# THE BASE CHANGE $L$-FUNCTION FOR MODULAR FORMS AND BEYOND ENDOSCOPY

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Abstract. Following ideas of Langlands and Sarnak on *Beyond Endoscopy*, we introduce a method to study the base change $L$-function of a modular form via the trace formula. We complete the analysis in the case of quadratic base change for real fields, to deduce a new proof of the analytic continuation of the base change $L$-function to the left of the line $\Re(s) = 1$.

## 1. Introduction

Langlands [Lan04] has proposed a new program, entitled *Beyond Endoscopy*, for attacking cases of Langlands functoriality which are not accessible via current methods. The crucial idea is to apply the Arthur-Selberg trace formula to detect the existence of poles at $s = 1$ of certain $L$-functions attached to automorphic representations. Incorporating additional ideas of Sarnak [Sar01], progress has been made. Specifically, Venkatesh [Ven04] studied the the symmetric square and Herman [Her12c, Her12a] has studied both the symmetric cube and the Asai $L$-function in the context of quadratic base change. The ideas of Beyond Endoscopy, can also be applied directly to the study of $L$-functions. This was observed by Sarnak [Sar01] who applied a version of the trace formula to obtain a new proof of the analytic continuation to $\mathbf{C}$ of the $L$-function associated to a modular form. These ideas have been extended by Herman [Her12b] to deduce the functional equation of the $L$-function. In this article, we shall apply these ideas to study the $L$-functions arising from base change in the case of modular forms.

In order to study the $L$-functions arising from base change, we introduce in Section 2 a related Dirichlet series, denoted $D(s, \pi, \omega, E/F)$, which is more amenable to studying via the Petersson trace formula. We show that proving the analytic continuation of $D(s, \pi, \omega, E/F)$ to the right of the line $\Re(s) = \frac{1}{2}$ implies the analytic continuation of the base change $L$-function to the right of the line $\Re(s) = \frac{1}{2}$. The key idea is then to apply the Petersson trace formula to study a linear combination of the Dirichlet series $D(s, \pi, \omega, E/F)$ associated to a family of automorphic representations $\pi$. We are able to deduce from this sum information about a single $D(s, \pi, \omega, E/F)$. The analysis in a first case is carried out in Section 3. This gives the main result of the article (cf. Theorem 3.14).

**Theorem 1.** *Let $k \geq 6$, let $N \in \mathbf{N}$, and let $\phi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ be a primitive Dirichlet character. Let $\pi \in \mathcal{A}_k(N, \phi)$ (see Section 3.1.1) be a cuspidal automorphic representation $GL_2(\mathbf{A}_\mathbf{Q})$ such that $\pi_p$ is not unramified for all rational primes $p < 70$. Let $E = \mathbf{Q}(\sqrt{D})$ be a quadratic real extension such that $D = 2, 3 \mod 4$. Then the base change $L$-function $\Lambda(s, \mathrm{BC}_{E/\mathbf{Q}}(\pi))$*

*Date*: December 28, 2012.

- *converges absolutely in the right half plane* $\Re(s) > 1$, *and*
- *continues to a holomorphic function in the right half plane* $\Re(s) > \frac{3}{4} + \frac{5}{2(k-3)}$.

*Remark* 1.1. The conditions imposed on $\pi$ and on the extension $E$ (with the exception of the condition that $E$ be a quadratic extension of $\mathbf{Q}$) are made exclusively for convenience and should be able to be removed.

*Remark* 1.2. In the simple setting of this theorem there exists an old proof of a stronger result. Specifically, the $L$-function decomposes as a product

$$\Lambda(s, \mathrm{BC}_{E/\mathbf{Q}}(\pi)) = \Lambda(s, \pi)\Lambda(s, \pi \cdot \chi_{E/F})$$

where $\chi_{E/F} : \mathbf{Q}^\times \backslash \mathbf{A}_Q^\times \to \mathbf{C}^\times$ denotes the Hecke character corresponding to the extension $E/\mathbf{Q}$ via class field theory. It follows from the theory of $L$-functions associated to cuspidal automorphic representations of $GL_2(\mathbf{A}_\mathbf{Q})$ that the $L$-function $\Lambda(s, \mathrm{BC}_{E/\mathbf{Q}}(\pi))$ continues to a holomorphic function of the complex plane.

We are hopeful that the methods of this article can be extended to study the base change $L$-function in new cases for example in the case of non-solvable base change for Hilbert modular forms. We hope to attack this problem in a future article.

1.1. **Acknowledgments.** I am extremely grateful to D.R. Heath-Brown for his help in resolving analytic questions. I am thankful to P. E. Herman and P. Sarnak for providing helpful comments on this article.

## 2. The $L$-function

In this section, we recall the base change $L$-function for Hilbert modular forms and introduce a related Dirichlet series. We show that the analytic properties of the latter influence the analytic properties of the former.

Let $\psi_\mathbf{Q} = \otimes_\nu \psi_{\mathbf{Q},\nu} : \mathbf{Q} \backslash \mathbf{A}_\mathbf{Q} \to \mathbf{C}^\times$ denote the additive character which is unramified at all finite places and whose restriction to $\mathbf{R}$ is the exponential function $e(x) = e^{2\pi i x}$. Let $F$ be a totally real number field. Let $\psi_F = \psi_\mathbf{Q} \circ \mathrm{Tr}_{F/\mathbf{Q}} : F \backslash \mathbf{A}_F \to \mathbf{C}^\times$ be the additive character where $\mathrm{Tr}_{F/\mathbf{Q}} : \mathbf{A}_F \to \mathbf{A}_\mathbf{Q}$ denotes the trace map.

To an irreducible admissible representation $\sigma$ of $GL_1(\mathbf{A}_F)$ or $GL_2(\mathbf{A}_F)$, one associates in the usual way

- the finite $L$-function $L(s, \sigma) = \prod_{\nu < \infty} L(s, \sigma_\nu)$,
- the completed $L$-function $\Lambda(s, \sigma) = \prod_\nu L(s, \sigma_\nu)$, and
- the epsilon factor $\epsilon(s, \sigma) = \prod_\nu \epsilon(s, \sigma, \psi_\nu)$

where the local Euler factors are those defined in [JL70]. The finite $L$-function can be written formally as a Dirichlet series

$$L(s, \sigma) = \sum_{\mathfrak{n}} \frac{\lambda_\sigma(\mathfrak{n})}{\mathrm{N}(\mathfrak{n})^s}$$

where $\mathfrak{n}$ runs through the integral ideals of $F$ and $\mathrm{N}(\mathfrak{n}) \in \mathbf{N}$ denotes the absolute norm of $\mathfrak{n}$.

In the cases of interest to us, the $L$-functions are explicitly described below.

- If $\mathfrak{p}$ is a non-archimedean place of $F$ and if $\chi : F_{\mathfrak{p}}^{\times} \to \mathbf{C}^{\times}$ is a character, then

$$L(s,\chi) = \begin{cases} 1 + \frac{\chi(\varpi_{\mathfrak{p}})}{\mathrm{N}(\mathfrak{p})^s} + \cdots + \frac{\chi(\varpi_{\mathfrak{p}})^n}{\mathrm{N}(\mathfrak{p})^{ns}} + \cdots & \text{if } \chi \text{ is unramified,} \\ 1 & \text{otherwise,} \end{cases}$$

where $\varpi_{\mathfrak{p}} \in \mathcal{O}_{F_\nu}$ denotes a chosen uniformizer.
- If $\nu$ is a non-archimedean place of $F$ and if $\pi$ is an irreducible admissible generic representation of $GL_2(F_\nu)$, then

$$L(s,\pi) = \begin{cases} 1 & \text{if } \pi \text{ is supercuspidal,} \\ L(s+\frac{1}{2},\chi) & \text{if } \sigma \simeq \mathrm{St}(\chi) = \chi \cdot \mathrm{St} \text{ is Steinberg,} \\ L(s,\chi_1)L(s,\chi_2) & \text{if } \sigma \simeq \mathrm{I}(\chi_1,\chi_2) \text{ where } \chi_1 \not\simeq \chi_2|\cdot|^{\pm 1} \text{ is a principal series.} \end{cases}$$

Let $\pi$ be a cuspidal automorphic representation of $GL_2(\mathbf{A}_F)$ such that

- $\pi_\infty$ is a discrete series representation, and
- either $[F : \mathbf{Q}]$ is odd or $k_\nu = k_{\nu'} \mod 2$ for all archimedean places $\nu$ and $\nu'$ of $F$ where $\pi_\nu$ is the discrete series representation of weight $k_\nu$.

The Ramanujan-Petersson conjecture is known for such $\pi$, which states that the local representations $\pi_\nu$ are tempered for all $\nu$ (cf. [Bla06, Theorem 1]).

Let $E/F$ be a Galois extension of totally real number fields. Let $\mathrm{BC}_{E/F}(\pi) = \otimes_\nu \mathrm{BC}_{E_\nu/F_\nu}(\pi_\nu)$ denote the (Langlands) base change of $\pi$ to $E$. This is the irreducible admissible representation of $GL_2(\mathbf{A}_E)$ which is determined by the condition that for all places $\nu$ of $E$,

$$\mathrm{rec}(\mathrm{BC}_{E_\nu/F_\nu}(\pi_\nu)) \simeq \mathrm{rec}(\pi_\nu)|_{W'_\nu}$$

as Weil-Deligne representations. Here rec denotes the local Langlands correspondence, normalised as in [HT01], that associates to an irreducible admissible representation of $GL_n(k)$ an $n$-dimensional Weil-Deligne representation of $W'_k$ where $k$ is a local field field of characteristic 0. The Langlands functoriality conjectures predict that $\mathrm{BC}_{E/F}(\pi)$ should be an automorphic representation, and as $\mathrm{BC}_{E/F}(\pi)_\infty$ is a discrete series representations one expects the following.

**Conjecture 2.1.** *The representation* $\mathrm{BC}_{E/F}(\pi)$ *is a cuspidal automorphic representation.*

*Remark* 2.2. If either $E/F$ is solvable or $F = \mathbf{Q}$, then the conjecture is known due to the work of Langlands [Lan80] and Dieulefait [Die12] respectively.

**Conjecture 2.3.** *For all unitary Hecke characters* $\omega : E^\times \backslash \mathbf{A}_E^\times \to \mathbf{C}^\times$ *the completed L-functions* $\Lambda(s, \mathrm{BC}_{E/F}(\pi) \otimes \omega)$ *and* $\Lambda(s, \mathrm{BC}_{E/F}(\pi^\vee) \otimes \omega^{-1})$

(i) *converge absolutely in some right half plane* $\Re(s) > M$*;*
(ii) *continue to meromorphic functions of* $\mathbf{C}$*;*
(iii) *satisfy the functional equation*

$$\Lambda(s, \mathrm{BC}_{E/F}(\pi) \otimes \omega) = \epsilon(s, \mathrm{BC}_{E/F}(\pi) \otimes \omega)\Lambda(s, \mathrm{BC}_{E/F}(\pi^\vee) \otimes \omega^{-1});$$

*and if* $\omega$ *is unramified at all finite places, the completed L-function* $\Lambda(s, \mathrm{BC}_{E/F}(\pi) \otimes \omega)$ *continues to an analytic function of* $\mathbf{C}$ *that is bounded in vertical strips.*

**Lemma 2.4.** *Conjecture 2.1 and Conjecture 2.3 are equivalent.*

*Proof.* The contragredient of the base change is isomorphic to the base change of the contragredient that is

$$\mathrm{BC}_{E/F}(\pi^\vee) \simeq \mathrm{BC}_{E/F}(\pi)^\vee$$

as irreducible admissible representations of $GL_2(\mathbf{A}_E)$. It follows from well known properties of the $L$-functions associated to cuspidal automorphic representations (cf. [JL70, Theorem 11.1]) that Conjecture 2.1 implies Conjecture 2.3. The fact that Conjecture 2.3 implies Conjecture 2.1 follows from the Booker-Krishnamurthy [BK12, Theorem 1.1] improvement of the Weil converse theorem. □

Let $\omega : E^\times \backslash \mathbf{A}_E^\times \to \mathbf{C}^\times$ be a unitary Hecke character that is unramified at all finite places. We wish to make progress towards Conjecture 2.3 by studying the $L$-function

$$\Lambda(s, \mathrm{BC}_{E/F}(\pi) \otimes \omega).$$

We are unable to study this $L$-function directly, and shall instead study the related Dirichlet series

$$D(s, \pi, \omega, E/F) = \sum_{\mathfrak{n}} \frac{\lambda_\pi(\mathrm{N}_{E/F}(\mathfrak{n}))\lambda_\omega(\mathfrak{n})}{\mathrm{N}(\mathfrak{n})^s}$$

where $\mathfrak{n}$ runs through the integral ideals of $E$ and $\mathrm{N}_{E/F}$ denotes the relative norm map.

**Lemma 2.5.** *The Dirichlet series $D(s, \pi, \omega, E/F)$ converges absolutely in the right half plane $\Re(s) > 1$.*

*Proof.* The representation $\pi$ is tempered at all finite places. Consequently for all $\delta > 0$, we have the bound

$$\lambda_\pi(\mathfrak{a}) \ll \mathrm{N}(\mathfrak{a})^\delta$$

where $\mathfrak{a}$ denotes an integral ideal of $F$. The Hecke character $\omega$ is unitary and unramified at all finite places, which implies that $|\lambda_\omega(\mathfrak{n})| = 1$ for all integral ideals $\mathfrak{n}$ of $E$. It follows that $D(s, \pi, \omega, E/F)$ converges absolutely in the right half plane $\Re(s) > 1 + \delta$. □

**Lemma 2.6.** *The Dirichlet series $D(s, \pi, \omega, E/F)$ admits an Euler product expansion over $F$. That is formally,*

$$D(s, \pi, \omega, E/F) = \prod_{\mathfrak{p}} D_{\mathfrak{p}}(s, \pi, \omega, E/F)$$

*where $\mathfrak{p}$ runs through the prime ideals of $\mathcal{O}_F$ and*

$$D_{\mathfrak{p}}(s, \pi, \omega, E/F) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} \frac{\lambda_\pi(\mathfrak{p}^{(n_1+\cdots+n_r)f_{E/F,\mathfrak{p}}})\lambda_\omega(\mathfrak{q}_1^{n_1} \cdots \mathfrak{q}_r^{n_r})}{\mathrm{N}(\mathfrak{p})^{(n_1+\cdots+n_r)f_{E/F,\mathfrak{p}}s}}$$

*where $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ denote the distinct prime ideals of $\mathcal{O}_E$ lying above $\mathfrak{p}$ and $f_{E/F,\mathfrak{p}} = [\mathcal{O}_E/\mathfrak{q}_1 : \mathcal{O}_F/\mathfrak{p}]$ denotes the relative residue extension degree.*

*Proof.* If $\mathfrak{n}$ and $\mathfrak{n}'$ are integral ideals of $E$ such that $\mathrm{N}_{E/F}(\mathfrak{n})$ and $\mathrm{N}_{E/F}(\mathfrak{n}')$ are relatively prime as integral ideals of $F$, then

$$\lambda_\pi(\mathrm{N}_{E/F}(\mathfrak{n}\mathfrak{n}')) = \lambda_\pi(\mathrm{N}_{E/F}(\mathfrak{n})\mathrm{N}_{E/F}(\mathfrak{n}')) = \lambda_\pi(\mathrm{N}_{E/F}(\mathfrak{n}))\lambda_\pi(\mathrm{N}_{E/F}(\mathfrak{n}')).$$

Such integral ideals $\mathfrak{n}$ and $\mathfrak{n}'$ are relatively prime, and we see that

$$\lambda_\omega(\mathfrak{n}\mathfrak{n}') = \lambda_\omega(\mathfrak{n})\lambda_\omega(\mathfrak{n}').$$

It follows that the Dirichlet series $D(s,\pi,\omega,E/F)$ admits an Euler product expansion over $F$ where

$$D_{\mathfrak{p}}(s,\pi,\omega,E/F)=\sum_{\mathfrak{n}}\frac{\lambda_\pi(\mathrm{N}_{E/F}(\mathfrak{n}))\lambda_\omega(\mathfrak{n})}{\mathrm{N}(\mathfrak{n})^s}$$

where $\mathfrak{n}$ runs through the integral ideals of $E$ such that $\mathrm{N}_{E/F}(\mathfrak{n})=\mathfrak{p}^n$ for some $n\geq 0$. The integral ideals $\mathfrak{n}$ are of the form $\mathfrak{q}_1^{n_1}\cdots\mathfrak{q}_r^{n_r}$ where $n_1,\ldots,n_r\geq 0$. As the extension $E/F$ is Galois $\mathrm{N}_{E/F}(\mathfrak{q}_i)=\mathfrak{p}^{f_{E/F,\mathfrak{p}}}$ for all $i=1,\ldots,r$. □

**Lemma 2.7.** *Let $\mathfrak{p}$ denote a prime ideal of $\mathcal{O}_F$, then the local Euler factor*

$$D_{\mathfrak{p}}(s,\pi,\omega,E/F)$$

*converges absolutely in the right half plane $\Re(s)>0$.*

*Proof.* The definition of the local Euler factor states that

$$D_{\mathfrak{p}}(s,\pi,\omega,E/F)=\sum_{n_1=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}\frac{\lambda_\pi(\mathfrak{p}^{(n_1+\cdots+n_r)f_{E/F,\mathfrak{p}}})\lambda_\omega(\mathfrak{q}_1^{n_1}\cdots\mathfrak{q}_r^{n_r})}{\mathrm{N}(\mathfrak{p})^{(n_1+\cdots+n_r)f_{E/F,\mathfrak{p}}s}}$$

As the representation $\pi$ is tempered at all finite places and the Hecke character $\omega$ is unitary, we have the bound, valid for all $\delta>0$,

$$\lambda_\omega(\mathfrak{a})\lambda_\pi(\mathrm{N}_{E/F}(\mathfrak{a}))\ll\mathrm{N}(\mathfrak{a})^\delta$$

where $\mathfrak{a}$ denotes an integral ideal of $E$. It follows that $D_{\mathfrak{p}}(s,\pi,\omega,E/F)$ converges absolutely in the right half plane $\Re(s)>\delta$. □

**Lemma 2.8.** *Let $\mathfrak{n}=\mathfrak{p}_1\cdots\mathfrak{p}_r$ where the $\mathfrak{p}_i$ are prime ideals of $\mathcal{O}_E$ that are totally split over distinct prime ideals of $\mathcal{O}_F$. If $\pi$ is unramified at the places $\mathrm{N}_{E/F}(\mathfrak{p}_i)$ for all $i=1,\ldots,r$, then*

$$\lambda_\pi(\mathrm{N}_{E/F}(\mathfrak{n}))\lambda_\omega(\mathfrak{n})=\lambda_{\mathrm{BC}_{E/F}(\pi)\otimes\omega}(\mathfrak{n}).$$

*Proof.* As the Hecke character $\omega$ is unramified at all finite places, it follows that

$$\lambda_{\mathrm{BC}_{E/F}(\pi)\otimes\omega}(\mathfrak{n})=\lambda_{\mathrm{BC}_{E/F}(\pi)}(\mathfrak{n})\lambda_\omega(\mathfrak{n}).$$

As the prime $\mathfrak{p}_i$ is totally split above $F$, we see that

$$\mathrm{BC}_{E/F}(\pi)_{\mathfrak{p}_i}\simeq\pi_{\mathrm{N}_{E/F}(\mathfrak{p}_i)}$$

for all $i=1,\ldots,r$. Finally as the $\mathfrak{p}_i$ lie above distinct prime ideals of $\mathcal{O}_F$, we see that

$$\lambda_\pi(\mathrm{N}_{E/F}(\mathfrak{n}))=\prod_{i=1}^{r}\lambda_\pi(\mathrm{N}_{E/F}(\mathfrak{p}_i)).$$

The result follows. □

**Lemma 2.9.** *Let $A>0$. Assume that the Dirichlet series $D(s,\pi,\omega,E/F)$ has an analytic continuation to the right half plane $\Re(s)>\frac{1}{2}+A$ and that the Euler factors $D_{\mathfrak{p}}(s,\pi,\omega,E/F)$ are non-zero in the right half plane $\Re(s)>\frac{1}{2}+A$ for all prime ideals $\mathfrak{p}$ of $\mathcal{O}_F$. Then the $L$-function*

$$L(s,\mathrm{BC}_{E/F}(\pi)\otimes\omega)$$

*converges absolutely in the right half plane $\Re(s)>1$ and has an analytic continuation to the right half plane $\Re(s)>\frac{1}{2}+A$.*

*Proof.* By Lemma 2.6, we can write formally

$$L(s, \mathrm{BC}_{E/F}(\pi)\otimes\omega) = D(s,\pi,\omega,E/F)\prod_{\mathfrak{p}} \frac{L_{\mathfrak{p}}(s, \mathrm{BC}_{E/F}(\pi)\otimes\omega)}{D_{\mathfrak{p}}(s,\pi,\omega,E/F)}$$

where $\mathfrak{p}$ runs through the prime ideals of $\mathcal{O}_F$ and

$$L_{\mathfrak{p}}(s, \mathrm{BC}_{E/F}(\pi)\otimes\omega) = \prod_{i=1}^{r} L(s, (\mathrm{BC}_{E/F}(\pi)\otimes\omega)_{\mathfrak{q}_i})$$

where $\mathfrak{q}_1,\ldots,\mathfrak{q}_r$ are the distinct primes of $\mathcal{O}_E$ lying above a prime ideal $\mathfrak{p}$. The result will follow if we can show that the Dirichlet series

$$\prod_{\mathfrak{p}} \frac{L_{\mathfrak{p}}(s, \mathrm{BC}_{E/F}(\pi)\otimes\omega)}{D_{\mathfrak{p}}(s,\pi,\omega,E/F)} \tag{2.1}$$

converges to an analytic function for $\Re(s) > \frac{1}{2} + A$.

As the representation $\pi$ is tempered at all finite places, we have the bound, valid for all $\delta > 0$,

$$\lambda_\pi(\mathfrak{a}) \ll \mathrm{N}(\mathfrak{a})^{\delta}$$

where $\mathfrak{a}$ denotes an integral ideal of $F$. If $\Re(s) > \delta$, then this gives the bound

$$\begin{aligned} D_{\mathfrak{p}}(s,\pi,\omega,E/F) &= 1 + \sum_{\substack{n_1,\ldots,n_r=0 \\ n_1+\cdots+n_r\geq 1}}^{\infty} \frac{\lambda_\pi(\mathfrak{p}^{(n_1+\cdots+n_r)f_{E/F,\mathfrak{p}}})\lambda_\omega(\mathfrak{q}_1^{n_1}\cdots\mathfrak{q}_r^{n_r})}{\mathrm{N}(\mathfrak{p})^{(n_1+\cdots+n_r)f_{E,\mathfrak{p}}s}} \\ &= 1 + \mathrm{O}(\mathrm{N}(\mathfrak{p})^{f_{E/F,\mathfrak{p}}(\delta-\Re(s))}). \end{aligned}$$

If $\Re(s) > \frac{1}{2} + A$, then by assumption $D_{\mathfrak{p}}(s,\pi,\omega,E/F) \neq 0$, and we obtain the bound

$$D_{\mathfrak{p}}(s,\pi,\omega,E/F)^{-1} \ll 1.$$

The Langlands base change of a tempered representation remains tempered. It follows that the representation $\mathrm{BC}_{E/F}(\pi)\otimes\omega$ is tempered at all places, and we have the bound

$$\lambda_{\mathrm{BC}_{E/F}(\pi)\otimes\omega}(\mathfrak{a}) \ll \mathrm{N}(\mathfrak{a})^{\delta}$$

where $\mathfrak{a}$ denotes an integral ideal of $E$. (We remark that this bound implies that the $L$-function $L(s, \mathrm{BC}_{E/F}(\pi)\otimes\omega)$ converges absolutely in the right half plane $\Re(s) > 1+\delta$.) If $\mathfrak{p}$ is such that $f_{E/F,\mathfrak{p}} \geq 2$, then arguing as before, we obtain the bound

$$L_{\mathfrak{p}}(s, \mathrm{BC}_{E/F}(\pi)\otimes\omega) - D_{\mathfrak{p}}(s,\pi,\omega,E/F) \ll \mathrm{N}(\mathfrak{p})^{2(\delta-\Re(s))}.$$

If $\mathfrak{p}$ is totally split in $E$ and $\pi_{\mathfrak{p}}$ is unramified, then we can apply Lemma 2.8 to obtain the bound

$$L_{\mathfrak{p}}(s, \mathrm{BC}_{E/F}(\pi)\otimes\omega) - D_{\mathfrak{p}}(s,\pi,\omega,E/F) \ll \mathrm{N}(\mathfrak{p})^{2(\delta-\Re(s))}.$$

We can now combine our obtained bounds assuming that $\mathfrak{p}$ is either totally split in $E$ and $\pi_{\mathfrak{p}}$ is unramified, or $f_{E/F,\mathfrak{p}} \geq 2$. If $\Re(s) > \frac{1}{2} + A$, then we obtain the bound

$$\frac{L_{\mathfrak{p}}(s, \mathrm{BC}_{E/F}(\pi)\otimes\omega)}{D_{\mathfrak{p}}(s,\pi,\omega,E/F)} = 1 + \frac{L_{\mathfrak{p}}(s, \mathrm{BC}_{E/F}(\pi)\otimes\omega) - D_{\mathfrak{p}}(s,\pi,\omega,E/F)}{D_{\mathfrak{p}}(s,\pi,\omega,E/F)} \ll \mathrm{N}(\mathfrak{p})^{2(\delta-\Re(s))}.$$

We are left to consider the finite number of remaining $\mathfrak{p}$. In this case we remark that $L_{\mathfrak{p}}(s, \mathrm{BC}_{E/F}(\pi) \otimes \omega)$ converges absolutely if $\Re(s) > \delta$. Thus

$$\frac{L_{\mathfrak{p}}(s, \mathrm{BC}_{E/F}(\pi) \otimes \omega)}{D_{\mathfrak{p}}(s, \pi, \omega, E/F)}$$

defines an analytic function in the right half plane $\Re(s) > \frac{1}{2} + A$.

Combing our results, we deduce that the Dirichlet series (2.1) defines an analytic function in the right half plane defined by $\Re(s) > \frac{1}{2} + A$ and $2(\delta - \Re(s)) < -1$, that is $\Re(s) > \frac{1}{2} + \max(A, \delta)$. The result follows. □

**Lemma 2.10.** *If $\pi$ is not unramified at a prime ideal $\mathfrak{p}$, then $D_{\mathfrak{p}}(s, \pi, \omega, E/F)$ converges and is non-zero in the right half plane $\Re(s) > 0$.*

*Proof.* The representation $\pi_{\mathfrak{p}}$ is a tempered representation that is not unramified. Consequently, $\pi_{\mathfrak{p}}$ will be either supercuspidal, Steinberg $\mathrm{St}(\chi)$ where $\chi$ is a unitary character, or a principal series $\mathrm{I}(\chi_1, \chi_2)$ where $\chi_1$ and $\chi_2$ are unitary characters and $\chi_1$ is not unramified. By considering the possible cases, we observe that the local $L$-function $L(s, \pi_{\mathfrak{p}})$ will either be equal to 1 or of the form $L(s + \frac{1}{2}, \chi)$ or $L(s, \chi)$ where $\chi : F_{\mathfrak{p}}^{\times} \to \mathbf{C}^{\times}$ is a unitary character. In the first case $D_{\mathfrak{p}}(s, \pi, \omega, E/F) = 1$, in the second case $D_{\mathfrak{p}}(s, \pi, \omega, E/F) = \prod_{\mathfrak{q}|\mathfrak{p}} L(s + \frac{1}{2}, (\chi \circ \mathrm{N}_{E_{\mathfrak{q}}/F_{\mathfrak{p}}}) \otimes \omega_{\mathfrak{q}})$ and in the third case $D_{\mathfrak{p}}(s, \pi, \omega, E/F) = \prod_{\mathfrak{q}|\mathfrak{p}} L(s, (\chi \circ \mathrm{N}_{E_{\mathfrak{q}}/F_{\mathfrak{p}}}) \otimes \omega_{\mathfrak{q}})$. The local $L$-function of a unitary character is non-zero in the open half plane $\Re(s) > 0$. □

**Lemma 2.11.** *Let $V$ denote the finite set of prime ideals of $\mathcal{O}_F$*

$$V = \left\{ \mathfrak{p} : \sum_{n=0}^{\infty} (n+1)^{d-1} (dn+1) \mathrm{N}(\mathfrak{p})^{-\frac{n}{2}} > 2 \right\}$$

*where $d = [E : F]$. Then for all $\mathfrak{p} \notin V$, $D_{\mathfrak{p}}(s, \pi, \omega, E/F)$ is non-zero in the right half plane $\Re(s) > \frac{1}{2}$.*

*Proof.* Let $\mathfrak{p} \notin V$. If $\pi_{\mathfrak{p}}$ is not unramified, then the result follows by Lemma 2.10. We are left to consider the case that $\pi_{\mathfrak{p}}$ is unramified tempered, that is $\pi_{\mathfrak{p}} \simeq \mathrm{I}(\chi_1, \chi_2)$ is a principal series where both $\chi_1$ and $\chi_2$ are unitary unramified characters. In this case $L(s, \pi_{\mathfrak{p}}) = L(s, \chi_1) L(s, \chi_2)$. It follows that

$$\lambda_{\pi}(\mathfrak{p}^n) = \sum_{i+j=n} \lambda_{\chi_1}(\mathfrak{p}^i) \lambda_{\chi_2}(\mathfrak{p}^j).$$

for all $n \geq 0$. This gives the bound $|\lambda_{\pi}(\mathfrak{p}^n)| \leq n + 1$ for all $n \geq 0$. We must study the local Euler factor

$$D_{\mathfrak{p}}(s, \pi, \omega, E/F) = 1 + \sum_{\substack{n_1, \ldots, n_r = 0 \\ n_1 + \cdots + n_r \geq 1}}^{\infty} \frac{\lambda_{\pi}(\mathfrak{p}^{(n_1 + \cdots + n_r) f_{E/F, \mathfrak{p}}}) \lambda_{\omega}(\mathfrak{q}_1^{n_1} \cdots \mathfrak{q}_r^{n_r})}{\mathrm{N}(\mathfrak{p})^{(n_1 + \cdots + n_r) f_{E/F, \mathfrak{p}} s}}.$$

The local Euler factor will be non-zero if we can show that for $\Re(s) > \frac{1}{2}$,

$$\sum_{n_1, \ldots, n_r = 0}^{\infty} \left| \frac{\lambda_{\pi}(\mathfrak{p}^{(n_1 + \cdots + n_r) f_{E/F, \mathfrak{p}}}) \lambda_{\omega}(\mathfrak{q}_1^{n_1} \cdots \mathfrak{q}_r^{n_r})}{\mathrm{N}(\mathfrak{p})^{(n_1 + \cdots + n_r) f_{E/F, \mathfrak{p}} s}} \right| < 2.$$

We observe that

$$\sum_{n_1,\ldots,n_r=0}^{\infty}\left|\frac{\lambda_\pi(\mathfrak{p}^{(n_1+\cdots+n_r)f_{E/F,\mathfrak{p}}})\lambda_\omega(\mathfrak{q}_1^{n_1}\cdots\mathfrak{q}_r^{n_r})}{\mathrm{N}(\mathfrak{p})^{(n_1+\cdots+n_r)f_{E/F,\mathfrak{p}}s}}\right|$$
$$< \sum_{n_1,\ldots,n_r=0}^{\infty}(n_1+\cdots+n_r)f_{E/F,\mathfrak{p}}+1)\mathrm{N}(\mathfrak{p})^{-\frac{n_1+\cdots+n_r}{2}f_{E/F,\mathfrak{p}}}$$
$$\leq \sum_{n_1,\ldots,n_r=0}^{\infty}((n_1+\cdots+n_r)d+1)\mathrm{N}(\mathfrak{p})^{-\frac{n_1+\cdots+n_r}{2}}$$
$$\leq \sum_{n=0}^{\infty}(n+1)^{r-1}(dn+1)\mathrm{N}(\mathfrak{p})^{-\frac{n}{2}}$$
$$\leq \sum_{n=0}^{\infty}(n+1)^{d-1}(dn+1)\mathrm{N}(\mathfrak{p})^{-\frac{n}{2}}.$$

□

We can collect the results of this section to obtain the desired lemma.

**Lemma 2.12.** *Let $E/F$ be a Galois extension of totally real number fields. Let $\pi$ be a cuspidal automorphic representation of $GL_2(\mathbf{A}_F)$ such that*

- *$\pi_\infty$ is a discrete series representation, and*
- *either $[F:\mathbf{Q}]$ is odd or $k_\nu = k_{\nu'} \mod 2$ for all archimedean places $\nu$ and $\nu'$ of $F$ where $\pi_\nu$ is the discrete series representation of weight $k_\nu$.*

*Let $\omega : E^\times\backslash\mathbf{A}_E^\times \to \mathbf{C}^\times$ be a unitary Hecke character that is unramified at all finite places. Assume that the Dirichlet series $D(s,\pi,\omega,E/F)$ has an analytic continuations to the right half plane $\Re(s) > \frac{1}{2}+A$ for some $A>0$ and that $\pi$ is not unramified for all $\mathfrak{p}\in V$ (see Lemma 2.11). Then the L-function $\Lambda(s,\mathrm{BC}_{E/F}(\pi)\otimes\omega)$ converges absolutely in the right half plane $\Re(s)>1$ and has an analytic continuation to the right half plane $\Re(s)>\frac{1}{2}+A$.*

*Proof.* We begin by noting that the Euler factors $L(s,(\mathrm{BC}_{E/F}(\pi)\otimes\omega)_\nu)$ at archimedean places $\nu$ of $E$ are analytic in the closed right half plane $\Re(s)\geq\frac{1}{2}$ (cf. [BK11, §4.3]). The result now follows by Lemma 2.9 whose hypotheses are satisfied by Lemma 2.10 and Lemma 2.11. □

## 3. Analysis of the Dirichlet series $D(s,\pi,\omega,E/F)$

In this section, we shall apply the Petersson trace formula to study the analytic properties of the Dirichlet series $D(s,\pi,\omega,E/F)$. Let $E/F$ be a Galois extension of totally real number fields. Let $\pi_1,\ldots,\pi_r$ be distinct cuspidal automorphic representations of $GL_2(\mathbf{A}_F)$. Let $\omega : E^\times\backslash\mathbf{A}_E^\times\to\mathbf{C}^\times$ be a unitary Hecke character that is unramified at all finite places.

**Lemma 3.1.** *Assume that there exist non-zero constants $c_{\pi_i}\in\mathbf{C}^\times$ for $i=1,\ldots,r$ and that there exists a $\delta\in\mathbf{R}$ such that for almost all prime ideals $\mathfrak{p}$ of $\mathcal{O}_F$, we have the bound*

$$\sum_{i=1}^{r}c_{\pi_i}\overline{\lambda_{\pi_i}(\mathfrak{p})}\sum_{\mathfrak{n}}g(\frac{\mathrm{N}(\mathfrak{n})}{X})\lambda_{\pi_i}(\mathrm{N}_{E/F}(\mathfrak{n}))\lambda_\omega(\mathfrak{n})\ll X^\delta$$

*where* $\mathfrak{n}$ *runs through the integral ideals of* $E$ *and* $g \in \mathcal{C}_c^\infty(\mathbf{R})$ *is a smooth non-negative test function with compact support in* $(0,\infty)$*. Then for all* $i = 1, \dots, r$*, the Dirichlet series*

$$D(s, \pi_i, \omega, E/F) = \sum_{\mathfrak{n}} \frac{\lambda_{\pi_i}(\mathrm{N}_{E/F}(\mathfrak{n}))\lambda_\omega(\mathfrak{n})}{\mathrm{N}(\mathfrak{n})^s}$$

*converges to an analytic function in the right half plane* $\Re(s) > \delta$*.*

*Proof.* By strong multiplicity 1 for cuspidal automorphic representations of $GL_2(\mathbf{A}_F)$, we can find prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ of $\mathcal{O}_F$ such that the matrix $(\lambda_{\pi_i}(\mathfrak{p}_j))$ is non singular. Let $A = (\overline{\lambda_{\pi_i}(\mathfrak{p}_j)})$ which is also non-singular, and we shall write $A^{-1} = (a_{i,j})$ for the inverse of $A$. Then for $i = 1, \dots, r$, we have the bound

$$c_{\pi_i} \sum_{\mathfrak{n}} g(\frac{\mathrm{N}(\mathfrak{n})}{X})\lambda_{\pi_i}(\mathrm{N}_{E/F}(\mathfrak{n}))\lambda_\omega(\mathfrak{n}) \ll a_{i,1}X^\delta + \dots + a_{i,r}X^\delta \ll X^\delta.$$

This implies that for $i = 1, \dots, r$, the Dirichlet series $D(s, \pi_i, \omega, E/F)$ converges to an analytic function in the right half plane $\Re(s) > \delta$. □

3.1. **A first case.** To provide support to our approach, we shall analyse the case where $E/F = \mathbf{Q}(\sqrt{D})/\mathbf{Q}$ is a real quadratic extension and $\omega : E^\times \backslash \mathbf{A}_E^\times \to \mathbf{C}^\times$ the identity Hecke character.

3.1.1. *The Petersson trace formula.* For $N \in \mathbf{N}$, we define the groups

$$K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{\mathbf{Z}}) : c \in N\widehat{\mathbf{Z}} \right\}$$

and

$$K_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N) : d \in 1 + N\widehat{\mathbf{Z}} \right\}.$$

Let $k > 2$ be an integer, let $N \in \mathbf{N}$, and let $\phi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ be a primitive Dirichlet character. Let $\mathcal{A}_k(N, \phi)$ denote the set of equivalence classes of cuspidal automorphic representations $\pi$ of $GL_2(\mathbf{A}_\mathbf{Q})$ such that

- the central character of $\pi$, $\omega_\pi : \mathbf{Q}^\times \backslash \mathbf{A}_\mathbf{Q}^\times \to \mathbf{C}^\times$ is equal to the composite
$$\mathbf{A}_\mathbf{Q}^\times \to \widehat{\mathbf{Z}}^\times \to (\mathbf{Z}/N\mathbf{Z})^\times \xrightarrow{\phi} \mathbf{C}^\times$$
where the first two maps are the natural projection maps from the decomposition $\mathbf{A}_\mathbf{Q}^\times = \mathbf{Q}^\times(\mathbf{R}_+^\times \times \widehat{\mathbf{Z}}^\times)$,
- $\pi_\infty$ is the discrete series representation of weight $k$, and
- $\pi_f^{K_1(N)} \neq 0$.

We shall further assume that $\phi(-1) = (-1)^k$ as otherwise the set of automorphic representations $\mathcal{A}_k(N, \phi) = \emptyset$ will be trivial. The classical Petersson trace formula (see for example [KL06, Corollary 3.12]) can be rephrased to give the following statement.

**Lemma 3.2.** *For all* $m, n \in \mathbf{N}$*,*

$$\frac{\psi(N)^{-1}(k-2)!}{(4\pi)^{k-1}} \sum_{\pi \in \mathcal{A}_k(N,\phi)} \overline{\lambda_\pi(m)}\lambda_\pi(n) = \delta_{m,n} + \frac{2\pi}{i^k} \sum_{\substack{c>0 \\ c=0 \mod N}} \frac{1}{c} S_\phi(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

*where*

- $\psi(N) = [GL_2(\widehat{\mathbf{Z}}) : K_0(N)]$,

- $\delta_{m,n} = \begin{cases} 1 & \textit{if } m = n, \\ 0 & \textit{otherwise}, \end{cases}$
- $S_\phi(m, n; c) = \sum_{x \in (\mathbf{Z}/c\mathbf{Z})^\times} \phi^{-1}(x) e(\frac{mx + n\overline{x}}{c})$ *is the Kloosterman sum, and*
- $J_{k-1}$ *is the Bessel function.*

3.1.2. *The analysis.* Let $D > 1$ be a square free integer and let $E/F = \mathbf{Q}(\sqrt{D})/\mathbf{Q}$ be the corresponding real quadratic extension. Let $g \in \mathcal{C}_c^\infty(\mathbf{R})$ be a smooth non-negative test function with compact support in $(0, \infty)$. For $m \in \mathbf{N}$, we shall be interested in the smoothly weighted sum

$$S(m, X) = \sum_{\pi \in \mathcal{A}_k(N,\phi)} \overline{\lambda_\pi(m)} \sum_{\mathfrak{n}} g\left(\frac{\mathrm{N}(\mathfrak{n})}{X}\right) \lambda_\pi(\mathrm{N}(\mathfrak{n}))$$

where $\mathfrak{n}$ runs through the integral ideals of $E = \mathbf{Q}(\sqrt{D})$. We shall first split the sum via the ideal class group of $E$. Let $\xi_1, \ldots, \xi_h$ be a set of integral ideals of $E$ that form a set of representatives of the class group $\mathrm{Cl}(E)$ of $E$. We can write

$$S(m, X) = S(\xi_1, m, X) + \cdots + S(\xi_h, m, X)$$

where for all $i = 1, \ldots, h$,

$$S(\xi_i, m, X) = \sum_{\pi \in \mathcal{A}_k(N,\phi)} \overline{\lambda_\pi(m)} \sum_{\alpha \in \mathcal{O}_E/\mathcal{O}_E^\times} g\left(\frac{\mathrm{N}((\alpha) \cdot \xi_i)}{X}\right) \lambda_\pi(\mathrm{N}((\alpha) \cdot \xi_i)).$$

**Lemma 3.3.** *Assume that* $D = 2, 3 \mod 4$. *If* $k \geq 6$, *then we have the bound*

$$S(m, X) \ll X^{\frac{3}{4} + \frac{5}{2(k-3)}}.$$

*Remark* 3.4. The assumption that $D = 2, 3$ is wholly unnecessary and can easily be removed. It was added to simplify notation in the proof of the lemma.

*Remark* 3.5. The trivial bound coming from the Ramanujan-Petersson conjecture for the $\pi \in \mathcal{A}_k(N, \phi)$ is $S(m, X) \ll X^{1+\delta}$ valid for all $\delta > 0$.

*Proof.* Let $i \in \{1, \ldots, h\}$. It will suffice to obtain the bound $S(\xi_i, m, X) \ll X^{\frac{3}{4} + \frac{5}{2(k-3)}}$. Rearranging the terms in the sum, we have that

$$S(\xi_i, m, X) = \sum_{\alpha \in \mathcal{O}_E/\mathcal{O}_E^\times} g\left(\frac{\mathrm{N}((\alpha) \cdot \xi_i)}{X}\right) \sum_{\pi \in \mathcal{A}_k(N,\phi)} \overline{\lambda_\pi(m)} \lambda_\pi(\mathrm{N}((\alpha) \cdot \xi_i)).$$

Applying the Petersson trace formula (cf. Lemma 3.2), we can write

$$\begin{aligned}
S(\xi_i, m, X) &= \frac{(4\pi)^{k-1}}{\psi(N)^{-1}(k-2)!} \sum_{\alpha \in \mathcal{O}_E/\mathcal{O}_E^\times} g\left(\frac{\mathrm{N}((\alpha)\cdot\xi_i)}{X}\right) \\
&\quad \left(\delta_{m,\mathrm{N}((\alpha)\cdot\xi_i)} + \frac{2\pi}{i^k} \sum_{\substack{c>0 \\ c=0 \mod N}} \frac{1}{c} S_\phi(m, \mathrm{N}((\alpha)\cdot\xi_i); c) J_{k-1}(\frac{4\pi\sqrt{m\mathrm{N}((\alpha)\cdot\xi_i)}}{c})\right) \\
&\ll \sum_{\alpha \in \mathcal{O}_E/\mathcal{O}_E^\times} \sum_{\substack{c>0 \\ c=0 \mod N}} g\left(\frac{\mathrm{N}((\alpha)\cdot\xi_i)}{X}\right) \frac{1}{c} S_\phi(m, \mathrm{N}((\alpha)\cdot\xi_i); c) J_{k-1}(\frac{4\pi\sqrt{m\mathrm{N}((\alpha)\cdot\xi_i)}}{c}) \\
&\ll X^{\frac{3}{4}} + \sum_{\alpha \in \mathcal{O}_E/\mathcal{O}_E^\times} \sum_{\substack{X^{\frac{1}{2}+B} \geq c>0 \\ c=0 \mod N}} g\left(\frac{\mathrm{N}((\alpha)\cdot\xi_i)}{X}\right) \frac{1}{c} S_\phi(m, \mathrm{N}((\alpha)\cdot\xi_i); c) J_{k-1}(\frac{4\pi\sqrt{m\mathrm{N}((\alpha)\cdot\xi_i)}}{c})
\end{aligned}$$

where $B = \frac{5}{4(k-3)}$ and the last line follows from an application of Lemma 3.6. We note that $B < \frac{1}{2}$ as $k \geq 6$.

We shall convert the sum over $\mathcal{O}_E/\mathcal{O}_E^\times$ to a smoothly weighted sum over $\mathbf{Z} \times \mathbf{Z}$. To do so, we recall that for all $t \in \mathbf{R}$ and for all principal ideals $\mathfrak{n}$ of $\mathcal{O}_E$, there exists a unique representative $\alpha = a + b\sqrt{D}$ such that

- $a + b\sqrt{D} > 0$, and
- $t < \log|a + b\sqrt{D}| \leq t + R$ where $R$ is the regulator of $\mathbf{Q}(\sqrt{D})$.

Let $f \in \mathcal{C}_c^\infty(\mathbf{R})$ be a smooth test function with compact support whose Fourier transform at 0 is equal to $\widehat{F}(0) = \int_{\mathbf{R}} f(t)\ dt = 1$. Then

$$\begin{aligned}
L_X &:= \sum_{\alpha \in \mathcal{O}_E/\mathcal{O}_E^\times} \sum_{\substack{X^{\frac{1}{2}+B} \geq c>0 \\ c=0 \mod N}} g\left(\frac{\mathrm{N}((\alpha)\cdot\xi_i)}{X}\right) \frac{1}{c} S_\phi(m, \mathrm{N}((\alpha)\cdot\xi_i); c) J_{k-1}(\frac{4\pi\sqrt{m\mathrm{N}((\alpha)\cdot\xi_i)}}{c}) \\
&= \widehat{F}(0) \sum_{\substack{X^{\frac{1}{2}+B} \geq c>0 \\ c=0 \mod N}} \sum_{\alpha \in \mathcal{O}_E/\mathcal{O}_E^\times} g\left(\frac{\mathrm{N}((\alpha)\cdot\xi_i)}{X}\right) \frac{1}{c} S_\phi(m, \mathrm{N}((\alpha)\cdot\xi_i); c) J_{k-1}(\frac{4\pi\sqrt{m\mathrm{N}((\alpha)\cdot\xi_i)}}{c}) \\
&= \sum_{\substack{X^{\frac{1}{2}+B} \geq c>0 \\ c=0 \mod N}} \sum_{a,b \in \mathbf{Z}} H(a + b\sqrt{D}) g\left(\frac{\mathrm{N}(\xi_i)|a^2 - Db^2|}{X}\right) \\
&\qquad \frac{1}{c} S_\phi(m, \mathrm{N}(\xi_i)|a^2 - Db^2|; c) J_{k-1}(\frac{4\pi\sqrt{m\mathrm{N}(\xi_i)|a^2 - Db^2|}}{c})
\end{aligned}$$

where

$$H(z) = \begin{cases} 0 & \text{if } z \leq 0, \\ \int_{\log|z|-R}^{\log|z|} f(t)\ dt & \text{otherwise.} \end{cases}$$

The function $H$ is easily seen to be a smooth function with compact support in $(0,\infty)$. As both the functions $g$ and $H$ have support in $(0,\infty)$, we see that

$$\begin{aligned} L_X &= \sum_{\substack{X^{\frac{1}{2}+B}\geq c>0 \\ c=0 \mod N}} \sum_{a,b\in\mathbf{Z}} H(a+b\sqrt{D})g\left(\frac{\mathrm{N}(\xi_i)(a^2-Db^2)}{X}\right) \\ &\qquad \frac{1}{c}S_\phi(m,\mathrm{N}(\xi_i)(a^2-Db^2);c)J_{k-1}(\frac{4\pi\sqrt{m\mathrm{N}(\xi_i)(a^2-Db^2)}}{c}) \\ &= \sum_{\substack{X^{\frac{1}{2}+B}\geq c>0 \\ c=0 \mod N}} \sum_{a,b\in\mathbf{Z}} c^{-1}I_{\xi_i,c}(a,b)S_\phi(m,\mathrm{N}(\xi_i)(a^2-Db^2);c) \end{aligned}$$

where $I_{\xi_i,c}(a,b) = H(a+b\sqrt{2})g\left(\frac{\mathrm{N}(\xi_i)(a^2-Db^2)}{X}\right)J_{k-1}(\frac{4\pi\sqrt{m\mathrm{N}(\xi_i)(a^2-Db^2)}}{c})$.

We shall expand out the Kloosterman sum and then perform Poisson summation on the variables $a$ and $b$ modulo $c$. This gives the following

$$\begin{aligned} L_X &= \sum_{\substack{X^{\frac{1}{2}+B}\geq c>0 \\ c=0 \mod N}} \sum_{x\in(\mathbf{Z}/c\mathbf{Z})^\times} \phi^{-1}(x) \sum_{a,b\in\mathbf{Z}} c^{-1}I_{\xi_i,c}(a,b)e(\frac{mx+\mathrm{N}(\xi_i)(a^2-Db^2)\overline{x}}{c}) \\ &= \sum_{\substack{X^{\frac{1}{2}+B}\geq c>0 \\ c=0 \mod N}} \sum_{x\in(\mathbf{Z}/c\mathbf{Z})^\times} \phi^{-1}(x) \sum_{a_0,b_0\in(\mathbf{Z}/c\mathbf{Z})} e(\frac{mx+\mathrm{N}(\xi_i)(a_0^2-Db_0^2)\overline{x}}{c}) \sum_{\substack{a=a_0 \mod c \\ b=b_0 \mod c}} c^{-1}I_{\xi_i,c}(a,b) \\ &= \sum_{\substack{X^{\frac{1}{2}+B}\geq c>0 \\ c=0 \mod N}} \sum_{x\in(\mathbf{Z}/c\mathbf{Z})^\times} \phi^{-1}(x) \sum_{a_0,b_0\in(\mathbf{Z}/c\mathbf{Z})} e(\frac{mx+\mathrm{N}(\xi_i)(a_0^2-Db_0^2)\overline{x}}{c}) \sum_{r,s\in\mathbf{Z}} c^{-3}\widehat{I}_{\xi_i,c}(\frac{r}{c},\frac{s}{c})e(\frac{ra_0+sb_0}{c}) \end{aligned}$$

where

$$\widehat{I}_{\xi_i,c}(\frac{r}{c},\frac{s}{c}) = \int_{\mathbf{R}}\int_{\mathbf{R}} I_{\xi_i,c}(t_1,t_2)e(-\frac{r}{c}t_1-\frac{s}{c}t_2)\ dt_1dt_2.$$

Thus we are left to study the sum

$$L_X = \sum_{\substack{X^{\frac{1}{2}+B}\geq c>0 \\ c=0 \mod N}} \sum_{r,s\in\mathbf{Z}} c^{-3}\widehat{I}_{\xi_i,c}(\frac{r}{c},\frac{s}{c})\mathcal{A}_\phi(m,\mathrm{N}(\xi_i),-D\mathrm{N}(\xi_i),r,s,c)$$

where

$$\mathcal{A}_\phi(m,\mathrm{N}(\xi_i),-D\mathrm{N}(\xi_i),r,s,c) = \sum_{x\in(\mathbf{Z}/c\mathbf{Z})^\times} \phi^{-1}(x) \sum_{a_0,b_0\in(\mathbf{Z}/c\mathbf{Z})} e(\frac{mx+\mathrm{N}(\xi_i)(a_0^2-Db_0^2)\overline{x}+ra_0+sb_0}{c}).$$

Let $\delta>0$ and let

$$\mathcal{D} = \{(r,s)\in\mathbf{Z}\times\mathbf{Z} : |s-\sqrt{D}r| < X^{-\frac{1}{2}+B+\delta} \text{ and } |s+\sqrt{D}r| < X^{\frac{1}{2}+B+\delta}\}.$$

We shall study the two sums

$$L'_X = \sum_{\substack{X^{\frac{1}{2}+B}\geq c>0 \\ c=0 \mod N}} \sum_{(r,s)\in\mathcal{D}} c^{-3}\widehat{I}_{\xi_i,c}(\frac{r}{c},\frac{s}{c})\mathcal{A}_\phi(m,\mathrm{N}(\xi_i),-D\mathrm{N}(\xi_i),r,s,c)$$

and

$$L''_X = \sum_{\substack{X^{\frac{1}{2}+B}\geq c>0 \\ c=0 \mod N}} \sum_{(r,s)\notin\mathcal{D}} c^{-3}\widehat{I}_{\xi_i,c}(\frac{r}{c},\frac{s}{c})\mathcal{A}_\phi(m,\mathrm{N}(\xi_i),-D\mathrm{N}(\xi_i),r,s,c)$$

By applying the trivial bound $|\mathcal{A}_\phi(m,\mathrm{N}(\xi_i),-D\mathrm{N}(\xi_i),r,s,c)| \le c^3$ along with Lemma 3.7, we can bound the sum $L''_X$ absolutely as

$$L''_X \ll X^T$$

which is valid for all $T \in \mathbf{R}$. It follows that $L_X \ll L'_X$ and we are left to bound the sum

$$L'_X = \sum_{(r,s)\in\mathcal{D}} \sum_{\substack{X^{\frac{1}{2}+B} \ge c > 0 \\ c=0 \mod N}} c^{-3}\widehat{I}_{\xi_i,c}(\frac{r}{c},\frac{s}{c})\mathcal{A}_\phi(m,\mathrm{N}(\xi_i),-D\mathrm{N}(\xi_i),r,s,c).$$

Let $r,s \in \mathbf{Z}$ such that $-4D\mathrm{N}(\xi_i)^2 m + \mathrm{N}(\xi_i)Dr^2 - \mathrm{N}(\xi_i)s^2 \neq 0$. We note that all elements $(r,s) \in \mathcal{D}$ satisfy this property when $X$ is sufficiently large. Applying the respective definitions, we have that

$$\begin{aligned}
&\sum_{\substack{X^{\frac{1}{2}+B} \ge c > 0 \\ c=0 \mod N}} c^{-3}\widehat{I}_{\xi_i,c}(\frac{r}{c},\frac{s}{c})\mathcal{A}_\phi(m,\mathrm{N}(\xi_i),-D\mathrm{N}(\xi_i),r,s,c) \\
&= \sum_{\substack{X^{\frac{1}{2}+B} \ge c > 0 \\ c=0 \mod N}} c^{-3}\mathcal{A}_\phi(m,\mathrm{N}(\xi_i),-D\mathrm{N}(\xi_i),r,s,c) \int_{\mathbf{R}}\int_{\mathbf{R}} H(t_1+t_2\sqrt{D}) \\
&\qquad g\left(\frac{\mathrm{N}(\xi_i)(t_1^2-Dt_2^2)}{X}\right) J_{k-1}(\frac{4\pi\sqrt{m\mathrm{N}(\xi_i)(t_1^2-Dt_2^2)}}{c}) e(-\frac{r}{c}t_1 - \frac{s}{c}t_2)\ dt_1 dt_2.
\end{aligned}$$

If we perform the change of variables $u_1 = t_1 + t_2\sqrt{D}$, $u_2 = c^{-2}(t_2 - t_2\sqrt{D})$ and bring the $c$-sum inside the double integral, we obtain the following absolute bound

$$\begin{aligned}
&\frac{1}{2\sqrt{D}}\int_{\mathbf{R}}\int_{\mathbf{R}} |H(u_1)J_{k-1}\left(4\pi\sqrt{m\mathrm{N}(\xi_i)u_1u_2}\right)| \\
&\sum_{\substack{X^{\frac{1}{2}+B} \ge c > 0 \\ c=0 \mod N}} c^{-1}|\mathcal{A}_\phi(m,\mathrm{N}(\xi_i),-D\mathrm{N}(\xi_i),r,s,c) g\left(\frac{\mathrm{N}(\xi_i)u_1u_2c^2}{X}\right)| du_2 du_1
\end{aligned} \tag{3.1}$$

By Lemma 3.13, the $L$-function

$$\sum_{\substack{c=1 \\ c=0 \mod N}}^{\infty} c^{-s}c^{-1}\mathcal{A}_\phi(m,\mathrm{N}(\xi_i),-D\mathrm{N}(\xi_i),r,s,c)$$

converges absolutely and is bounded in vertical strips in the right half plane $\Re(s) > 1$. We may contour shift to obtain the bound (cf. [Ven02, §6.5.1])

$$\sum_{\substack{X^{\frac{1}{2}+B} \ge c > 0 \\ c=0 \mod N}} c^{-1}\mathcal{A}_\phi(m,\mathrm{N}(\xi_i),-D\mathrm{N}(\xi_i),r,s,c) g\left(\frac{\mathrm{N}(\xi_i)u_1u_2c^2}{X}\right) \ll \left(\frac{X}{u_1u_2\mathrm{N}(\xi_i)}\right)^A$$

which is valid for all $A > \frac{1}{2}$. We shall apply this bound for $A = \frac{3}{4} + \delta$ since using the bound $J_{k-1}(z) \ll \min(1, z^{-\frac{1}{2}})$ this ensures that the double integral of (3.1) converges. This gives a bound for (3.1) of $\mathrm{O}(X^{\frac{3}{4}+\delta})$. That is we have bounded the contribution of a single $(r,s) \in \mathcal{D}$ by $\mathrm{O}(X^{\frac{3}{4}+\delta})$. By Lemma 3.8, the cardinality of the set $|\mathcal{D}| = \mathrm{O}(X^{2B+2\delta})$. Putting everything together, we have the bound

$$L'_X \ll X^{\frac{3}{4}+2B+3\delta}$$

valid for all $\delta > 0$.

$\square$

3.1.3. *Auxillary lemmas.* We have collected the Lemmas needed in Section 3.1.2. Unless otherwise specified, we shall keep the notations and assumptions of Section 3.1.2.

**Lemma 3.6.** *Let $B > 0$. Then if $k > 3$, we have the bound*

$$\sum_{\alpha \in \mathcal{O}_E/\mathcal{O}_E^\times} \sum_{\substack{c > X^{\frac{1}{2}+B} \\ c=0 \mod N}} \left| g\left(\frac{\mathrm{N}((\alpha)\cdot\xi_i)}{X}\right) \frac{1}{c} S_\phi(m, \mathrm{N}((\alpha)\cdot\xi_i); c) J_{k-1}\left(\frac{4\pi\sqrt{m\mathrm{N}((\alpha)\cdot\xi_i)}}{c}\right) \right| \ll X^{2+B(3-k)}$$

*for all $i = 1, \dots, h$.*

*Proof.* By applying the trivial bound for the Kloosterman sum $|S_\phi(m, \mathrm{N}(\mathfrak{n}); c)| \leq c^2$ along with Bessel function asymptotic $J_{k-1}(z) \ll z^{k-1}$ for $z \ll 1$, we see that

$$\begin{aligned}
&\sum_{\alpha \in \mathcal{O}_E/\mathcal{O}_E^\times} \sum_{\substack{c > X^{\frac{1}{2}+B} \\ c=0 \mod N}} \left| g\left(\frac{\mathrm{N}((\alpha)\cdot\xi_i)}{X}\right) \frac{1}{c} S_\phi(m, \mathrm{N}((\alpha)\cdot\xi_i); c) J_{k-1}\left(\frac{4\pi\sqrt{m\mathrm{N}((\alpha)\cdot\xi_i)}}{c}\right) \right| \\
&\ll \sum_{\alpha \in \mathcal{O}_E/\mathcal{O}_E^\times} |g\left(\frac{\mathrm{N}((\alpha)\cdot\xi_i)}{X}\right)| \sum_{\substack{c > X^{\frac{1}{2}+B} \\ c=0 \mod N}} c\left(\frac{\sqrt{\mathrm{N}((\alpha)\cdot\xi_i)}}{c}\right)^{k-1} \\
&\ll \sum_{\alpha \in \mathcal{O}_E/\mathcal{O}_E^\times} |g\left(\frac{\mathrm{N}((\alpha)\cdot\xi_i)}{X}\right)| X^{\frac{k-1}{2}} \sum_{\substack{c > X^{\frac{1}{2}+B} \\ c=0 \mod N}} c^{2-k} \\
&\ll \sum_{\alpha \in \mathcal{O}_E/\mathcal{O}_E^\times} |g\left(\frac{\mathrm{N}((\alpha)\cdot\xi_i)}{X}\right)| X^{\frac{k-1}{2}} X^{(\frac{1}{2}+B)(3-k)} \\
&\ll X X^{\frac{k-1}{2}} X^{(\frac{1}{2}+B)(3-k)} = X^{2+B(3-k)}
\end{aligned}$$

where the last line follows from the fact the the number of principal ideals $(\alpha)$ of $\mathcal{O}_E$ for which $\mathrm{N}((\alpha)) = \mathrm{O}(X)$ is $\mathrm{O}(X)$. $\square$

**Lemma 3.7.** *Assume that we are not in the case where $r = s = 0$. Then for all integers $K \in \mathbf{N}$, we have the bounds*

$$\widehat{I}_{\xi_i,c}\left(\frac{r}{c}, \frac{s}{c}\right) \ll (s - \sqrt{D}r)^{-K} X^{1-K(\frac{1}{2}-B)}$$

*and*

$$\widehat{I}_{\xi_i,c}\left(\frac{r}{c}, \frac{s}{c}\right) \ll (s + \sqrt{D}r)^{-K} X^{1+K(\frac{1}{2}+B)}$$

*for all $c \leq X^{\frac{1}{2}+B}$ and for all $i = 1, \dots h$.*

*Proof.* We begin with the definition

$$\widehat{I}_{\xi_i,c}\left(\frac{r}{c}, \frac{s}{c}\right) = \int_{\mathbf{R}} \int_{\mathbf{R}} H(t_1 + t_2\sqrt{D}) g\left(\frac{\mathrm{N}(\xi_i)(t_1^2 - Dt_2^2)}{X}\right) J_{k-1}\left(\frac{4\pi\sqrt{m\mathrm{N}(\xi_i)(t_1^2 - Dt_2^2)}}{c}\right) e\left(-\frac{r}{c}t_1 - \frac{s}{c}t_2\right) dt_1 dt_2.$$

We now perform a change of variables $u_1 = t_1 + t_2\sqrt{D}$ and $u_2 = t_1 - t_2\sqrt{D}$. This gives that

$$\widehat{I}_{\xi_i,c}(\frac{r}{c},\frac{s}{c}) = -\frac{1}{2\sqrt{D}}\int_{\mathbf{R}}\int_{\mathbf{R}} H(u_1)g\left(\frac{\mathrm{N}(\xi_i)u_1u_2}{X}\right)J_{k-1}(\frac{4\pi\sqrt{m\mathrm{N}(\xi_i)u_1u_2}}{c})e(\frac{\beta}{c}u_2+\frac{\alpha}{c}u_1)\,du_2du_1$$

where $\alpha = -\frac{1}{2\sqrt{D}}(\sqrt{D}r+s)$ and $\beta = -\frac{1}{2\sqrt{D}}(-\sqrt{D}r+s)$. After performing another change of variables $u_2 \mapsto u_2X$, we see that

$$\widehat{I}_{\xi_i,c}(\frac{r}{c},\frac{s}{c}) = -\frac{X}{2\sqrt{D}}\int_{\mathbf{R}}\int_{\mathbf{R}} H(u_1)g\left(\mathrm{N}(\xi_i)u_1u_2\right)J_{k-1}(\frac{4\pi\sqrt{m\mathrm{N}(\xi_i)Xu_1u_2}}{c})e(\frac{\beta X}{c}u_2+\frac{\alpha}{c}u_1)\,du_2du_1$$

The first (resp. second) bound is now obtained by integrating by parts $K$-times on the $u_2$-variable (resp. $u_1$-variable). Let us sketch the argument in the case of the first bound. After integrating by parts $K$-times on the $u_2$-variable, we observe that

$$\widehat{I}_{\xi_i,c}(\frac{r}{c},\frac{s}{c}) = -\frac{X}{2\sqrt{D}}\left(\frac{c}{\beta X}\right)^K\int_{\mathbf{R}}\int_{\mathbf{R}} e(\frac{\beta X}{c}u_2+\frac{\alpha}{c}u_1)\frac{\partial^K}{\partial u_2^K}\left(H(u_1)g\left(\mathrm{N}(\xi_i)u_1u_2\right)J_{k-1}(\frac{4\pi\sqrt{m\mathrm{N}(\xi_i)Xu_1u_2}}{c})\right)du_2$$

The partial derivative is a sum of terms of the form

$$\left(\frac{\sqrt{X}}{c}\right)^{\alpha_1}u_1^{\alpha_2}u_2^{\alpha_3}H^{(\alpha_4)}(u_1)g^{(\alpha_5)}(\mathrm{N}(\xi_i)u_1u_2)J_{k-1}^{(\alpha_6)}(\frac{4\pi\sqrt{m\mathrm{N}(\xi_i)Xu_1u_2}}{c})$$

where $\alpha_1,\alpha_4,\alpha_5,\alpha_6 \in \mathbf{N}^0$ and $\alpha_2,\alpha_3$ are (half)-integers such that $|\alpha_j| \le K$ for all $j = 1,\ldots,6$. As both $H$ and $g$ are smooth functions with compact support in $(0,\infty)$, we obtain the bound

$$\widehat{I}_{\xi_i,c}(\frac{r}{c},\frac{s}{c}) \ll X\left(\frac{c}{\beta X}\right)^K\max\left(1,\left(\frac{\sqrt{X}}{c}\right)^K\right) = \beta^{-K}X^{1-K}\max(c^K,X^{\frac{K}{2}}) \ll \beta^{-K}X^{1-K}X^{K(\frac{1}{2}+B)}.$$

$\square$

**Lemma 3.8.** *Let $0 \le A < \frac{1}{2}$, and let*

$$\mathcal{D} = \{(r,s)\in\mathbf{Z}\times\mathbf{Z} : |s-\sqrt{D}r| < X^{-\frac{1}{2}+A} \text{ and } |s+\sqrt{D}r| < X^{\frac{1}{2}+A}\}.$$

*Then the cardinality of the set $\mathcal{D}$ is $\mathrm{O}(X^{2A})$.*

*Proof.* By writing $s = \frac{1}{2}\left(s-\sqrt{D}r+s+\sqrt{D}r\right)$ and $r = -\frac{1}{2\sqrt{D}}\left(s-\sqrt{D}r-(s+\sqrt{D}r)\right)$, we obtain the bounds $r,s \ll X^{\frac{1}{2}+A}$. In order for the condition $|s-\sqrt{D}r| < X^{-\frac{1}{2}+A}$ to be satisfied, one requires that either

$$\{r\sqrt{D}\} < X^{-\frac{1}{2}+A} \text{ or } \{r\sqrt{D}\} > 1-X^{-\frac{1}{2}+A}$$

where $\{z\}$ denotes the non-integer part of a $z\in\mathbf{R}$. Furthermore, for any such $r$ there exists at most a single $s$ for which the condition is satisfied. By the Weyl equidistribution theorem, the number of such $r$ is $\mathrm{O}(X^{-\frac{1}{2}+A}X^{\frac{1}{2}+A}) = \mathrm{O}(X^{2A})$.

$\square$

We shall have need of the following type of exponential sum.

$$\mathcal{A}_\phi(m,A,B,r,s,c) = \sum_{x\in(\mathbf{Z}/c\mathbf{Z})^\times}\phi^{-1}(x)e(\frac{mx}{c})\sum_{a_0,b_0\in(\mathbf{Z}/c\mathbf{Z})}e(\frac{A\overline{x}a_0^2+ra_0+B\overline{x}b_0^2+sb_0}{c}).$$

where $m, A, B, r, s \in \mathbf{Z}$, $\phi$ is a Dirichlet character modulo some integer $N$, and $N|c$. We can rewrite this sum in terms of quadratic Gauss sums to give

$$\mathcal{A}_\phi(m, A, B, r, s, c) = \sum_{x \in (\mathbf{Z}/c\mathbf{Z})^\times} \phi^{-1}(x) e(\frac{mx}{c}) G(A\overline{x}, r, c) G(B\overline{x}, s, c)$$

where $G(\alpha, \beta, \gamma) = \sum_{z \in (\mathbf{Z}/\gamma\mathbf{Z})} e(\frac{\alpha z^2 + \beta z}{\gamma})$ denotes the usual quadratic Gauss sum defined for $\alpha, \beta \in \mathbf{Z}$ and $\gamma \in \mathbf{N}$.

**Lemma 3.9.** *The exponential sum $\mathcal{A}_\phi(m, A, B, r, s, c)$ is multiplicative in the sense that writing $c = uv$ where $\gcd(u, v) = 1$, we have that*

$$\mathcal{A}_\phi(m, A, B, r, s, c) = \mathcal{A}_{\phi_u}(mv', Av', Bv', rv', sv', u) \mathcal{A}_{\phi_v}(mu', Au', Bu', ru', su', v)$$

*where $u', v' \in \mathbf{Z}$ such that $vv' + uu' = 1$ and we have decomposed the Dirichlet character $\phi = \phi_u \times \phi_v : (\mathbf{Z}/u\mathbf{Z})^\times \times (\mathbf{Z}/v\mathbf{Z})^\times \to \mathbf{C}^\times$.*

*Proof.* This follows directly from the definition. □

**Lemma 3.10.** *We have the bound $|\mathcal{A}_\epsilon(m, A, B, r, s, c)| \leq |\gcd(A, c) \gcd(B, c)| c^2$.*

*Proof.* This follows from the quadratic Gauss sum bounds $|G(A\overline{x}, r, c)| \leq \gcd(A, c)\sqrt{c}$ and $|G(B\overline{x}, s, c)| \leq \gcd(B, c)\sqrt{c}$. □

**Lemma 3.11.** *Assume that $\gamma \in \mathbf{N}$ is odd and that $\gcd(\alpha, \gamma) = 1$. Then for all $\beta \in \mathbf{Z}$,*

$$G(\alpha, \beta, \gamma) = e(\frac{-\overline{4\alpha}\beta^2}{\gamma}) \left(\frac{\alpha}{\gamma}\right) \epsilon_\gamma \sqrt{\gamma}$$

*where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Jacboi symbol, and $\epsilon_\gamma = 1$ if $\gamma = 1 \mod 4$ and $\epsilon_\gamma = i$ otherwise.*

*Proof.* The quadratic Gauss sum is evaluated here by completing the square to give

$$G(\alpha, \beta, \gamma) = e(\frac{-\overline{4\alpha}\beta^2}{\gamma}) G(\alpha, 0, \gamma).$$

The result follows from the explicit evaluation of $G(\alpha, 0, \gamma)$. □

**Lemma 3.12.** *Assume that $c$ is odd, $\gcd(A, c) = \gcd(B, c) = 1$, and $\phi = 1$ the identity character. Then*

$$\mathcal{A}_1(m, A, B, r, s, c) = c\epsilon_c^2 \left(\frac{AB}{c}\right) \sum_{x \in (\mathbf{Z}/c\mathbf{Z})^\times} e(\frac{(4ABm - Br^2 - As^2)x}{c})$$

*Proof.* By applying Lemma 3.11, we see that

$$\begin{aligned}
\mathcal{A}_1(m, A, B, r, s, c) &= \sum_{x \in (\mathbf{Z}/c\mathbf{Z})^\times} e(\frac{mx}{c}) G(A\overline{x}, r, c) G(B\overline{x}, s, c) \\
&= c\epsilon_c^2 \sum_{x \in (\mathbf{Z}/c\mathbf{Z})^\times} \left(\frac{A\overline{x}}{c}\right) \left(\frac{B\overline{x}}{c}\right) e(\frac{mx - \overline{4A}r^2 x - \overline{4B}s^2 x}{c}) \\
&= c\epsilon_c^2 \left(\frac{AB}{c}\right) \sum_{x \in (\mathbf{Z}/c\mathbf{Z})^\times} e(\frac{mx - \overline{4A}r^2 x - \overline{4B}s^2 x}{c}) \\
&= c\epsilon_c^2 \left(\frac{AB}{c}\right) \sum_{x \in (\mathbf{Z}/c\mathbf{Z})^\times} e(\frac{(4ABm - Br^2 - As^2)x}{c})
\end{aligned}$$

where the last line follows as $\gcd(4AB, c) = 1$. □

**Lemma 3.13.** *If $4ABm - Br^2 - As^2 \neq 0$, then the $L$-function*

$$\sum_{\substack{c=0 \\ c=1 \mod N}}^{\infty} \frac{\mathcal{A}_\phi(m, A, B, r, s, c)}{c^s}$$

*converges absolutely and is bounded in vertical strips in right half plane $\Re(s) > 2$.*

*Proof.* Let $p_1, \ldots, p_r$ be a finite set of rational primes including the prime 2 such that for all rational primes $p \neq p_1, \ldots, p_r$, $\gcd(A, p) = \gcd(B, p) = \gcd(N, p) = 1$. We shall decompose $c = \alpha\beta$ where $\alpha = p_1^{l_1} \cdots p_r^{l_r}$ for some $l_1, \ldots, l_r \in \mathbf{N}^0$ and $\gcd(p_i, \beta) = 1$ for all $i = 1, \ldots, r$. By Lemma 3.9, we can write

$$\mathcal{A}_\phi(m, A, B, r, s) = \mathcal{A}_\phi(m\beta', A\beta', B\beta', r\beta', s\beta', \alpha)\mathcal{A}_1(m\alpha', A\alpha', B\alpha', r\alpha', s\alpha', \beta).$$

where $\alpha', \beta' \in \mathbf{Z}$ such that $\alpha\alpha' + \beta\beta' = 1$. By Lemma 3.10, the first exponential sum is bounded

$$|\mathcal{A}_\phi(m\beta', A\beta', B\beta', r\beta', s\beta', \alpha)| \leq |AB|\alpha^2.$$

By Lemma 3.12, the second sum is a Ramanujan sum which is bounded by

$$\mathcal{A}_1(m\alpha', A\alpha', B\alpha', r\alpha', s\alpha', \beta) \ll \beta \cdot \beta^\delta$$

valid for all $\delta > 0$. It follows that the $L$-function

$$\sum_{\substack{c=0 \\ c=1 \mod N}}^{\infty} \frac{\mathcal{A}_\phi(m, A, B, r, s, c)}{c^s}$$

converges absolutely and is bounded in vertical strips in the right half plane $\Re(s) > 2 + \delta$. □

3.1.4. *The result.*

**Theorem 3.14.** *Let $k \geq 6$, let $N \in \mathbf{N}$, and let $\phi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ be a primitive Dirichlet character. Let $\pi \in \mathcal{A}_k(N, \phi)$ (see Section 3.1.1) be a cuspidal automorphic representation $GL_2(\mathbf{A_Q})$ such that $\pi_p$ is not unramified for all rational primes $p < 70$. Let $E = \mathbf{Q}(\sqrt{D})$ be a quadratic real extension such that $D = 2, 3 \mod 4$. Then the base change $L$-function $\Lambda(s, \mathrm{BC}_{E/\mathbf{Q}}(\pi))$*

- *converges absolutely in the right half plane $\Re(s) > 1$, and*
- *continues to a holomorphic function in the right half plane $\Re(s) > \frac{3}{4} + \frac{5}{2(k-3)}$.*

*Remark* 3.15. Let us remark that in the simple setting of this theorem, the $L$-function decomposes as

$$\Lambda(s, \mathrm{BC}_{E/\mathbf{Q}}(\pi)) = \Lambda(s, \pi)\Lambda(s, \pi \cdot \chi_{E/F})$$

where $\chi_{E/F} : \mathbf{Q}^\times \backslash \mathbf{A}_Q^\times \to \mathbf{C}^\times$ denotes the Hecke character corresponding to the extension $E/\mathbf{Q}$ via class field theory. It then follows from the theory of $L$-functions associated to cuspidal automorphic representations of $GL_2(\mathbf{A_Q})$ that the $L$-function $\Lambda(s, \mathrm{BC}_{E/\mathbf{Q}}(\pi))$ continues to a holomorphic function of the complex plane. We stress however that the aim of the method detailed in this article is to eventually obtain results in situations where the known results are no longer available.

*Remark* 3.16. As has already been remarked, the condition that $D = 2, 3 \mod 4$ can be removed without any difficulties and was imposed only to simplify notation in the proof of Lemma 3.3.

*Remark* 3.17. We have chosen throughout this article to work with the Petersson trace formula as opposed to more general Petersson-Kuznetsov trace formula which incorporates the contributions of both Maass forms and Eisenstein series. The reason being that the spectral side of the Petersson-Kuznetsov is no longer finite and we can no longer appeal to Lemma 3.1 to separate the Dirichlet series. If one were willing to admit certain hypotheses (see [Her12a, Hypothesis 1.3] for the case of the symmetric cube) then one could still separate the Dirichlet series. In this case, we expect that our analysis could be improved to show the analytic continuation of the $L$-function $\Lambda(s, \mathrm{BC}_{E/\mathbf{Q}}(\pi))$ to the right half plane $\Re(s) > \frac{1}{2}$ for a general cuspidal automorphic representation of $GL_2(\mathbf{A_Q})$. The reason being that in the Petersson-Kuznetsov trace formula the Bessel function $J_{k-1}$ is replaced with a smooth function of compact support which would allow the bound of Lemma 3.3 to be improved.

*Proof.* By Lemma 3.1 and Lemma 3.3, we see that the Dirichlet series $D(s, \pi, 1, E/\mathbf{Q})$ continues to an analytic function in the right half plane $\Re(s) > \frac{3}{4} + \frac{5}{2(k-3)}$. The result follows by applying Lemma 2.12 and observing that

$$V = \left\{ p : \sum_{n=0}^{\infty} (n+1)(2n+1) p^{-\frac{n}{2}} > 2 \right\} = \{p : p < 70\}.$$

□

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