramanujan´s Lost Notebook, Part I*, Springer, New York, 2005.

[3] G. E. Andrews and B. C. Berndt, *Ramanujan´s Lost Notebook, Part II*, Springer, New York, 2009.

[4] W. N. Bailey, A note on certain q-identities, *Quart. J. Math.* (Oxford) **19** (1941), 157-166.

[5] B. C. Berndt, *Ramanujan Rediscovered: Proceedings of a Conference on Elliptic Functions, Partitions and q-series in memory of K. Venkatachaliengar*. Bangalore, 1-5 June, 2009.

[6] L. Euler, *Introductio in Analysin Infinitorum*, Marcum-Michaelem Bousquet, Lausannae, 1748, 237-240.

[7] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd. ed., Cambridge University Press, Cambridge, 2004.

[8] C. F. Gauss, *Werke*, Vol. 3, K. Gesell,Wiss., Göttingen, 1866.

[9] H. Gollnitz, Partitionen mit Differenzenbedingungen, *J. Reine Angew. Math.* **225** (1967), 154-190.

[10] B. Gordon, Some continued fractions of the Rogers-Ramanujan type, *Duke Math J.* **32** (1965), 741-748.

[11] E. Heine, Untersuchungen über die Reihe

$$1 + \frac{(1-q^\alpha)(1-q^\beta)}{(1-q)(1-q^\gamma)}.x + \frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q)(1-q^2)(1-q^\gamma)(1-q^{\gamma+1})}.x^2 + ...,$$

J. Reine Angew. Math. **34** (1847), 285-328.

[12] V. A. Lebesgue, Sommation de quelques series, *J. Math. Pure. Appl.* **5** (1840), 42-71.

[13] L. J. Rogers, Second memoir on the expansion of certain infinite products, *Proc. London. Math. Soc.* **16** (1917), 315-336.

[14] G. N. Watson, The mock theta functions (2), *Proc. London Math. Soc.* **42** (1937), 274-304.