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Asymptotic Behavior of Block Toeplitz Matrices and Determinants. II

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1. INTRODUCTION

The use of Hankel operators in the study of Toeplitz operators goes back at least to the fundamental paper [6] of Gohberg and Krein who used them to show that certain Toeplitz operators are Fredholm. There is a simple identity relating Toeplitz and Hankel operators that makes this crystal-clear, i.e.,

$$T[\varphi\psi] - T[\varphi]\,T[\psi] = H[\varphi]\,H[\tilde{\psi}]. \tag{1.1}$$

The notation here is as follows. If φ is a function defined on the unit circle and with Fourier coefficients φ_k, then $T[\varphi]$, $H[\varphi]$ are, respectively, the semi-infinite Toeplitz and Hankel matrices

$$T[\varphi] = (\varphi_{i-j}) \qquad 0 \leqslant i, j < \infty$$
$$H[\varphi] = (\varphi_{i+j+1}) \qquad 0 \leqslant i, j < \infty.$$

If φ is bounded these may be thought of as operators on l_2 of the non-negative integers. In addition we write

$$\tilde{\varphi}(z) = \varphi(z^{-1}).$$

Identity (1.1) is trivial. The left side has i, j entry

$$\sum_{k=-\infty}^{\infty} \varphi_{i-k}\psi_{k-j} - \sum_{k=0}^{\infty} \varphi_{i-k}\psi_{k-j} = \sum_{k=-\infty}^{-1} \varphi_{i-k}\psi_{k-j} = \sum_{k=0}^{\infty} \varphi_{i+k+1}\psi_{-k-j-1},$$

which is the i, j entry of the right side of (1.1).

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Two applications of the identity give (I = identity matrix):

$$T[\varphi]\, T[\varphi^{-1}] = I - H[\varphi]\, H[\tilde{\varphi}^{-1}] \tag{1.2}$$

$$T[\varphi^{-1}]\, T[\varphi] = I - H[\varphi^{-1}]\, H[\tilde{\varphi}]. \tag{1.3}$$

Thus, if the two products of Hankel operators are compact $T[\varphi^{-1}]$ is an inverse of $T[\varphi]$ modulo the compact operators and therefore, $T[\varphi]$ is Fredholm.

A theorem of Hartman [9] gives a necessary and sufficient condition that $H[\varphi]$ be compact, i.e., that there exist a continuous function ψ such that $\varphi_k = \psi_k$ for $k > 0$. It follows that if φ is a nonzero continuous function then all four Hankel operators are compact. More generally, if φ and φ^{-1} belong to $H^\infty + C$ then $H[\tilde{\varphi}]$ and $H[\tilde{\varphi}^{-1}]$ are compact and $H[\varphi]$ and $H[\varphi^{-1}]$ are bounded, the two products of Hankel operators are compact and $T[\varphi]$ is Fredholm. This fact is well-known [3, Corollary 7.34].

It turns out there is an analogue of (1.1) for finite Toeplitz matrices

$$T_n[\varphi] = (\varphi_{i-j}) \qquad 0 \leqslant i, j \leqslant n.$$

It reads

$$T_n[\varphi\psi] - T_n[\varphi]\, T_n[\psi] = P_n H[\varphi]\, H[\tilde{\psi}] P_n + Q_n H[\tilde{\varphi}]\, H[\psi] Q_n\,. \tag{1.4}$$

Here P_n and Q_n are defined by

$$P_n(f_0, f_1, \ldots) = (f_0, \ldots, f_n, 0, \ldots)$$

$$Q_n(f_0, f_1, \ldots) = (f_n, \ldots, f_0, 0, \ldots)$$

and $T_n[\varphi]$ is identified with $P_n T[\varphi]\, P_n$ in the usual way. The proof of (1.4) is similar to that of (1.1) and is left to the reader. This identity will be exploited to obtain simple proofs under quite general assumptions of asymptotic results for finite Toeplitz matrices.

The results also will hold for block Toeplitz matrices. These are of the same form (φ_{i-j}) but each φ_i is itself a matrix of fixed order r; the corresponding function φ is then $r \times r$ matrix-valued. Identities (1.1) and (1.4) hold without change in the matrix case.

The Szegö limit theorem (in the scalar case $r = 1$) states that under certain conditions one has for the Toeplitz determinants

$$D_n[\varphi] = \det T_n[\varphi]$$

the asymptotic formula

$$D_n[\varphi] \sim G[\varphi]^{n+1} E[\varphi] \tag{1.5}$$

where

$$G[\varphi] = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log \varphi(e^{i\theta})\, d\theta\right\} \tag{1.6}$$

$$E[\varphi] = \exp \sum_{k=1}^{\infty} k(\log \varphi)_k (\log \varphi)_{-k}\,. \tag{1.7}$$

This holds for non-negative integrable φ whenever $G[\varphi]$ is nonzero and $E[\varphi]$ is finite [8], but what happens for nonreal φ is less clear. Sufficient conditions for the validity of (1.5) have been given by many authors. We mention here only the conditions

$$\begin{gathered} \sum_{k=-\infty}^{\infty} |\varphi_k| + \sum_{k=-\infty}^{\infty} |k|\, |\varphi_k|^2 < \infty \\ \varphi(e^{i\theta}) \neq 0, \qquad \Delta_{0\leqslant\theta\leqslant 2\pi} \arg \varphi(e^{i\theta}) = 0 \end{gathered} \tag{1.8}$$

the sufficiency of which was established by Hirschman [11].

Recently [16], relation (1.5) was extended to the matrix case under the analogue of these conditions, i.e.,

$$\begin{gathered} \sum_{k=-\infty}^{\infty} \| \varphi_k \| + \sum_{k=-\infty}^{\infty} |k|\, \| \varphi_k \|^2 < \infty \\ \det \varphi(e^{i\theta}) \neq 0, \qquad \Delta_{0\leqslant\theta\leqslant 2\pi} \arg \det \varphi(e^{i\theta}) = 0. \end{gathered} \tag{1.9}$$

Here $\| \varphi_k \|$ denotes the Hilbert–Schmidt norm of the matrix φ_k. Then (1.5) holds with

$$G[\varphi] = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log \det \varphi(e^{i\theta})\, d\theta\right\} \tag{1.10}$$

$$E[\varphi] = \det T[\varphi]\, T[\varphi^{-1}] \tag{1.11}$$

where the last det refers to the determinant defined for operators on Hilbert space differing from the identity by an operator of trace class [7, Chap. 4]. The equality of the two expressions for $E[\varphi]$ in the scalar case was established directly in [16] and will be established again here at the beginning of the proof of Theorem 7.1.

All proofs of (1.5), even in the scalar case and with the most generous assumptions, are indirect, to put it mildly. However, the expression (1.11) for $E[\varphi]$ suggests that there ought to be a quick direct proof. There is; it is based on identity (1.4) and works very nicely under the assumptions (1.8). The matrix case, under assumptions (1.9), is practically as easy as the scalar case. These are presented in Section 3. (Section 2 contains some general remarks about operators on Hilbert space.)

After another preliminary section the asymptotic inversion of $T_n[\varphi]$ is taken up in Section 5. It will be shown that if φ belongs to $H_r^\infty + C_r$ (the $r \times r$ matrix analogue of $H^\infty + C$; this sort of notation will be used consistently) and if $T[\varphi]$ and $T[\tilde{\varphi}]$ are invertible as operators on the space of l_2 sequences of r-vectors, then $T_n[\varphi]$ is invertible for sufficiently large n. In addition, an asymptotic formula for $T_n[\varphi]^{-1}$ will be derived. (An equivalent formula was derived in [15] under stronger assumptions.) It will follow that under these conditions $T_n[\varphi]^{-1}$ converges strongly to $T[\varphi]^{-1}$ and a consequence of this is that

$$\lim_{n\to\infty} D_n[\varphi]/D_{n-1}[\varphi] = G[\varphi] \tag{1.12}$$

where $G[\varphi]$ is given by (1.10) or a suitable modification if φ is discontinuous. The asymptotic inversion formula will be a consequence of (1.4) and general facts about compact operators.

Section 6 begins with a derivation of (1.5) under weaker conditions, the weakest general conditions (even in the scalar case) to date. We shall use the matrix analogue of a Banach algebra introduced by Krein [12], which we call K_r for this reason. This consists of those $r \times r$ matrix-valued functions φ that satisfy

$$\| \varphi \| = \operatorname{ess\,sup} \| \varphi(e^{i\theta}) \| + \left\{ \sum_{k=-\infty}^{\infty} | k | \, \| \varphi_k \|^2 \right\}^{1/2} < \infty.$$

That K_r is a Banach algebra follows without difficulty from the easily established identity

$$\sum_{k=-\infty}^{\infty} | k | \, \| \varphi_k \|^2 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\| \varphi(e^{i\theta_1}) - \varphi(e^{i\theta_2}) \|^2}{| e^{i\theta_1} - e^{i\theta_2} |^2} \, d\theta_1 \, d\theta_2 \tag{1.13}$$

which was used in the scalar case by Devinatz [2] in his investigation of the Szegö limit theorem. Krein proved that in the scalar case (1.5) holds if φ is an invertible element of K and $\varphi > 0$. Theorem 6.1 states that (1.5) holds if φ is an invertible element of K_r of index zero. (The meaning of *index zero* will be explained in Section 4. It is equivalent to $T[\varphi]$ having index zero as a Fredholm operator, and for φ continuous, it is equivalent to $\det \varphi$ having zero winding number.) The proof, although not as simple as that given in Section 3 under assumptions (1.9), is still not bad. The major ingredients are (1.12) and what can only be called a trick.

The asymptotic evaluation of $D_n[\varphi]$ may be thought of as a special case of the asymptotic evaluation of sums of the form

$$\sum f(\lambda_i) \tag{1.14}$$

where $\lambda_1, \ldots, \lambda_{(n+1)r}$ are the eigenvalues of $T_n[\varphi]$ and f is a suitable function. (For the determinant $f = \log$). Since the derivative of the logarithm is the inverse and any analytic function can be obtained via Cauchy's formula from inverses, it is possible to evaluate (1.14) asymptotically for suitable analytic functions f. This is the content of Theorem 6.2.

The final section reinterprets the results of Section 6 in the scalar case. The equality of the two expressions for $E[\varphi]$ given by (1.7) and (1.11) will be established and Theorem 6.2 will yield a generalization to the non-Hermitian case of a formula of Libkind [13].

As the reader will have noticed, the results of this paper are for the most part not terribly new; they push old results perhaps as epsilon further. The main point is to show how easy it can all be made if one uses some trivial identities and elementary operator theory.

2. Operator-Theoretic Preliminaries

Proofs of the facts stated but not proven in this section can be found in [7].

If A is a compact operator on Hilbert space, then $\| A \|_p$ denotes the p norm of the sequence of eigenvalues $(A^*A)^{1/2}$. Here $1 \leqslant p \leqslant \infty$. The ∞ norm is the usual operator norm and is so defined even if A is not compact. The 2 norm is the Hilbert–Schmidt norm and the 1 norm is the trace norm. The set of compact operators with finite p norm is denoted by $\mathscr{S}_p$. It is a Banach space under the p norm in which the finite rank operators are dense. The only spaces of interest to us will be $\mathscr{S}_1$ (the operators of trace class), $\mathscr{S}_2$ (the Hilbert–Schmidt operators), and $\mathscr{S}_\infty$ (the compact operators). Hölder's inequality holds for the p norms. In particular

$$\| BAC \|_p \leqslant \| A \|_p \| B \|_\infty \| C \|_\infty , \qquad \| AB \|_1 \leqslant \| A \|_2 \| B \|_2 . \tag{2.1}$$

PROPOSITION 2.1. *Suppose $\{B_n\}$ and $\{C_n\}$ are two sequences of bounded operators satisfying*

$$B_n \to B \text{ strongly}, \qquad C_n^* \to C^* \text{ strongly}. \tag{2.2}$$

Then if $A \in \mathscr{S}_p$ $(1 \leqslant p \leqslant \infty)$

$$\lim_{n\to\infty} \| B_n A C_n - BAC \|_p = 0.$$

The convergence is uniform with respect to any parameter for which $\| B_n \|_\infty$, $\| C_n \|_\infty$ are uniformly bounded, the convergence (2.2) *is uniform, and the operators A lie in a compact subset of $\mathscr{S}_p$.*

Proof. Let $\mathfrak{A}$ be a compact subset of $\mathscr{S}_p$. Cover $\mathfrak{A}$ by a finite set of balls of radius ϵ. Take one operator from each ball and approximate it to within ϵ by a finite rank operator. We obtain a finite set $F_1, \dots, F_k$ of finite rank operators such that each operator A of $\mathfrak{A}$ is within 2ϵ of one of them.

Take one of these rank operators F. It has the form

$$Fx = \sum (x, y_i) z_i \tag{2.3}$$

where $\{y_i\}$, $\{z_i\}$ are finite sets of vectors. Then

$$\begin{aligned}(B_n F C_n - BFC)x &= (B_n - B) F C_n x + BF(C_n - C)x \\ &= \sum (C_n x, y_i)(B_n - B) z_i + \sum ((C_n - C)x, y_i) B z_i \\ &= \sum (x, C_n^* y_i)(B_n - B) z_i + \sum (x, (C_n^* - C^*) y_i) B z_i .\end{aligned}$$

For an operator of the form (2.3):

$$\| F \|_p \leqslant \| F \|_\infty \leqslant \sum \| y_i \| \| z_i \|.$$

Applying this inequality to the finite rank operators

$$x \to \sum (x, C_n^* y_i)(B_n - B) z_i$$
$$x \to \sum (x, (C_n^* - C^*) y_i) B z_i$$

shows that $\| B_n F C_n - BFC \|_p \to 0$ uniformly. Hence, if n is sufficiently large

$$\| B_n F_i C_n - BF_i C \|_p \leqslant \epsilon \qquad i = 1, \dots, k$$

for all values of the parameter, and for these n and all values of the parameter

$$\| B_n A C_n - BAC \|_p \leqslant \epsilon + 4\epsilon \sup \| B_n \|_\infty \| C_n \|_\infty .$$

This completes the proof.

PROPOSITION 2.2. *Let $\{P_n\}$ be a sequence of projections converging strongly to I and A be a compact operator such that $I + A$ is invertible. Then there exists an n_0 such that $n \geqslant n_0$ implies that $P_n(I + A) P_n$ is invertible (as an operator on the range of P_n) and*

$$\lim_{n \to \infty} \| \{P_n(I + A)P_n\}^{-1} - P_n(I + A)^{-1}P_n \|_\infty = 0.$$

If $\mathcal{C}$ is a compact subset of $\mathcal{S}_\infty$ such that every operator of $I + \mathcal{C}$ is invertible then the conclusions hold uniformly for $A \in \mathcal{C}$.

Proof. We have

$$\begin{aligned} P_n(I + A) P_n(I + A)^{-1}P_n &= P_n + P_n(I + A)(P_n - I)(I + A)^{-1}P_n \\ &= P_n + A(P_n - I)(I + A)^{-1}P_n . \end{aligned}$$

By Proposition 2.1, $\| A(P_n - I)\|_\infty \to 0$ uniformly in $\mathcal{C}$, and $\|(I + A)^{-1}\|_\infty$ is uniformly bounded. Hence

$$\| P_n(I + A) P_n(I + A)^{-1}P_n - P_n \|_\infty \to 0$$

uniformly, and similarly

$$\| P_n(I + A)^{-1}P_n(I + A) P_n - P_n \|_\infty \to 0$$

uniformly. The conclusions of the proposition follow.

Finally, we mention that trace and determinant, tr A and $\det(I + A)$, are defined (and continuous) for $A \in \mathcal{S}_1$. They are, respectively,

$$\sum \lambda_i \quad \text{and} \quad \prod (1 + \lambda_i),$$

where λ_i are the eigenvalues of A repeated according to algebraic multiplicity (dimension of the generalized eigenspace). If P is a finite-dimensional projection then

$$\det(I + PAP) = \det P(I + A)P \tag{2.4}$$

where the det on the right refers to the ordinary finite-dimensional determinant for operators defined on the range of P.

3. Szegö's Theorem for φ Smooth

Consider first the scalar case. If A denotes the set of functions with absolutely convergent Fourier series then $A \cap K$ is a Banach algebra under the norm

$$\| \varphi \| = \sum_{k=-\infty}^{\infty} | \varphi_k | + \left\{ \sum_{k=-\infty}^{\infty} | k | \, | \varphi_k |^2 \right\}^{1/2}$$

and the trigonometric polynomials are dense. It follows that if

$$\varphi(e^{i\theta}) \neq 0, \quad \underset{0 \leqslant \theta \leqslant 2\pi}{\Delta} \arg \varphi(e^{i\theta}) = 0,$$

then any continuously defined $\log \varphi$ belongs to $A \cap K$, as do also

$$\varphi_+(z) = \exp \left\{ \sum_{k=0}^{\infty} (\log \varphi)_k z^k \right\}, \qquad \varphi_-(z) = \exp \left\{ \sum_{k=-\infty}^{-1} (\log \varphi)_k z^k \right\}.$$

Because $\| H[\varphi] \|_2^2 = \sum_{k=1}^{\infty} k \, | \varphi_k |^2$, the operator $H[\varphi]$ is Hilbert–Schmidt whenever $\varphi \in K$, and thus, the product of any two such is of trace class.

Apply (1.4) with ψ replaced by φ^{-1}. Since $H[\varphi_-^{-1}] = 0$, what results is

$$T_n[\varphi_+] - T_n[\varphi] \, T_n[\varphi_-^{-1}] = P_n H[\varphi] \, H[\tilde{\varphi}_-^{-1}] \, P_n$$

and thus

$$T_n[\varphi_+] \, T_n[\varphi_+^{-1}] - T_n[\varphi] \, T_n[\varphi_-^{-1}] \, T_n[\varphi_+^{-1}] = P_n H[\varphi] \, H[\tilde{\varphi}_-^{-1}] \, P_n T_n[\varphi_+^{-1}].$$

Another application of (1.4) together with the fact that

$$H[\tilde{\varphi}_+] = H[\tilde{\varphi}_+^{-1}] = 0$$

gives $T_n[\varphi_+] \, T_n[\varphi_+^{-1}] = I_n$, the identity $n \times n$ matrix. Thus,

$$\begin{aligned} &T_n[\varphi] \, T_n[\varphi_-^{-1}] \, T_n[\varphi_+^{-1}] \\ &\quad = I_n - P_n H[\varphi] \, H[\tilde{\varphi}_-^{-1}] \, P_n T_n[\varphi_+^{-1}] = P_n(I - H[\varphi] \, H[\tilde{\varphi}_-^{-1}] \, P_n T[\varphi_+^{-1}]) \, P_n . \end{aligned}$$

Since $T_n[\varphi_\pm^{-1}]$ are triangular matrices, one sees that the left side has determinant exactly

$$D_n[\varphi] \, G[\varphi_-^{-1}]^{n+1} \, G[\varphi_+^{-1}]^{n+1} = D_n[\varphi] / G[\varphi]^{n+1}.$$

For the right side, we have

$$(P_n T[\varphi_+^{-1}])^* = T[\varphi_+^{-1}]^* P_n \to T[\varphi_+^{-1}]^*$$

strongly. Therefore, by (2.4), Proposition 2.1, and the continuity of det in $\mathscr{S}_1$, the determinant of the right side has limit

$$\det(I - H[\varphi]\, H[\tilde{\varphi}_-^{-1}]\, T[\varphi_+^{-1}]).$$

Now, we use another identity similar to (1.1), i.e.,

$$H[\varphi]\, T[\tilde{\psi}] = H[\varphi\psi] - T[\varphi]\, H[\psi] \tag{3.1}$$

whose proof is also straightforward. This gives

$$H[\tilde{\varphi}_-^{-1}]\, T[\varphi_+^{-1}] = H[\tilde{\varphi}_-^{-1}\tilde{\varphi}_+^{-1}] - T[\tilde{\varphi}_-^{-1}]\, H[\tilde{\varphi}_+^{-1}]$$

which is just $H[\tilde{\varphi}^{-1}]$ since $H[\tilde{\varphi}_+^{-1}] = 0$. Hence

$$I - H[\varphi]\, H[\tilde{\varphi}_-^{-1}]\, T[\varphi_+^{-1}] = I - H[\varphi]\, H[\tilde{\varphi}^{-1}]$$

and by (1.2) this is $T[\varphi]\, T[\varphi^{-1}]$. This completes the proof in the scalar case.

It is clear that the above argument extends to any matrix-valued function in $A_r \cap K_r$ that has a factorization

$$\varphi = \varphi_+ \varphi_-$$

where $\varphi_\pm$ are invertible in $A_r \cap K_r$ and $\varphi_+^{\pm 1}$ resp. $\varphi_-^{\pm 1}$ have Fourier coefficients that vanish for negative resp. positive indices. Unfortunately, (1.9) does not guarantee the existence of such a factorization. What is true is that there is a factorization

$$\varphi(z) = \varphi_+(z) \begin{bmatrix} z^{\kappa_1} & & 0 \\ & \ddots & \\ 0 & & z^{\kappa_r} \end{bmatrix} \varphi_-(z)$$

where $\varphi_\pm$ have the desired properties and the κ_i are integers. See [5, p. 188] for example. All the κ_i are zero if and only if the Toeplitz operator $T[\tilde{\varphi}]$ is invertible; all that one can say under assumption (1.9) is that $T[\tilde{\varphi}]$ is a Fredholm operator of index zero. If $T[\tilde{\varphi}]$ is invertible, then the factors $\varphi_\pm$ can be determined in terms of $T[\tilde{\varphi}]^{-1}$. In fact, $T[\tilde{\varphi}]$

is necessarily also invertible when thought of as an operator on l_2 sequences of $r \times r$ matrices, and $\tilde{\varphi}_-^{-1}$ is the matrix-valued function whose sequence of non-negative Fourier coefficients is

$$T[\tilde{\varphi}]^{-1}(I, 0, 0, \ldots).$$

Now, it is known [17] that if $T[\tilde{\varphi}]$ is Fredholm of index zero, which is the case when (1.9) holds, then there is a ψ with only finitely many non-vanishing Fourier coefficients such that $T[\tilde{\varphi} + \epsilon\tilde{\psi}]$ is invertible for all sufficiently small nonzero ϵ. Therefore

$$\lim_{n \to \infty} D_n[\varphi + \epsilon\psi]/G[\varphi + \epsilon\psi]^{n+1} = \det T[\varphi + \epsilon\psi]\, T[(\varphi + \epsilon\psi)^{-1}] \qquad (3.2)$$

for all ϵ belonging to some punctured disk with center $\epsilon = 0$. Moreover, the convergence is uniform for ϵ on the boundary of the disk. This follows from the fact that the factors $(\varphi + \epsilon\psi)_\pm$ can be chosen so that they are continuous in ϵ (recall the explanation given above of how the factor φ_- can be obtained) and the second assertion of Proposition 2.1. Since all functions are analytic in the full disk, relation (3.2) also holds for $\epsilon = 0$ and this is what was wanted.

4. $H^\infty + C$

The reader who only cares about continuous functions can go directly to the next section, replacing the algebra $H_r^\infty + C_r$ by C_r and in the following section replacing K_r by $C_r \cap K_r$.

The facts about $H_r^\infty + C$ stated here can be found in [3] or, for the matrix case $H_r^\infty + C_r$, in [4].

$H_r^\infty + C_r$ is actually a Banach algebra under the uniform norm and the subspace

$$\bigcup_{k=0}^{\infty} z^{-k} H_r^\infty$$

is dense. The harmonic extension of $\varphi \in H_r^\infty + C_r$ is defined by

$$\varphi_\rho(e^{i\theta}) = \sum_{k=-\infty}^{\infty} \rho^{|k|} \varphi_k\, e^{ik\theta} \qquad (0 \leqslant \rho < 1).$$

A necessary and sufficient condition that φ be invertible in $H_r^\infty + C_r$ is that $\det \varphi_\rho$ be bounded away from zero in some annulus $\rho_0 < \rho < 1$. Therefore, such an invertible element has a well-defined index

$$\frac{1}{2\pi} \underset{0 \le \theta \le 2\pi}{\Delta} \arg \det \varphi_\rho(e^{i\theta})$$

which is independent of $\rho > \rho_0$.

If $\varphi \in H_r^\infty + C_r$, then a necessary as well as sufficient condition that $T[\varphi]$ be Fredholm is that φ be invertible in $H_r^\infty + C_r$. For such φ the index of $T[\varphi]$ as a Fredholm operator is the negative of the index of φ.

If φ has index zero there is a continuously defined

$$\log \det \varphi_\rho(e^{i\theta})$$

in the annulus $\rho_0 < \rho < 1$ which as $\rho \to 1$ converges in L_1 to a determination of

$$\log \det \varphi(e^{i\theta}).$$

This is easily checked if $\varphi \in z^{-k} H_r^\infty$ for some k and follows for general φ by the uniform density of the union of these spaces. For such φ we define

$$G[\varphi] = \lim_{\rho \to 1} \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log \det \varphi_\rho(e^{i\theta})\, d\theta \right).$$

This clearly agrees with (1.10) if φ is continuous.

5. Asymptotic Inversion of $T_n[\varphi]$

The main result of this section is an asymptotic inversion formula for $T_n[\varphi]$ under the assumptions that $\varphi \in H_r^\infty + C_r$ and that $T[\varphi]$ and $T[\tilde{\varphi}]$ are both invertible. First, we prove a couple of lemmas:

Lemma 1. *If φ, $\varphi^{-1} \in L_r^\infty$ and $T[\varphi]$ has a nontrivial null space then so does $T[\tilde{\varphi}^{-1}]$.*

Proof. The nontriviality of the null space of $T[\varphi]$ is equivalent to the existence of nonzero functions f, g in r-vector valued H^2 such that $\varphi f = z^{-1}\tilde{g}$. This implies $\tilde{\varphi}^{-1} g = z^{-1}\tilde{f}$ and the conclusion follows.

Lemma 2. *Suppose $\varphi \in H_r^\infty + C_r$ and $T[\varphi]$ and $T[\tilde{\varphi}]$ are invertible. Then $T[\varphi^{-1}]$, $T[\tilde{\varphi}^{-1}]$ are also invertible.*

Proof. It follows from Lemma 1 that the operators in question have trivial null spaces.

Since $H[\tilde{\varphi}]$ is compact (by Hartman's theorem) and $H[\varphi^{-1}]$ is bounded, identity (1.3) implies that $T[\varphi^{-1}]\, T[\varphi]$ is Fredholm of index zero. Since it has trivial null space, it must be invertible. Then $T[\varphi^{-1}]$ is also invertible.

Since φ is invertible in $H_r^\infty + C_r$, $H[\tilde{\varphi}^{-1}]$ is compact. Therefore, we can conclude from (1.3) with φ replaced by $\tilde{\varphi}$ that $T[\tilde{\varphi}^{-1}]\, T[\tilde{\varphi}]$ is Fredholm of index zero, and this implies the invertibility of $T[\tilde{\varphi}^{-1}]$.

DEFINITION. $\mathscr{I}_r = \{\varphi \in L_r^\infty \colon T[\varphi] \text{ and } T[\tilde{\varphi}] \text{ are invertible}\}$.

Notation. $o(1)$ denotes any sequence of operators whose ∞ norms have limit zero.

THEOREM 5.1. *If $\varphi \in (H_r^\infty + C_r) \cap \mathscr{I}_r$, then $T_n[\varphi]$ is invertible for sufficiently large n and*

$$\begin{aligned} T_n[\varphi]^{-1} = T_n[\varphi^{-1}] &+ P_n(T[\varphi]^{-1} - T[\varphi^{-1}])\, P_n \\ &+ Q_n(T[\tilde{\varphi}]^{-1} - T[\tilde{\varphi}^{-1}])\, Q_n + o(1). \end{aligned}$$

This holds uniformly for φ belonging to any compact subset of $(H_r^\infty + C_r) \cap \mathscr{I}_r$.

Proof. The assumption also implies the invertibility of $T[\varphi^{-1}]$ and $T[\tilde{\varphi}^{-1}]$ by Lemma 2.

By identity (1.4) we have (I_n = identity $n \times n$ matrix)

$$T_n[\varphi]\, T_n[\varphi^{-1}] = I_n - P_n H[\varphi]\, H[\tilde{\varphi}^{-1}]\, P_n - Q_n H[\tilde{\varphi}]\, H[\varphi^{-1}]\, Q_n\,.$$

Since φ is invertible in $H_r^\infty + C_r$, both products of Hankel operators are compact. Since $Q_n \to 0$ weakly

$$Q_n H[\tilde{\varphi}]\, H[\varphi^{-1}]\, Q_n \to 0$$

strongly and this convergence is uniform for φ belonging to any compact set of invertible elements of $H_r^\infty + C_r$. It follows from Proposition 2.1 that

$$Q_n H[\tilde{\varphi}]\, H[\varphi^{-1}]\, Q_n H[\varphi]\, H[\tilde{\varphi}^{-1}]\, P_n = o(1)$$

uniformly in φ, and thus,

$$\begin{aligned} T_n[\varphi]\, T_n[\varphi^{-1}] &= Q_n\{I - H[\tilde{\varphi}]\, H[\varphi^{-1}]\}\, Q_n\{I - H[\varphi]\, H[\tilde{\varphi}^{-1}]\}\, P_n + o(1) \\ &= Q_n T[\tilde{\varphi}]\, T[\tilde{\varphi}^{-1}]\, Q_n T[\varphi]\, T[\varphi^{-1}]\, P_n + o(1). \end{aligned} \tag{5.1}$$

Since the two products of Toeplitz operators are invertible and differ from I by compact operators, and since Q_n is P_n times a commuting unitary, we deduce from Proposition 2.2 that $T_n[\varphi]\ T_n[\varphi^{-1}]$ is invertible for sufficiently large n and that

$$T_n[\varphi^{-1}]^{-1}\ T_n[\varphi]^{-1} = P_n T[\varphi^{-1}]^{-1}\ T[\varphi]^{-1} Q_n T[\tilde{\varphi}^{-1}]^{-1}\ T[\tilde{\varphi}]^{-1} Q_n + o(1)$$

uniformly in φ.

Reversing the argument that led to (5.1) and using the fact that if an invertible operator differs from I by a compact operator then so does its inverse, give

$$\begin{aligned} T_n[\varphi^{-1}]^{-1}\ T_n[\varphi]^{-1} &= I_n + P_n\{T[\varphi^{-1}]^{-1}\ T[\varphi]^{-1} - I\} P_n \\ &\quad + Q_n\{T[\tilde{\varphi}^{-1}]^{-1}\ T[\tilde{\varphi}]^{-1} - I\} Q_n + o(1). \end{aligned}$$

From which, left multiplying by $T_n[\varphi^{-1}]$,

$$\begin{aligned} T_n[\varphi]^{-1} &= T_n[\varphi^{-1}] + P_n T_n[\varphi^{-1}]\{T[\varphi^{-1}]^{-1}\ T[\varphi]^{-1} - I\} P_n \\ &\quad + Q_n T_n[\tilde{\varphi}^{-1}]\{T[\tilde{\varphi}^{-1}]^{-1}\ T[\tilde{\varphi}]^{-1} - I\} Q_n + o(1). \end{aligned}$$

Here we have used the identity

$$T_n[\varphi]\, Q_n = Q_n T_n[\tilde{\varphi}]$$

applied to φ^{-1}.

Finally, we apply Proposition 2.1 once again to conclude that

$$\begin{aligned} T_n[\varphi^{-1}]\{T[\varphi^{-1}]^{-1}\ T[\varphi]^{-1} - I\} &= T[\varphi^{-1}]\{T[\varphi^{-1}]^{-1}\ T[\varphi]^{-1} - I\} + o(1) \\ &= T[\varphi]^{-1} - T[\varphi^{-1}] + o(1) \end{aligned}$$

and that similarly

$$T_n[\tilde{\varphi}^{-1}]\{T[\tilde{\varphi}^{-1}]^{-1}\ T[\tilde{\varphi}]^{-1} - I\} = T[\tilde{\varphi}]^{-1} - T[\tilde{\varphi}^{-1}] + o(1)$$

uniformly in φ.

COROLLARY 1. *If* $\varphi \in (H_r^\infty + C_r) \cap \mathscr{I}_r$, *then* $T_n[\varphi]^{-1}$ *converges strongly to* $T[\varphi]^{-1}$. *If* φ *belongs to any compact subset of* $(H_r^\infty + C_r) \cap \mathscr{I}_r$, *the convergence is uniform and* $\| T_n[\varphi]^{-1} \|_\infty$ *are uniformly bounded.*

Proof. The operator $T[\tilde{\varphi}]^{-1} - T[\tilde{\varphi}^{-1}]$ is compact and $Q_n \to 0$ weakly. Therefore

$$Q_n(T[\tilde{\varphi}]^{-1} - T[\tilde{\varphi}^{-1}])\, Q_n \to 0$$

strongly (uniformly in φ). Moreover

$$T_n[\varphi^{-1}] + P_n(T[\varphi]^{-1} - T[\varphi^{-1}])\, P_n = P_n T[\varphi]^{-1} P_n$$

which converges strongly to $T[\varphi]^{-1}$ (uniformly in φ). The assertions of the corollary follow.

COROLLARY 2. *If* $\varphi \in (H_r^\infty + C_r) \cap \mathscr{I}_r$ *then*

$$\lim_{n\to\infty} D_n[\varphi]/D_{n-1}[\varphi] = G[\varphi].$$

The convergence is uniform on compact subsets of $(H_r^\infty + C_r) \cap \mathscr{I}_r$.

Proof. By Cramer's rule if $r = 1$ and Jacobi's theorem on minors of the inverse matrix [1, p. 98] for general r, the 0, 0 entry of $T_n[\varphi]^{-1}$ is an $r \times r$ matrix with determinant

$$D_{n-1}[\varphi]/D_n[\varphi].$$

By Corollary 1, the limit of this 0, 0 entry is the 0, 0 entry of $T[\varphi]^{-1}$. Therefore, what must be proven is

$$\det(T[\varphi]^{-1})_{0,0} = G[\varphi]^{-1}. \tag{5.2}$$

First, suppose that in addition $\varphi \in A_r$. Then there is a factorization

$$\varphi = \varphi_- \varphi_+$$

having the usual properties. It follows from (1.1) that

$$T[\varphi] = T[\varphi_-]\, T[\varphi_+], \qquad T[\varphi_\pm]^{-1} = T[\varphi_\pm^{-1}].$$

Therefore, $T[\varphi]^{-1} = T[\varphi_+^{-1}]\, T[\varphi_-^{-1}]$. The factors $T[\varphi_+^{-1}]$, $T[\varphi_-^{-1}]$ are lower (block) triangular and upper triangular, respectively, so that their product $T[\varphi]^{-1}$ has 0, 0 entry

$$(\varphi_+^{-1})_0\, (\varphi_-^{-1})_0$$

which has determinant

$$G[\varphi_+^{-1}]\, G[\varphi_-^{-1}] = G[\varphi]^{-1}.$$

Now take a general φ and denote its harmonic extension by φ_ρ as in Section 4. Since

$$G[\varphi] = \lim_{\rho\to 1} G[\varphi_\rho]$$

by definition, and since (5.2) holds with φ replaced by φ_ρ, to obtain (5.2) for φ itself it suffices to show that $T[\varphi_\rho]$ is invertible for ρ sufficiently close to 1 and that $T[\varphi_\rho]^{-1}$ converges strongly to $T[\varphi]^{-1}$. We use (1.2) again to obtain

$$T[\varphi_\rho]\,T[\varphi_\rho^{-1}] = I - H[\varphi_\rho]\,H[\tilde{\varphi}_\rho^{-1}].$$

Clearly, $H[\varphi_\rho] \to H[\varphi]$ strongly. Accept for the moment that

$$H[\tilde{\varphi}_\rho^{-1}] - H[\tilde{\varphi}^{-1}] = o(1). \tag{5.3}$$

Write

$$H[\varphi_\rho]\,H[\tilde{\varphi}_\rho^{-1}] - H[\varphi]\,H[\tilde{\varphi}^{-1}]$$
$$= H[\varphi_\rho](H[\tilde{\varphi}_\rho^{-1}] - H[\tilde{\varphi}^{-1}]) + (H[\varphi_\rho] - H[\varphi])\,H[\tilde{\varphi}^{-1}].$$

The first term on the right is $o(1)$ by (5.3) and the boundedness of $\|\varphi_\rho\|_\infty$. Since φ is invertible in $H_r^\infty + C_r$, the operator $H[\tilde{\varphi}^{-1}]$ is compact and thus the second term on the right is $o(1)$ by Proposition 2.1. Therefore

$$H[\varphi_\rho]\,H[\tilde{\varphi}_\rho^{-1}] = H[\varphi]\,H[\tilde{\varphi}^{-1}] + o(1)$$

or equivalently

$$T[\varphi_\rho]\,T[\varphi_\rho^{-1}] = T[\varphi]\,T[\varphi^{-1}] + o(1).$$

Since $T[\varphi]$ and $T[\varphi^{-1}]$ are invertible we deduce that $T[\varphi_\rho]\,T[\varphi_\rho^{-1}]$ is invertible for ρ sufficiently close to 1 and that

$$(T[\varphi_\rho]\,T[\varphi_\rho^{-1}])^{-1} = T[\varphi^{-1}]^{-1}\,T[\varphi]^{-1} + o(1). \tag{5.4}$$

Since $T[\varphi_\rho]$ and $T[\varphi_\rho^{-1}]$ are both Fredholm of index zero we deduce the invertibility of each from the invertibility of the product. Hence, (5.4) gives

$$T[\varphi_\rho]^{-1} = T[\varphi_\rho^{-1}](T[\varphi^{-1}]^{-1}\,T[\varphi]^{-1} + o(1))$$

which converges strongly to $T[\varphi]^{-1}$ since $T[\varphi_\rho^{-1}]$ converges strongly to $T[\varphi^{-1}]$.

It remains to verify (5.3). By the asymptotic multiplicativity of the mapping $\varphi \to \varphi_\rho$ [3, Lemma 6.44] we have

$$\|(\varphi^{-1})_\rho - (\varphi_\rho)^{-1}\|_\infty \to 0$$

and thus, (5.3) is equivalent to

$$H[(\tilde{\varphi}^{-1})_\rho] - H[\tilde{\varphi}^{-1}] = o(1).$$

Replacing φ^{-1} by ψ we see that what has to be shown is that

$$H[\tilde{\psi}_\rho] - H[\tilde{\psi}] = o(1).$$

This is trivial if $\psi \in z^{-k}H_r^\infty$ for some k and follows for all $\psi \in H_r^\infty + C_r$ by the density of the union of these subspaces.

This completes the proof of the corollary except for the asserted uniformity which the reader will have no difficulty in verifying.

It should be mentioned that the condition $\varphi \in \mathscr{I}_r$ is necessary as well as sufficient for the conclusion of Corollary 1 to hold. In fact

$$\liminf_{n\to\infty} \| T_n[\varphi]^{-1} \|_\infty < \infty \tag{5.5}$$

implies the invertibility of $T[\varphi]$ (see [5, p. 73]). Since $T_n[\tilde{\varphi}]$ and $T_n[\varphi]$ are unitarily equivalent (5.5) implies

$$\liminf_{n\to\infty} \| T_n[\tilde{\varphi}]^{-1} \|_\infty < \infty$$

and thus, also the invertibility of $T[\tilde{\varphi}]$.

It should also be mentioned that the assumption $\varphi \in H_r^\infty + C_r$ used throughout could be replaced by $\tilde{\varphi} \in H_r^\infty + C_r$. Only minor modifications of the proofs are required.

6. Szegö's Theorem for $\varphi \in K_r$

Since $\varphi \in K_r$ implies that both $H[\varphi]$ and $H[\tilde{\varphi}]$ are Hilbert–Schmidt (and therefore compact) Hartman's theorem implies that both φ and $\tilde{\varphi}$ belong to $H_r^\infty + C_r$. In particular, an invertible element of K_r has a well-defined index and if φ has index zero $G[\varphi]$ is defined. Moreover, if φ is invertible in K_r then $\tilde{\varphi}$ is also and

$$I - T[\varphi]\, T[\varphi^{-1}] = H[\varphi]\, H[\tilde{\varphi}^{-1}]$$

is of trace class; thus, det $T[\varphi]\, T[\varphi^{-1}]$ is defined.

THEOREM 6.1. *If φ is an invertible element of K_r of index zero then*

$$\lim_{n\to\infty} D_n[\varphi]/G[\varphi]^{n+1} = \det T[\varphi]\, T[\varphi^{-1}].$$

Proof. Assume first that in addition $\varphi \in \mathscr{I}_r$ and write the block matrix $T_{m+n+1}[\varphi]$ in block form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A is a square block matrix of order $n+1$ and D is a square block matrix of order $m+1$. The entries of this 2×2 block matrix are given by

$$A = T_n[\varphi], \qquad D = T_m[\varphi],$$
$$B = Q_n H[\tilde{\varphi}]\, P_m\,, \qquad C = P_m H[\varphi]\, Q_n\,.$$

Since $\varphi \in \mathscr{I}_r$, Theorem 5.1 says that A and D are invertible for sufficiently large m, n. Therefore, we can factor out the matrix

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

on the left and take determinants. This gives

$$\begin{aligned} D_{m+n+1}[\varphi] &= D_m[\varphi]\, D_n[\varphi] \det \begin{bmatrix} I & A^{-1}B \\ D^{-1}C & I \end{bmatrix} \\ &= D_m[\varphi]\, D_n[\varphi] \det(I_m - D^{-1}CA^{-1}B). \end{aligned}$$

We have

$$\begin{aligned} D^{-1}CA^{-1}B &= T_m[\varphi]^{-1}\, P_m H[\varphi]\, Q_n T_n[\varphi]^{-1}\, Q_n H[\tilde{\varphi}]\, P_m \\ &= T_m[\varphi]^{-1}\, P_m H[\varphi]\; T_n[\tilde{\varphi}]^{-1}\, H[\tilde{\varphi}]\, P_m\,. \end{aligned}$$

By Corollary 1 of Theorem 5.1:

$$T_m[\varphi]^{-1} \to T[\varphi]^{-1}, \qquad T_n[\tilde{\varphi}]^{-1} \to T[\tilde{\varphi}]^{-1}$$

strongly as $m, n \to \infty$, and thus, by Proposition 2.1 with $p = 2$ and the second inequality of (2.1)

$$\| D^{-1}CA^{-1}B - P_m T[\varphi]^{-1} H[\varphi]\; T[\tilde{\varphi}]^{-1}\, H[\tilde{\varphi}]\, P_m \|_1 \to 0.$$

Thus

$$\lim_{m,n\to\infty} \det(I_m - D^{-1}CA^{-1}B) = \det(I - T[\varphi]^{-1}\, H[\varphi]\; T[\tilde{\varphi}]^{-1}\, H[\tilde{\varphi}]). \qquad (6.1)$$

Once again we use (3.1), this time with φ replaced by $\tilde{\varphi}$ and ψ by $\tilde{\varphi}^{-1}$. Since $H[\tilde{\varphi}\tilde{\varphi}^{-1}] = 0$, what results is

$$H[\tilde{\varphi}]\, T[\varphi^{-1}] = -T[\tilde{\varphi}]\, H[\tilde{\varphi}^{-1}]$$

from which we conclude

$$T[\tilde{\varphi}]^{-1}\, H[\tilde{\varphi}] = -H[\tilde{\varphi}^{-1}]\, T[\varphi^{-1}]^{-1}.$$

Thus, the determinant on the right side of (6.1) is

$$\begin{aligned}\det(I + T[\varphi]^{-1}\, H[\varphi]\, H[\tilde{\varphi}^{-1}]\, T[\varphi^{-1}]^{-1})\\ = \det\{I + T[\varphi]^{-1}(I - T[\varphi]\, T[\varphi^{-1}])\, T[\varphi^{-1}]^{-1}\}\end{aligned}$$

[here we have used (1.2) again], and this is

$$\det T[\varphi]^{-1}\, T[\varphi^{-1}]^{-1}.$$

Since det is a similarity invariant this is also equal to

$$\det T[\varphi^{-1}]^{-1}\, T[\varphi]^{-1} = (\det T[\varphi]\, T[\varphi^{-1}])^{-1}.$$

If we set

$$\alpha_n = D_n[\varphi]/\{G[\varphi]^{n+1} \det T[\varphi]\, T[\varphi^{-1}]\}$$

then what we have just shown is

$$\lim_{m,n\to\infty} \frac{\alpha_m \alpha_n}{\alpha_{m+n+1}} = 1$$

and what we want to show is

$$\lim_{n\to\infty} \alpha_n = 1.$$

Now we know that

$$\lim_{m\to\infty} \alpha_m/\alpha_{m+1} = 1$$

(this is Corollary 2 of Theorem 5.1) and it follows that

$$\lim_{m\to\infty} \alpha_m/\alpha_{m+n+1} = 1$$

for each fixed n. This implies that there is a sequence $m_n \to \infty$ such that

$$\lim_{n\to\infty} \alpha_{m_n}/\alpha_{m_n+n+1} = 1.$$

Consequently

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{\alpha_{m_n}\alpha_n/\alpha_{m_n+n+1}}{\alpha_{m_n}/\alpha_{m_n+n+1}} = 1$$

as desired.

To remove the extra assumption $\varphi \in \mathscr{I}_r$, we use the theorem of [17] quoted in Section 3 which implies that if $T[\varphi]$ is Fredholm of index 0 then there is a ψ with only finitely many nonvanishing Fourier coefficients such that $T[\varphi + \epsilon\psi]$ is invertible for all sufficiently small nonvanishing ϵ. In particular, the set of $\varphi \in K_r$ for which $T[\varphi]$ is invertible is a (necessarily open) dense subset of the set of invertible elements of K_r of index zero. Similarly, so is the set of φ for which $T[\tilde{\varphi}]$ is invertible, and so also is their intersection $K_r \cap \mathscr{I}_r$.

Observe next that if φ is of index zero then $\varphi \in \mathscr{I}_r$ is equivalent to the simultaneous invertibility of $T[\varphi]$ and $T[\varphi^{-1}]$ (cf., Lemma 1 of the preceding section), which in turn is equivalent to the invertibility of their product. Therefore, $\varphi \in \mathscr{I}_r$ is equivalent to

$$\det T[\varphi]\, T[\varphi^{-1}] \neq 0.$$

Now take the given φ of index zero and choose ϵ such that $\| \varphi - \psi \| < \epsilon$ implies ψ invertible in K_r with index zero. In this neighborhood there is a $\psi \in \mathscr{I}_r$. With this ψ write

$$\varphi_\zeta = (1 - \zeta)\varphi + \zeta\psi.$$

Then

$$\det T[\varphi_\zeta]\, T[\varphi_\zeta^{-1}]$$

is analytic in an open set containing the disk $| \zeta | \leqslant 1$ and at $\zeta = 1$ it is nonzero since $\psi \in \mathscr{I}_r$. Therefore, the set of zeros is discrete, and thus, we can find a ρ with $0 < \rho < 1$ such that

$$\det T[\varphi_\zeta]\, T[\varphi_\zeta^{-1}] \neq 0, \qquad | \zeta | = \rho.$$

Therefore $\varphi_\zeta \in \mathscr{I}_r$ for all ζ on the circle with radius ρ.

By what has already been shown

$$\lim_{n\to\infty} D_n[\varphi_\zeta]/G[\varphi_\zeta]^{n+1} = \det T[\varphi_\zeta]\, T[\varphi_\zeta^{-1}] \tag{6.2}$$

for $| \zeta | = \rho$. Moreover, a check of the proof shows that the uniformity assertions in Proposition 2.1 and the corollaries to Theorem 5.1 imply that this holds uniformly on the circle. It follows that (6.2) holds throughout the disk $|\zeta| \leqslant 1$, and in particular at $\zeta = 0$. This completes the proof of the theorem.

In the case $\varphi \in \mathscr{I}_r$ Theorem 6.1 is equivalent to the assertion

$$\lim_{n\to\infty} \left\{\sum \log \lambda_i - (n+1) \log G[\varphi]\right\} = \log \det T[\varphi]\, T[\varphi^{-1}]$$

with the logarithms appropriately defined. Here $\lambda_1, \ldots, \lambda_{(n+1)r}$ are the eigenvalues of $T_n[\varphi]$ repeated according to their algebraic multiplicity. As was mentioned in the introduction, one can replace the logarithm by more general analytic functions.

As usual we use σ to denote spectrum; $\sigma(\varphi)$ is the spectrum of φ as an element of K_r. If f is analytic on $\sigma(\varphi)$ then one defines

$$f(\varphi) = -\frac{1}{2\pi i} \int_{\partial\Omega} f(\lambda)(\varphi - \lambda)^{-1}\, d\lambda$$

where Ω is any bounded open set (with rectifiable boundary) that contains $\sigma(\varphi)$ and on the closure of which f is analytic.

If $\varphi \in K_r$ and $T[\varphi]$ is invertible then of course $\varphi^{-1} \in L_r^\infty$. It is easy to deduce from this, using (1.13), that $\varphi^{-1} \in K_r$. Consequently

$$\sigma(\varphi) \subset \sigma(T[\varphi]).$$

Corollary 1 of Theorem 5.1 implies that for n large enough $\sigma(T_n[\varphi])$ will lie in any given open subset of

$$\sigma(T[\varphi]) \cup \sigma(T[\tilde{\varphi}]).$$

In particular, if f is analytic on this set then $f(\lambda_i)$ as well as $f(\varphi)$ make sense.

THEOREM 6.2. *Let $\varphi \in K_r$ and assume f is analytic on*

$$\sigma(T[\varphi]) \cup \sigma(T[\tilde{\varphi}]).$$

Then

$$\lim_{n\to\infty} \left\{\sum f(\lambda_i) - \frac{n+1}{2\pi} \int_0^{2\pi} \operatorname{tr} f(\varphi)(e^{i\theta})\, d\theta\right\}$$
$$= -\frac{1}{2\pi i} \int_{\partial\Omega} f(\lambda) \frac{d}{d\lambda} \log \det T[\varphi - \lambda]\, T[(\varphi - \lambda)^{-1}]\, d\lambda$$

where Ω is any bounded open set containing $\sigma(T[\varphi]) \cup \sigma(T[\tilde{\varphi}])$ and on the closure of which f is analytic.

Proof. For $\lambda \notin \sigma(T[\varphi]) \cup \sigma(T[\tilde{\varphi}])$

$$\lim_{n\to\infty} \{\log \det T_n[\varphi - \lambda] - (n + 1) \log G[\varphi - \lambda]\}$$
$$= \log \det T[\varphi - \lambda]\, T[(\varphi - \lambda)^{-1}],$$

and as in the proof of Theorem 6.1 this holds uniformly for λ belonging to a neighborhood of $\partial\Omega$. Hence, we can differentiate both sides with respect to λ, multiply by $f(\lambda)$, and integrate over $\partial\Omega$. The assertion of the theorem will follow once we check that

$$\frac{d}{d\lambda} \log \det G[\varphi - \lambda] = -\frac{1}{2\pi} \int_0^{2\pi} \operatorname{tr}(\varphi(e^{i\theta}) - \lambda)^{-1}\, d\theta$$

holds whenever $\varphi - \lambda$ has index zero.

If φ is continuous then

$$\log G[\varphi - \lambda] = \frac{1}{2\pi} \int_0^{2\pi} \log \det(\varphi(e^{i\theta}) - \lambda)\, d\theta$$

and the result follows from the well-known formula [7, p. 158]:

$$\frac{d}{d\lambda} \log \det(\varphi - \lambda) = -\operatorname{tr}(\varphi - \lambda)^{-1}$$

which holds for any matrix φ.

For general φ we use the harmonic extension φ_ρ to obtain

$$\frac{d}{d\lambda} \log G[\varphi - \lambda] = \frac{d}{d\lambda} \lim_{\rho\to 1} \log G[\varphi_\rho - \lambda]$$
$$= \lim_{\rho\to 1} \frac{d}{d\lambda} \log G[\varphi_\rho - \lambda] = -\lim_{\rho\to 1} \frac{1}{2\pi} \int_0^{2\pi} \operatorname{tr}(\varphi_\rho(e^{i\theta}) - \lambda)^{-1}\, d\theta$$
$$= -\frac{1}{2\pi} \int_0^{2\pi} \operatorname{tr}(\varphi(e^{i\theta}) - \lambda)^{-1}\, d\theta.$$

The interchanges of $\lim_{\rho\to 1}$ with $d/d\lambda$ and $\int \ldots d\theta$ are justified by the uniform boundedness and analyticity in λ of

$$\log G[\varphi_\rho - \lambda],\ \operatorname{tr}(\varphi_\rho(e^{i\theta}) - \lambda)^{-1}$$

for ρ near 1.

7. The Scalar Case

In this section all functions will be assumed scalar-valued and the index r will be dropped from the notation for the various algebras. The main fact is Theorem 7.1 which asserts the equality of the two expressions (1.7) and (1.11) if φ is an invertible element of K of index zero.

Before anything like this can be established sense must be made of (1.7), i.e., a determination of $\log \varphi$ must be produced such that the series in (1.7) converges. With the notation

$$|||\psi||| = |\psi_0| + \left\{\sum_{k=-\infty}^{\infty} |k|\,|\psi_k|^2\right\}^{1/2}$$

what is needed is a $\log \varphi$ with $|||\log \varphi||| < \infty$.

Lemma 1. *Suppose $\psi(\rho\, e^{i\theta})$ belongs to C^1 of the annulus $\rho_0 < \rho < 1$ and that*

$$\int_0^{2\pi}\int_{\rho_0}^1 \|\operatorname{grad}\psi\|^2\, d\rho\, d\theta < \infty.$$

Then $\psi(\rho e^{i\theta})$ is $|||\ |||$ convergent as $\rho \to 1$.

Proof. If $\psi(\rho e^{i\theta}) = \sum_{k=-\infty}^{\infty} \psi_k(\rho)\, e^{ik\theta}$ then the assumption is equivalent to

$$\sum_{k=-\infty}^{\infty} \int_{\rho_0}^1 |\psi_k'(\rho)|^2\, d\rho < \infty, \tag{7.1}$$

$$\sum_{k=-\infty}^{\infty} k^2 \int_{\rho_0}^1 |\psi_k(\rho)|^2\, d\rho < \infty. \tag{7.2}$$

In particular, for each k and with $\rho_0 < \rho_1 < \rho_2 < 1$

$$|\psi_k(\rho_2) - \psi_k(\rho_1)| = \left|\int_{\rho_1}^{\rho_2} \psi_k'(\rho)\, d\rho\right| \leqslant (\rho_2-\rho_1)^{1/2} \int_{\rho_0}^1 |\psi_k'(\rho)|^2\, d\rho$$

which tends to zero as $\rho_1, \rho_2 \to 1$. Therefore

$$\psi_k = \lim_{\rho\to 1} \psi_k(\rho)$$

exists for each k. We shall show that

$$\sum_{k=-\infty}^{\infty} |k| \, |\psi_k|^2 < \infty$$

and that if ψ is the function with Fourier coefficients ψ_k then

$$\lim_{\rho \to 1} |||\psi(\rho e^{i\theta}) - \psi(e^{i\theta})||| = 0.$$

We have

$$|\psi_k|^2 - |\psi_k(\rho)|^2 = \int_\rho^1 \frac{d}{dr} |\psi_k(r)|^2 \, dr = \int_\rho^1 2\mathscr{R}\overline{\psi_k(r)}\, \psi_k'(r)\, dr$$

and thus

$$\sum_{k=-\infty}^{\infty} |k| \, \big| |\psi_k|^2 - |\psi_k(\rho)|^2 \big|$$

$$\leqslant 2 \sum_{k=-\infty}^{\infty} |k| \left\{\int_\rho^1 |\psi_k(r)|^2 \, dr\right\}^{1/2} \left\{\int_\rho^1 |\psi_k'(r)|^2 \, dr\right\}^{1/2}$$

$$\leqslant 2 \left\{\sum_{k=-\infty}^{\infty} |k|^2 \int_\rho^1 |\psi_k(r)|^2 \, dr\right\}^{1/2} \left\{\sum_{k=-\infty}^{\infty} \int_\rho^1 |\psi_k'(r)|^2 \, dr\right\}^{1/2}$$

which tends to zero as $\rho \to 1$. It follows from this first that

$$\sum_{k=-\infty}^{\infty} |k| \, |\psi_k|^2 < \infty$$

and second that if k_0 is sufficiently large then

$$\sum_{|k|>k_0} |k| \, |\psi_k(\rho)|^2 < \epsilon$$

for all ρ sufficiently close to 1. Since $\psi_k(\rho) \to \psi_k$ for each k,

$$\lim_{\rho \to 1} \sum_{k=-\infty}^{\infty} |k| \, |\psi_k - \psi_k(\rho)|^2 = 0$$

follows, and thus, also $|||\psi(e^{i\theta}) - \psi(\rho e^{i\theta})||| \to 0$.

Lemma 2. *If $\varphi \in K$ is invertible and of index zero, then there is a determination of* $\log \varphi$, *unique up to an additive constant, satisfying* $|||\log \varphi||| < \infty$.

Proof. We apply the lemma to

$$\psi(\rho e^{i\theta}) = \log \varphi_\rho(e^{i\theta})$$

where φ_ρ is the harmonic extension of φ; here ψ may be any continuous determination of the logarithm, which exists in a sufficiently thin annulus $\rho_0 < \rho < 1$. Since

$$\operatorname{grad} \psi = \varphi_\rho^{-1} \operatorname{grad} \varphi_\rho$$

and φ_ρ^{-1} is bounded in the annulus, to establish the hypothesis of Lemma 1 it suffices to show that

$$\int_0^{2\pi} \int_{\rho_0}^1 \| \operatorname{grad} \varphi_\rho \|^2 \, d\rho \, d\theta < \infty.$$

But since $\varphi_\rho(e^{i\theta}) = \sum_{k=-\infty}^{\infty} \rho^{|k|} \varphi_k e^{ik\theta}$, this is equivalent [see (7.1), (7.2)] to

$$\sum_{k=-\infty}^{\infty} | k | \, | \varphi_k |^2 < \infty$$

which is true.

It follows from the lemma that there is a ψ satisfying $\|| \psi \|| < \infty$ and

$$\|| \psi(\rho e^{i\theta}) - \psi(e^{i\theta})\|| \to 0.$$

Since $\|| \; \||$ convergence implies a.e. convergence of a subsequence we have a.e.

$$\exp \psi(e^{i\theta}) = \lim_{\rho \to 1} \exp \log \varphi_\rho(e^{i\theta}) = \varphi(e^{i\theta}),$$

so ψ is a determination of $\log \varphi$.

It remains to show that $\log \varphi$ satisfying $\|| \log \varphi \|| < \infty$ is uniquely determined up to an additive constant or, equivalently, that if $\|| \psi \|| < \infty$ and $\psi/2\pi i$ takes only integral values then ψ is constant.

Consider the analytic function

$$\Psi(z) = \exp \left\{ 2 \sum_{k=1}^{\infty} \psi_k z^k \right\} \qquad | z | < 1.$$

Because $\sum_{k=1}^{\infty} k \mid \psi_k \mid^2 < \infty$, this function belongs to every H^p class with $p < \infty$ [18, p. 157, Example 6]. Since ψ is purely imaginary $\psi_{-k} = -\bar{\psi}_k$, and thus,

$$\sum_{k=1}^{\infty} \psi_k e^{ik\theta} - \sum_{k=1}^{\infty} \psi_{-k} e^{-ik\theta} = 2\mathscr{R} \sum_{k=1}^{\infty} \psi_k e^{ik\theta}.$$

It follows that $\Psi(z)$ has boundary function

$$\Psi(e^{i\theta}) = \exp \left\{ \psi(e^{i\theta}) - \psi_0 + 2\mathscr{R} \sum_{k=1}^{\infty} \psi_k e^{ik\theta} \right\}$$

which, since $\psi/2\pi i$ takes integral values, is a constant times a non-negative functions. It follows that Ψ is in fact constant, and therefore (since $\psi_{-k} = -\bar{\psi}_k$), ψ is also.

We need one more application of Lemma 1.

LEMMA 3. *If φ is an invertible element of K then*

$$\lim_{\rho \to 1} \| \varphi_\rho^{-1} - \varphi^{-1} \| = 0.$$

Proof. We have

$$\operatorname{grad} \varphi_\rho^{-1} = -\varphi_\rho^{-2} \operatorname{grad} \varphi_\rho$$

and the result follows as in the proof of Lemma 2.

THEOREM 7.1. *If φ is an invertible element of K of index zero and* $\log \varphi$ *is as given by Lemma 2 then*

$$\det T[\varphi]\, T[\varphi^{-1}] = \exp \sum_{k=1}^{\infty} k(\log \varphi)_k (\log \varphi)_{-k}\,.$$

Proof. Suppose first that $\varphi \in A \cap K$ and that

$$\varphi(e^{i\theta}) \neq 0, \qquad \underset{0 \leqslant \theta \leqslant 2\pi}{\Delta} \arg \varphi(e^{i\theta}) = 0.$$

Then we have the factorization $\varphi = \varphi_+ \varphi_-$ as in Section 3. A few applications of the fact that

$$T[\varphi\psi] = T[\varphi]\, T[\psi] \qquad \text{if} \quad H[\varphi] = 0 \quad \text{or} \quad H[\tilde{\psi}] = 0,$$

which follows from (1.1), gives

$$\begin{aligned} T[\varphi]\, T[\varphi^{-1}] &= T[\varphi_-\varphi_+]\, T[\varphi_-^{-1}\varphi_+^{-1}] \\ &= T[\varphi_-]\, T[\varphi_+]\, T[\varphi_-]^{-1}\, T[\varphi_+]^{-1} \\ &= \exp T[\log \varphi_-] \exp T[\log \varphi_+] \exp\{-T[\log \varphi_-]\} \exp\{-T[\log \varphi_+]\}. \end{aligned}$$

We apply the formula

$$\det e^A e^B e^{-A} e^{-B} = \exp \operatorname{tr}(AB - BA) \tag{7.3}$$

which holds if A and B are bounded operators with $AB - BA \in \mathscr{S}_1$ [10, 14]. In our case

$$A = T[\log \varphi_-], \qquad B = T[\log \varphi_+]$$

and

$$\begin{aligned} AB - BA &= T[\log \varphi_- \log \varphi_+] - T[\log \varphi_+]\, T[\log \varphi_-] \\ &= H[\log \varphi_+]\, H[(\log \varphi_-)^\sim] \end{aligned}$$

which belongs to $\mathscr{S}_1$ since $\log \varphi_\pm \in K$. Thus, the formula holds and we obtain

$$\begin{aligned} \det T[\varphi]\, T[\varphi^{-1}] &= \exp \operatorname{tr} H[\log \varphi_+]\, H[(\log \varphi_-)^\sim] \\ &= \exp \sum_{k=1}^{\infty} k(\log \varphi)_k (\log \varphi)_{-k}\,. \end{aligned}$$

Note that for any φ of index zero the Hankel operators that arise both belong to $\mathscr{S}_2$ since $|||\log \varphi||| < \infty$. The difficulty is that $\log \varphi_\pm$ need not belong to L^∞ so that A and B are not necessarily bounded operators. Rather than try to extend (7.3) to cover this case we use the usual approximation argument involving the harmonic extension φ_ρ. Since

$$\det T[\varphi_\rho]\, T[\varphi_\rho^{-1}] = \exp \sum_{k=1}^{\infty} k(\log \varphi_\rho)_k (\log \varphi_\rho)_{-k}$$

and since

$$|||\log \varphi_\rho - \log \varphi||| \to 0$$

(this was how $\log \varphi$ was determined in the proof of Lemma 2) it suffices to show

$$\lim_{\rho \to 1} \det T[\varphi_\rho]\, T[\varphi_\rho^{-1}] = \det T[\varphi]\, T[\varphi^{-1}].$$

This will follow by (1.2) and the continuity of det in trace norm, if we can show

$$\lim_{\rho\to 1} \| H[\varphi_\rho]\, H[\tilde{\varphi}_\rho^{-1}] - H[\varphi]\, H[\tilde{\varphi}^{-1}] \|_1 = 0. \tag{7.4}$$

But

$$\lim_{\rho\to 1} \|| \varphi_\rho - \varphi \|| = 0$$

trivially and

$$\lim_{\rho\to 1} \|| \varphi_\rho^{-1} - \varphi^{-1} \|| = 0$$

by Lemma 3. Hence

$$\lim_{\rho\to 1} \| H[\varphi_\rho] - H[\varphi] \|_2 = 0, \qquad \lim_{\rho\to 1} \| H[\tilde{\varphi}_\rho^{-1}] - H[\tilde{\varphi}^{-1}] \|_2 = 0$$

and (7.4) follows.

Finally, we give another formulation of Theorem 6.2 in the scalar case and with a stronger assumption on φ. The result is similar to one obtained by Libkind [13] for real-valued φ. Note that

$$\sigma(T[\varphi]) = \sigma(T[\tilde{\varphi}])$$

for $\varphi \in K$ since

$$\begin{aligned} T[\varphi] \text{ invertible} &\Rightarrow T[\varphi] \text{ Fredholm of index zero} \\ &\Rightarrow \varphi \text{ invertible in } K \text{ of index zero} \\ &\Rightarrow \tilde{\varphi} \text{ invertible in } K \text{ of index zero} \\ &\Rightarrow T[\tilde{\varphi}] \text{ Fredholm of index zero} \end{aligned}$$

and in the scalar case Fredholm Toeplitz operators of index zero are invertible [3, Corollary 7.25].

For convenience we shall write $\varphi(\theta)$ rather than $\varphi(e^{i\theta})$.

THEOREM 7.2. *Assume φ is absolutely continuous, $\|| \varphi \|| < \infty$, and f is analytic on $\sigma(T[\varphi])$. Then*

$$\begin{aligned} &\lim_{n\to\infty} \left\{ \sum f(\lambda_i) - \frac{n+1}{2\pi} \int_0^{2\pi} f(\varphi(\theta))\, d\theta \right\} \\ &= \frac{1}{4\pi^2} \sum_{k=1}^{\infty} \int_0^{2\pi} \int_0^{2\pi} \sin k(\theta_1 - \theta_2)\, \frac{f(\varphi(\theta_1)) - f(\varphi(\theta_2))}{\varphi(\theta_1) - \varphi(\theta_2)} \\ &\quad \times [\varphi'(\theta_2) - \varphi'(\theta_1)]\, d\theta_1\, d\theta_2 . \end{aligned}$$

Proof. By Theorems 6.2 and 7.1 the limit is equal to

$$-\int_{\partial\Omega} f(\lambda) \frac{d}{d\lambda} \sum_{k=1}^{\infty} k[\log(\varphi - \lambda)]_k[\log(\varphi - \lambda)]_{-k}$$

$$= -\sum_{k=1}^{\infty} k \int_{\partial\Omega} f(\lambda) \frac{d}{d\lambda}$$

$$\times \left\{ \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{ik(\theta_1-\theta_2)} \log[\varphi(\theta_1) - \lambda] \log[\varphi(\theta_2) - \lambda] \, d\theta_1 \, d\theta_2 \right\} d\lambda.$$

(The various interchanges of limiting operations in this proof are easily justified.) Now

$$-\int_0^{2\pi} \int_0^{2\pi} e^{ik(\theta_1-\theta_2)} \, d\theta_1 \, d\theta_2 \int_{\partial\Omega} f(\lambda) \frac{d}{d\lambda} \{\log[\varphi(\theta_1) - \lambda] \log[\varphi(\theta_2) - \lambda]\} \, d\lambda$$

$$= \int_0^{2\pi} \int_0^{2\pi} e^{ik(\theta_1-\theta_2)} \, d\theta_1 \, d\theta_2 \int_{\partial\Omega} f(\lambda) \frac{\log[\varphi(\theta_2) - \lambda]}{\varphi(\theta_1) - \lambda} \, d\lambda$$

$$+ \int_0^{2\pi} \int_0^{2\pi} e^{ik(\theta_1-\theta_2)} \, d\theta_1 \, d\theta_2 \int_{\partial\Omega} f(\lambda) \frac{\log[\varphi(\theta_1) - \lambda]}{\varphi(\theta_2) - \lambda} \, d\lambda.$$

In the first term we integrate $\int \dots d\theta_2$ by parts and in the second we integrate $\int \dots d\theta_1$ by parts. We obtain

$$(ik)^{-1} \int_0^{2\pi} \int_0^{2\pi} e^{ik(\theta_1-\theta_2)} \, d\theta_1 \, d\theta_2 \int_{\partial\Omega} f(\lambda) \frac{\varphi'(\theta_2)}{[\varphi(\theta_1) - \lambda][\varphi(\theta_2) - \lambda]} \, d\lambda$$

$$- (ik)^{-1} \int_0^{2\pi} \int_0^{2\pi} e^{ik(\theta_1-\theta_2)} \, d\theta_1 \, d\theta_2 \int_{\partial\Omega} f(\lambda) \frac{\varphi'(\theta_1)}{[\varphi(\theta_1) - \lambda][\varphi(\theta_2) - \lambda]} \, d\lambda$$

$$= (ik)^{-1} \int_0^{2\pi} \int_0^{2\pi} e^{ik(\theta_1-\theta_2)} \frac{f(\varphi(\theta_1)) - f(\varphi(\theta_2))}{\varphi(\theta_1) - \varphi(\theta_2)} \varphi'(\theta_2) \, d\theta_1 \, d\theta_2$$

$$- (ik)^{-1} \int_0^{2\pi} \int_0^{2\pi} e^{ik(\theta_1-\theta_2)} \frac{f(\varphi(\theta_1)) - f(\varphi(\theta_2))}{\varphi(\theta_1) - \varphi(\theta_2)} \varphi'(\theta_1) \, d\theta_1 \, d\theta_2$$

$$= (ik)^{-1} \int_0^{2\pi} \int_0^{2\pi} e^{ik(\theta_1-\theta_2)} \frac{f(\varphi(\theta_1)) - f(\varphi(\theta_2))}{\varphi(\theta_1) - \varphi(\theta_2)} [\varphi'(\theta_2) - \varphi'(\theta_1)] \, d\theta_1 \, d\theta_2 .$$

Since the factor multiplying $e^{ik(\theta_1-\theta_2)}$ in the integral changes sign if θ_1 and θ_2 are interchanged the exponential may be replaced by

$$i \sin k(\theta_1 - \theta_2)$$

and the result follows.

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