Course 214
Elliptic Functions
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David R. Wilkins

Contents

9 Elliptic Functions

9.1 Lattices in the Complex Plane

Definition A subset Λ of the complex plane $\mathbb{C}$ is said to be an *additive subgroup* of $\mathbb{C}$ if $0 \in \Lambda$, $w_1 + w_2 \in \Lambda$ and $-w_1 \in \Lambda$ for all $w_1, w_2 \in \Lambda$. An additive subgroup Λ of the complex plane is said to be a *lattice* if there exists some positive real number δ such that

$$\Lambda \cap \{z \in \mathbb{C} : |z| < \delta\} = \{0\}.$$

Definition The zero lattice $\{0\}$ is of dimension zero. A lattice in the complex plane is said to be *one-dimensional* if it is non-zero, but is contained in some line that passes through zero. A lattice in the complex plane, is said to be *two-dimensional* if it is not contained in any line that passes through zero.

The following proposition classifies all lattices in the complex plane.

Proposition 9.1 *The dimension of every lattice in the complex plane is zero, one or two. The only zero-dimensional lattice is the zero lattice $\{0\}$. A one-dimensional lattice is of the form $\{nw_1 : n \in \mathbb{Z}\}$, where w_1 is some non-zero element of the lattice. A two-dimensional lattice is of the form $\Lambda = \{mw_1 + nw_2 : m, n \in \mathbb{Z}\}$, where w_1 and w_2 are elements of the lattice that are linearly independent over the real numbers.*

Proof Suppose $\Lambda \neq \{0\}$. Then there exists a non-zero element w_1 of Λ with the property that $|w| \geq |w_1|$ for all non-zero elements w of Λ. Let $w \in \Lambda$ satisfy $w = tw_1$ for some real number t. Then there exists some integer n such that $0 \leq t - n < 1$. But then $w - nw_1 \in \Lambda$ and $|w - nw_1| = |t - n|\,|w_1| < |w_1|$, and therefore $w - nw_1 = 0$. This shows that if $w \in \Lambda$, and if w/w_1 is a real number, then $w = nw_1$ for some integer n. It follows from this that if the lattice Λ is one-dimensional, then $\Lambda = \{nw_1 : n \in \mathbb{Z}\}$.

It remains to consider the case when the lattice Λ is two-dimensional. We define $\alpha \colon \Lambda \to \mathbb{R}$ to be the function that sends $w \in \Lambda$ to the imaginary part of w/w_1, where w_1 is some non-zero element of Λ with the property that $|w| \geq |w_1|$ for all non-zero elements w of Λ. Let w be an element of the lattice Λ that is not an integer multiple of w_1. Then $w - nw_1$ is a non-zero element of Λ for all integers n, and therefore $|w - nw_1| \geq |w_1|$ for all integers n. Thus w/w_1 is a point of

$$\{z \in \mathbb{C} : |z - n| \geq 1 \text{ for all } n \in \mathbb{Z}\}.$$

This set is contained in $\{z \in \mathbb{C} : |\operatorname{Im}[z]| \geq \sqrt{3}/2\}$. It follows that $|\alpha(w)| \geq \sqrt{3}/2$ for all $w \in \Lambda \setminus \{nw_1 : n \in \mathbb{Z}\}$. Let $S = \{\alpha(w) : w \in \Lambda\}$. Then S is an additive subgroup of $\mathbb{R}$, and all non-zero elements s of S satisfy $|s| \geq \sqrt{3}/2$. But then any two distinct elements s_1 and s_2 of S satisfy $|s_1 - s_2| \geq \sqrt{3}/2$. It follows that any bounded interval of the real line can contain at most finitely many members of the set S, and we can therefore choose some positive element s_0 with the property that $|s| \geq s_0$ for all non-zero elements s of S. Now if $s \in S$ then there exists some natural number n such that $0 \leq s - ns_0 < s_0$. But then $s - ns_0 \in S$ and therefore $s - ns_0 = 0$. It follows that $S = \{ns_0 : n \in \mathbb{Z}\}$. Choose $w_2 \in \Lambda$ such that $\alpha(w_2) = s_0$. Then w_1 and w_2 are linearly independent over the real numbers. Moreover if $w \in \Lambda$ then $\alpha(w) = ns_0$ for some integer n. But then $\alpha(w - nw_2) = 0$, and therefore $w - nw_2 = mw_1$ for some $m \in \mathbb{Z}$. This proves that every element of the two-dimensional lattice Λ is of the form $mw_1 + nw_2$ for some integers m and n. This completes the classification of lattices in the complex plane. ∎

Let w_1 and w_2 be two complex numbers that are linearly independent over the real numbers. Then w_1 and w_2 are both non-zero, and they determine a two-dimensional lattice $\{mw_1 + nw_2 : m, n \in \mathbb{Z}\}$. We refer to this lattice as the lattice in the complex plane *generated* by w_1 and w_2.

Definition Let f be a meromorphic function on the complex plane, and let S be the set of poles of the function f. A complex number w is said to be a *period* of the function f if and only if $f(z + w) = f(z)$ for all $z \in \mathbb{C} \setminus S$.

Definition A meromorphic function on the complex plane is said to be doubly-periodic if there exists a two-dimensional lattice in the complex plane whose elements are periods of the function.

Lemma 9.2 *Any holomorphic doubly-periodic function defined throughout the complex plane is constant.*

Proof Let f be a holomorphic doubly-periodic function, let w_1 and w_2 be periods of f that are linearly independent over the real numbers, and let

$$K = \{sw_1 + tw_2 : 0 \leq s \leq 1 \text{ and } 0 \leq t \leq 1\}.$$

Then K is a bounded closed set in the complex plane. It follows that there exists some non-negative real number B such that $|f(z)| \leq B$ for all $z \in K$ (see Lemma 1.31). Now given any complex number z, there exist integers m and n such that $z - 2m\omega_1 - 2n\omega_2 \in K$. But then

$$|f(z)| = |f(z - 2m\omega_1 - 2n\omega_2)| \leq B.$$

We conclude therefore that any holomorphic doubly-periodic function f defined over the entire complex plane is bounded. It follows from Liouville's Theorem (Theorem 7.3) that f is constant, as required. ∎

We see from Lemma 9.2 that any non-constant doubly-periodic function must have poles.

Proposition 9.3 *A non-constant meromorphic function defined over the complex plane has only finitely many zeros and poles in any bounded region of the plane.*

Proof Suppose that a meromorphic function f had infinitely many zeros and poles in some bounded region of the complex plane. Then there would exist a bounded infinite sequence $z_1, z_2, z_3, \ldots$ of distinct complex numbers with the property that z_j is a zero or pole of f for each positive integer j. It would then follow from the Bolzano-Weierstrass Theorem that this sequence would have a subsequence converging to some complex number w. Moreover, given any positive real number δ, the open disk of radius δ about w would contain infinitely many zeros and poles of the function f. But this leads to a contradiction, for, given any complex number w, there must exist some positive real number δ_0 such that the punctured disk $\{z \in \mathbb{C} : 0 < |z - w| < \delta_0\}$ contains no zeros or poles of the meromorphic function f. (This is a consequence of the fact that $f(z) = (z - w)^m g(z)$ around w, where m is an integer and g is a holomorphic function of z, defined on a neighbourhood of w, which is non-zero at w, and is therefore non-zero throughout some sufficiently small open disk centred on w.) We conclude therefore that there cannot exist any bounded infinite sequence of complex numbers whose elements are distinct and are zeros or poles of the meromorphic function f, and therefore this function can have only finitely many zeros and poles in any bounded region of the complex plane, as required. ∎

Corollary 9.4 *The set of periods of a non-constant meromorphic function is a lattice in the complex plane.*

Proof If w_1 and w_2 are periods of a non-constant meromorphic function f, then so are $w_1 + w_2$ and $-w_1$. Thus the set of periods of f is an additive subgroup of $\mathbb{C}$. Let z_0 be a complex number that is not a pole of f. If w is a period of f then $z_0 + w$ is a zero of the meromorphic function that sends z to $f(z) - f(z_0)$ away from the poles of f. It follows from Proposition 9.3 that any bounded subset of the complex plane can contain only finitely many periods of the meromorphic function f. Therefore there exists some positive real number δ such that every non-zero period w of f satisfies $|w| \geq \delta >$

0. Therefore the set of periods of f is a lattice in the complex plane, as required. ∎

Definition An *elliptic function* is a non-constant doubly-periodic meromorphic function on the complex plane.

It follows from Corollary 9.4 that the set of periods of an elliptic function is a lattice in the complex plane. Moreover this lattice is two-dimensional.

Definition The *period lattice* Λ of an elliptic function is the two-dimensional lattice consisting of all the periods of the function.

Definition Let f be an elliptic function. A pair of *primitive periods* of the function f is a pair of non-zero complex numbers that are linearly independent over the real numbers and generate the period lattice of the function.

It follows from these definitions that if f is an elliptic function with primitive periods $2\omega_1$ and $2\omega_2$ then the period lattice is the set of all complex numbers that are of the form $2m\omega_1 + 2n\omega_2$ for some integers m and n.

Definition Let f be an elliptic function with period lattice Λ. A *fundamental region* X for f is a connected subset X of the complex plane with the property that, given any complex number z, there exists a unique element w of the period lattice Λ such that $z - w \in X$.

Let f be an elliptic function, and let $2\omega_1$ and $2\omega_2$ constitute a pair of primitive periods of f. The *primitive period-parallelogram* determined by this pair of primitive periods is the parallelogram in the complex plane whose vertices are at the points 0, $2\omega_1$, $2\omega_1+2\omega_2$ and $2\omega_2$. This period parallelogram forms the boundary of a fundamental region X for f, where

$$X = \{2s\omega_1 + 2t\omega_2 : 0 \leq s < 1 \text{ and } 0 \leq t < 1\}.$$

We shall refer to this fundamental region X as the fundamental region for f determined by the primitive periods $2\omega_1$ and $2\omega_2$. Note that two of the sides of the primitive period-parallelogram are contained in the fundamental region X. The other two sides lie outside this fundamental region.

Let f be an elliptic function, let $2\omega_1$ and $2\omega_2$ constitute a pair of primitive periods, and let X be a fundamental region for f. Then, given any complex number z, there exists a uniquely-determined point z_0 of the fundamental region X, and uniquely-determined integers m and n, such that $z = z_0 + 2m\omega_1 + 2n\omega_2$.

9.2 Basic Properties of Elliptic Functions

We first show that any elliptic function is determined, up to multiplication by a non-zero constant, by its periods, and by the locations and orders of its zeros and poles.

Proposition 9.5 *Let f and g be elliptic functions with the same period lattice. Suppose that the zeros and poles of g agree in location and order with those of f. Then there exists a non-zero complex number c such that $g(z) = cf(z)$ for all complex numbers that are not poles of f.*

Proof Let P be the set of zeros and poles of f. The condition that the zeros and poles of g agree in location and order with those of f ensures that the function sending each complex number z in $\mathbb{C} \setminus P$ to $g(z)/f(z)$ extends to a holomorphic function defined over the entire complex plane. Moreover this holomorphic function is doubly-periodic. But every holomorphic doubly-periodic function on the complex plane is constant (Lemma 9.2). It follows that there exists some non-zero complex number c such that $g(z) = cf(z)$ for all $z \in \mathbb{C} \setminus P$, as required. ∎

Proposition 9.6 *Let f be an elliptic function. Then, given any fundamental region X for f, the sum of the residues at f at those poles of f located in X is zero.*

Proof It is easy to see that the sum of the residues of f at the poles that lie in any fundamental region does not depend on the choice of fundamental region. We may therefore take

$$X = \{2s\omega_1 + 2t\omega_2 : 0 \leq s < 1 \text{ and } 0 \leq t < 1\}.$$

where $2\omega_1$ and $2\omega_2$ constitute a pair of primitive periods for f. The boundary of X is then the primitive period-lattice determined by $2\omega_1$ and $2\omega_2$.

First let us suppose that f has no poles on the boundary of the fundamental region X. We may order the primitive periods ω_1 and ω_2 so that ω_2/ω_1 has positive imaginary part. Now $f(z + 2\omega_1) = f(z)$ and $f(z + 2\omega_2) = f(z)$ for all complex numbers z that are not poles of f. On evaluating the path integral $\int_{\partial X} f(z)\, dz$ of f, taken in the anti-clockwise direction around the boundary of X, we find that

$$\begin{aligned}\int_{\partial X} f(z)\, dz &= \int_{[0,2\omega_1]} f(z)\, dz + \int_{[2\omega_1,2\omega_1+2\omega_2]} f(z)\, dz \\ &\quad + \int_{[2\omega_1+2\omega_2,2\omega_2]} f(z)\, dz + \int_{[2\omega_2,0]} f(z)\, dz\end{aligned}$$

$$
\begin{aligned}
&= \int_{[0,2\omega_1]} f(z)\,dz + \int_{[2\omega_1,2\omega_1+2\omega_2]} f(z)\,dz \\
&\qquad - \int_{[2\omega_2,2\omega_1+2\omega_2]} f(z)\,dz - \int_{[0,2\omega_2]} f(z)\,dz \\
&= \int_{[0,2\omega_1]} f(z)\,dz + \int_{[0,2\omega_2]} f(z+2\omega_1)\,dz \\
&\qquad - \int_{[0,2\omega_1]} f(z+2\omega_2)\,dz - \int_{[0,2\omega_2]} f(z)\,dz \\
&= \int_{[0,2\omega_2]} (f(z+2\omega_1) - f(z))\,dz \\
&\qquad - \int_{[0,2\omega_1]} (f(z+2\omega_2) - f(z))\,dz \\
&= 0.
\end{aligned}
$$

But it follows from Cauchy's Residue Theorem (Theorem 6.16) that

$$\int_{\partial X} f(z)\,dz = 2\pi i \sum_{j=1}^{k} c_j$$

where $c_1, c_2, \ldots, c_k$ are the residues of f at those poles of X that are located in the interior of the fundamental region X. It follows therefore that the sum $\sum_{j=1}^{k} c_j$ of the residues of f at those poles is zero.

Finally suppose that the function f has poles that lie on the boundary of X. Given a complex number c, let f_c be the meromorphic function defined such that $f_c(z) = f(z+c)$ for all complex numbers z for which $z+c$ is not a pole of f. Then f_c is an elliptic function with the same periods as f. We may choose the constant c so as to ensure that no pole of f_c lies on the boundary of X. Then the sum of the residues of f_c at the poles of this function located in X is zero. The corresponding result for the function f follows directly. ∎

Corollary 9.7 *Let f be an elliptic function, and let X be a fundamental region for f. Then the sum of the orders of the zeros of f that lie in X is equal to the sum of the orders of the poles of f that lie in X.*

Proof Let $g(z) = f'(z)/f(z)$ for all complex numbers z that are not zeros or poles of f. Then g is an elliptic function, and every period of f is a period of g. Let w be a complex number. If f has a zero of order m at w then g has a simple pole with residue m at w. If f has a pole of order m at w, then g has a simple pole with residue $-m$ at w. If w is neither a zero nor a pole of f

then the function g is holomorphic around w. It follows that the sum of the residues of g at poles of g that lie in the fundamental region X is equal to the sum of the orders of the zeros of f that lie in X, minus the sum of the orders of the poles of f that lie in X. But it follows from Proposition 9.6 that the sum of the residues of g at the poles of g that lie in X is zero. Therefore the sum of the orders of the zeros of f that lie in X is equal to the sum of the orders of the poles of f that lie in X. ∎

Definition Let f be an elliptic function, and let X be a fundamental region of f. The *degree* of an elliptic function is the sum of the orders of the poles of f that lie in X.

Corollary 9.8 *let f be an elliptic function, let c be a complex number, and let X be a fundamental region for f. Then the sum of the orders of the zeros of $f(z) - c$ that lie within X is equal to the degree of the elliptic function f.*

Proof Let $g(z) = f(z) - c$ for all complex numbers z that are not poles of f. Then the locations and orders of the poles of the elliptic function agree with those of f. It follows that the elliptic functions f and g have the same degree. The result therefore follows directly from Corollary 9.7. ∎

Corollary 9.9 *The degree of an elliptic function is greater than one.*

Proof If the elliptic function had just one pole in a fundamental region X, and if that pole were a simple pole, then the sum of the residues of the function at poles located in X would be non-zero, contradicting Proposition 9.6. It follows that the degree of the function must be at least two, as required. ∎

Proposition 9.10 *Let f be an elliptic function with period lattice Λ, let X be a fundamental region for f, let $z_1, z_2, \ldots, z_r$ be the zeros and poles of f that are located in the fundamental region X and let the integers $m_1, m_2, \ldots, m_r$ be determined so that $m_j = k$ when z_j is a zero of order k, and $m_j = -k$ when z_j is a pole of order k. Then $\sum_{j=1}^{r} m_j z_j \in \Lambda$.*

Proof The result does not depend on the choice of fundamental region. We may therefore take

$$X = \{2s\omega_1 + 2t\omega_2 : 0 \leq s < 1 \text{ and } 0 \leq t < 1\}.$$

where $2\omega_1$ and $2\omega_2$ constitute a pair of primitive periods for f. The boundary of X is then the primitive period-lattice determined by $2\omega_1$ and $2\omega_2$.

First let us suppose that $z_1, z_2, \ldots, z_r$ all lie in the interior of the fundamental region X that is bounded by the primitive period-parallelogram, so that the meromorphic function f has no zeros or poles on the boundary of X. We may order the primitive periods ω_1 and ω_2 so that ω_2/ω_1 has positive imaginary part. Let $g(z) = f'(z)/f(z)$ for all complex numbers z that are not zeros or poles of f. Then the function g is an elliptic function. Moreover the poles of g that lie in the fundamental region X are located at $z_1, z_2, \ldots, z_r$, and, for each integer j between 1 and r, the pole of g located at z_r is a simple pole with residue m_j. It follows from Cauchy's Residue Theorem (Theorem 6.16) that

$$\int_{\partial X} zg(z)\,dz = 2\pi i \sum_{j=1}^{k} z_j m_j,$$

where $\int_{\partial X} zg(z)\,dz$ is the path integral of $zg(z)$, taken in the anti-clockwise direction around the boundary of X. But the periodicity of f ensures that Now $g(z+2\omega_1) = g(z)$ and $g(z+2\omega_2) = g(z)$ for all complex numbers z that are not zeros or poles of f. On evaluating the path integral $\int_{\partial X} zg(z)\,dz$ of $zg(z)$, taken in the

$$\begin{aligned}
\int_{\partial X} zg(z)\,dz &= \int_{[0,2\omega_1]} zg(z)\,dz + \int_{[2\omega_1,2\omega_1+2\omega_2]} zg(z)\,dz \\
&\quad + \int_{[2\omega_1+2\omega_2,2\omega_2]} zg(z)\,dz + \int_{[2\omega_2,0]} zg(z)\,dz \\
&= \int_{[0,2\omega_1]} zg(z)\,dz + \int_{[2\omega_1,2\omega_1+2\omega_2]} zg(z)\,dz \\
&\quad - \int_{[2\omega_2,2\omega_1+2\omega_2]} zg(z)\,dz - \int_{[0,2\omega_2]} zg(z)\,dz \\
&= \int_{[0,2\omega_1]} zg(z)\,dz + \int_{[0,2\omega_2]} (z+2\omega_1)g(z+2\omega_1)\,dz \\
&\quad - \int_{[0,2\omega_1]} (z+2\omega_2)g(z+2\omega_2)\,dz - \int_{[0,2\omega_2]} zg(z)\,dz \\
&= 2\omega_1 \int_{[0,2\omega_2]} g(z)\,dz - 2\omega_2 \int_{[0,2\omega_1]} g(z)\,dz \\
&= 2\omega_1 \int_{[0,2\omega_2]} \frac{f'(z))\,dz}{f(z)} - 2\omega_2 \int_{[0,2\omega_1]} \frac{f'(z)\,dz}{f(z)} \\
&= 2\pi i\Big(2\omega_1 n(\gamma_2, 0) - 2\omega_2 n(\gamma_1, 0)\Big),
\end{aligned}$$

where $n(\gamma_1, 0)$ denotes the winding number about zero of the closed path $\gamma_1\colon [0,1] \to \mathbb{C}$ that is defined such that $\gamma_1(t) = f(2t\omega_1)$ for all $t \in [0,1]$, and

$n(\gamma_2, 0)$ denotes the winding number about zero of the closed path $\gamma_2\colon [0,1] \to \mathbb{C}$ that is defined such that $\gamma_2(t) = f(2t\omega_2)$ for all $t \in [0,1]$. Thus

$$\sum_{j=1}^{k} z_j m_j = 2\omega_1 n(\gamma_2, 0) - 2\omega_2 n(\gamma_1, 0).$$

But the winding numbers $n(\gamma_1, 0)$ and $n(\gamma_2, 0)$ are integers. It follows that $\sum\limits_{j=1}^{k} z_j m_j \in \Lambda$. This proves the result when f has no zeros or poles on the boundary of the fundamental region.

If f has a zero or pole on the boundary of the fundamental region X then one can choose some complex number c such that the function f_c has no zeros or poles on the boundary of X, where $f_c(z) = f(z + c)$ for all complex numbers z for which $f(z + c)$ is defined. A straightforward application of Corollary 9.7 shows that the sum $\sum\limits_{j=1}^{k} z_j m_j$ determined by the function f differs from the corresponding sum determined by the function f_c by an element of the period lattice Λ. This proves the result in the case when the function f has zeros or poles that lie on the boundary of the fundamental region. ∎

9.3 Summation over Lattices

Let Λ be a two-dimensional lattice in the complex plane. Then there exist elements c_1 and c_2 of Λ that generate the lattice Λ, where c_1 and c_2 are linearly independent over the real numbers. Let $\alpha\colon \Lambda \to \mathbb{C}$ be a function that associates to each element w of Λ a complex number $\alpha(w)$. Suppose that $\sum\limits_{m=-\infty}^{+\infty} \sum\limits_{n=-\infty}^{+\infty} |\alpha(mc_1 + nc_2)|$ converges. We can arrange the elements of the lattice Λ in an infinite sequence $w_1, w_2, w_3, \ldots$ so that each element of Λ occurs exactly once in the sequence. The infinite series $\sum\limits_{j=1}^{+\infty} |\alpha(w_j)|$ is then convergent. Standard properties of absolutely convergent infinite series then ensure that the infinite series $\sum\limits_{j=1}^{+\infty} \alpha(w_j)$ converges, and moreover the value of the sum of this series is unchanged under any rearrangement of the terms of the series. It follows that there is a well-defined complex number s with the property that $s = \sum\limits_{j=1}^{+\infty} \alpha(w_j)$ for any infinite sequence $w_1, w_2, w_3, \ldots$ of elements of the lattice Λ that includes every element of the lattice exactly once. We may therefore define

$$\sum_{w\in\Lambda}\alpha(w)=\sum_{j=1}^{+\infty}\alpha(w_j)$$

where $w_1, w_2, w_3, \ldots$ is any infinite series of elements of Λ in which every element of Λ occurs exactly once. Moreover, given any subset W of the lattice Λ, we may define

$$\sum_{w\in W}\alpha(w)=\sum_{w\in\Lambda}\beta(w),$$

where $\beta(w)=\alpha(w)$ for all $w\in W$, and $\beta(w)=0$ for all $w\in\Lambda\setminus W$.

Proposition 9.11 *Let W be a subset of a two-dimensional lattice Λ in the complex plane, let D be a subset of the complex plane, and for each $w\in W$, let $f_w\colon D\to\mathbb{C}$ be a continuous function on D. Suppose that one can associate a non-negative real number M_w to each element w of W so that $|f_w(z)|\leq M_w$ for all $z\in D$. Suppose also that $\sum_{w\in W} M_w$ converges. Then $\sum_{w\in W} f_w(z)$ converges for all $z\in D$, and defines a continuous function on D. Moreover if each of the functions f_w is holomorphic in the interior of D, then so is $\sum_{w\in W} f_w(z)$.*

Proof The result is immediate if the set W is finite. Suppose that the set W is infinite. Then there exists an infinite sequence $w_1, w_2, w_3, \ldots$ of elements of W in which every element of W occurs exactly once. Then $\sum_{w\in W} M_w=\sum_{j=1}^{+\infty} M_{w_j}$ and $\sum_{w\in W} f_w(z)=\sum_{j=1}^{+\infty} f_{w_j}(z)$. An application of the Weierstrass M-test (Proposition 2.8) shows that the infinite series $\sum_{j=1}^{+\infty} f_{w_j}(z)$ converges uniformly in D. Now the sum of any uniformly convergent series of continuous functions is itself a continuous function (see Theorem 1.20). Thus if $f(z)=\sum_{w\in W} f_w(z)$ then the function f is continuous on D.

Now suppose that each of the functions f_w is holomorphic in the interior of D. Then $\int_{\partial T} f_w(z)\,dz=0$ for all closed triangles T contained in the interior of D, where $\int_{\partial T} f_w(z)\,dz$ denotes the path integral of the function f_w taken around the boundary of the triangle in the anti-clockwise direction. But a standard result in the theory of integration ensures that the integral of a sum of a uniformly convergent infinite series of continuous functions on a bounded interval is the sum of the integrals of those functions. Therefore

$$\int_{\partial T} f(z)\,dz = \int_{\partial T} \left(\sum_{j=1}^{+\infty} f_{w_j}(z) \right) dz = \sum_{j=1}^{+\infty} \int_{\partial T} f_{w_j}(z)\,dz = 0.$$

It now follows from Morera's Theorem (Theorem 7.5) that the function f is holomorphic on the interior of D, as required. ∎

9.4 The Elliptic Functions of Weierstrass

Let Λ be a two-dimensional lattice in the complex plane, and let s be a complex number. One can show that $\sum_{w\in\Lambda\setminus\{0\}} |w|^{-s}$ converges if and only if $\mathrm{Re}[s] > 2$.

Let R be a positive real number, and let $W = \{w \in \Lambda : |w| \geq 2R\}$. If z is a complex number satisfying $|z| < R$ then $|z - w| > \frac{1}{2}|w|$ for all $w \in W$. A straightforward application of Proposition 9.11 shows that if k is an integer satisfying $k \geq 3$ then $\sum_{w\in W} (z - w)^k$ converges for all complex numbers z satisfying $|z| < R$, and defines a holomorphic function on the open disk $\{z \in \mathbb{C} : |z| < R\}$. Thus if we define $f_k(z) = \sum_{w\in\Lambda} (z - w)^{-k}$ for all $z \in \mathbb{C} \setminus \Lambda$ then the function f_k is meromorphic on the open disk $\{z \in \mathbb{C} : |z| < R\}$. This result holds no matter how large the value of R. We conclude therefore that, for each integer k satisfying $k \geq 3$, the function f_k is meromorphic over the entire complex plane, and its poles are of order k, and are located at the points of the lattice Λ. Moreover if $w_0 \in \Lambda$ then

$$f_k(z+w_0) = \sum_{w\in\Lambda}(z+w_0-w)^{-k} = \sum_{w\in\Lambda}(z+w_0-w)^{-k} = \sum_{w\in\Lambda}(z-w)^{-k} = f_k(z).$$

It follows that, for each integer k satisfying $k \geq 3$, the meromorphic function f_k is an elliptic function of degree k whose period lattice is the given lattice Λ.

We now proceed to construct an elliptic function of order 2. Let

$$\wp(z) = \frac{1}{z^2} + \sum_{w\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right),$$

for all $z \in \mathbb{C} \setminus \Lambda$, where Λ is a two-dimensional lattice in the complex plane. Let R be a positive real number. Then

$$\left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| = \left| \frac{2zw - z^2}{w^2(z-w)^2} \right| \leq \frac{8R|w| + 4R^2}{|w|^4}$$

when $|z| < R$ and $|w| \geq 2R$. A straightforward application of Proposition 9.11 shows that if $W = \{w \in \Lambda : |w| \geq 2R\}$ then

$$\sum_{w \in W} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

converges to a function that is holomorphic in the open disk $\{z \in \mathbb{C} : |z| < R\}$. It follows directly from this that the function $\wp$ is a meromorphic function whose poles are of order two, and are located at the points of the lattice Λ. Now

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z+w)^2} - \frac{1}{w^2} \right) \\
&= \frac{1}{(-z)^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{((-z)-w)^2} - \frac{1}{w^2} \right) = \wp(-z).
\end{aligned}$$

Thus the function $\wp$ is an even function.

Now any uniformly convergent series of holomorphic functions may be differentiated term-by-term. It follows that

$$\wp'(z) = \frac{-2}{z^3} + \sum_{w \in \Lambda \setminus \{0\}} \frac{-2}{(z-w)^3} = -2 f_3(z),$$

where $f_3(z) = \sum\limits_{w \in \Lambda} (z-w)^{-3}$. Thus the derivative of the function $\wp$ is an elliptic function of degree 3 whose period lattice is the lattice Λ. Therefore, given any $w_0 \in \Lambda$ the function that sends $z \in \mathbb{Z}$ to $\wp(z+w_0) - \wp(z)$ has zero derivative throughout $\mathbb{C} \setminus \Lambda$, and is therefore a constant value on this set. But the function $\wp$ is an even function, and therefore $\wp(z+w_0) - \wp(z)$ has the value zero when $z = -\frac{1}{2}w_0$. It follows that $\wp(z+w_0) = \wp(z)$ for all $z \in \mathbb{C} \setminus \Lambda$ and $w_0 \in \Lambda$. We conclude therefore that the function $\wp$ is an elliptic function of order 2 with period lattice Λ. This function is referred to as the *Weierstrass $\wp$-function* determined by the lattice Λ.

Note that

$$\begin{aligned}
\lim_{z \to 0} \left(\wp(z) - \frac{1}{z^2} \right) &= \lim_{z \to 0} \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right) \\
&= \sum_{w \in \Lambda \setminus \{0\}} \lim_{z \to 0} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right) = 0.
\end{aligned}$$

Lemma 9.12 *Let Λ be a two-dimensional lattice in the complex plane. Then the Weierstrass elliptic function $\wp(z)$ determined by this lattice is the unique elliptic function of degree 2 with period lattice Λ with the property that*

$$\lim_{z\to 0}\left(\wp(z) - \frac{1}{z^2}\right) = 0.$$

Proof Let f be an elliptic function of degree 2 with period lattice Λ which has the property that $\lim_{z\to 0}(f(z) - z^{-2}) = 0$. The sum of the orders of the poles of f that lie in any fundamental region must equal 2, and therefore the only poles of f are at the points of Λ. Thus the functions f and $\wp$ are both holomorphic throughout $\mathbb{C} \setminus \Lambda$. Also $\lim_{z\to 0}(f(z) - \wp(z)) = 0$. It follows that the function sending $z \in \mathbb{C} \setminus \Lambda$ to $f(z) - \wp(z)$ extends to a doubly-periodic function defined over the entire complex plane. But any doubly-periodic holomorphic function defined over the entire complex plane is constant (Lemma 9.2). Moreover the value of this constant function is the limit of $f(z)-\wp(z)$ as $z \to 0$ and is therefore zero. It follows that $f(z) = \wp(z)$ for all $z \in \mathbb{C} \setminus \{0\}$, as required. ∎

Let $2\omega_1$ and $2\omega_2$ be a pair of primitive periods that generate the two-dimensional lattice Λ in the complex plane, and let $\wp$ be the Weierstrass elliptic function determined by that lattice. Then $\wp(z) = \wp(-z) = \wp(2\omega_1 - z)$ for all $z \in \mathbb{C} \setminus \Lambda$. Similarly $\wp(z) = \wp(2\omega_2 - z)$ and $\wp(z) = \wp(2\omega_3 - z)$ for all $z \in \mathbb{C} \setminus \Lambda$, where $\omega_3 = -(\omega_1 + \omega_2)$. It follows that

$$\wp'(z) + \wp'(2\omega_1 - z) = \wp'(z) + \wp'(2\omega_2 - z) = \wp'(z) + \wp'(2\omega_3 - z) = 0$$

for all $z \in \mathbb{C} \setminus \Lambda$. In particular, $\wp'(\omega_1) = \wp'(\omega_2) = \wp'(\omega_3) = 0$. Thus if

$$e_1 = \wp(\omega_1), \quad e_2 = \wp(\omega_2), \quad e_3 = \wp(\omega_3)$$

then the function sending $z \in \mathbb{C} \setminus \Lambda$ to $\wp(z) - e_1$ has a zero of order at least 2 at ω_1. But the sum of the orders of the zeros of $\wp(z) - e_1$ that are contained in any fundamental region for this function is equal to the degree of this elliptic function (Corollary 9.8), and therefore has the value 2. It follows that the zeros of the elliptic function $\wp(z) - e_1$ are zeros of order 2 located at the points of the set $\Lambda + \omega_1$, where

$$\Lambda + \omega_1 = \{w + \omega_1 : w \in \Lambda\}$$

Similarly the zeros of the elliptic function $\wp(z) - e_2$ are zeros of order 2 located at the points of the set $\Lambda + \omega_2$, and the zeros of the elliptic function $\wp(z) - e_3$ are zeros of order 2 located at the points of the set $\Lambda + \omega_3$.

Proposition 9.13 *Let Λ be a two-dimensional lattice in the complex plane generated by $2\omega_1$ and $2\omega_2$, where ω_1 and ω_2 are complex numbers that are linearly independent over the real numbers, let $\omega_3 = -(\omega_1 + \omega_2)$, and let $\wp$ be the Weierstrass elliptic function determined by the lattice Λ. Then*

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

for all $z \in \mathbb{C} \setminus \Lambda$, where

$$e_1 = \wp(\omega_1), \quad e_2 = \wp(\omega_2), \quad \textit{and} \quad e_3 = \wp(\omega_3).$$

Proof The derivative $\wp'$ of the Weierstrass elliptic function $\wp$ is an elliptic function of order 3 which has zeros at the points belonging to the sets $\Lambda+\omega_1$, $\Lambda + \omega_2$ and $\Lambda + \omega_3$. But the sums of the orders of the zeros of this function that lie in any fundamental region has the value 3 (Corollary 9.8). It follows that $\wp'$ has no other zeros, and moreover all zeros of this function are of order 1.

Let
$$f(z) = (\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$
for all $z \in \mathbb{C} \setminus \Lambda$. Then f is an elliptic function of degree 6 which has poles of order 6 located at the points of the lattice Λ, and zeros of order 2 located at the points of the sets $\Lambda + \omega_1$, $\Lambda + \omega_2$ and $\Lambda + \omega_3$. Thus the locations and orders of the zeros and poles of the elliptic function f agree with those of the square $\wp'(z)^2$ of the derivative of the Weierstrass elliptic function $\wp$. It follows from Proposition 9.5 that $\wp'(z)^2 = cf(z)$ for some constant c. Now $\lim_{z\to 0} z^6\wp'(z)^2 = 4$ and $\lim_{z\to 0} z^6 f(z) = 1$. Therefore

$$\wp'(z)^2 = 4f(z) = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

for all $z \in \mathbb{C} \setminus \Lambda$, as required. ∎

Let $\gamma\colon [a, b] \to \mathbb{C}$ be a path in the complex plane that does not pass through any of the points e_1, e_2 and e_3. It can be shown that there exists a path Let $\beta\colon [a, b] \to \mathbb{C}$ such that $\gamma(t) = \wp(\beta(t))$ for all $t \in [a, b]$. Then

$$\begin{aligned} \int_\gamma \frac{dz}{\sqrt{(z-e_1)(z-e_2)(z-e_3)}} &= \int_\beta \frac{\wp'(z)\,dz}{\sqrt{(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)}} \\ &= \pm 2 \int_\beta dz = \pm 2(\beta(b) - \beta(a)). \end{aligned}$$

(The value of the square root $\sqrt{(z-e_1)(z-e_2)(z-e_3)}$ at each point of the path γ is chosen so that it varies continuously along the path.) This shows that the elliptic functions of Weierstrass can be used in order to integrate the reciprocal of the square root of a cubic polynomial.

9.5 Square Roots of Meromorphic Functions

Theorem 9.14 *Let D be a simply-connected open set in the complex plane, and let f be a meromorphic function on D. Suppose that the orders of the zeros and poles of f in D are all divisible by two. Then there exists a meromorphic function g on D which satisfies $g(z)^2 = f(z)$ for all elements z of D for which $f(z)$ is defined.*

Proof Suppose that the zeros and poles of f in D are located at the points $w_1, w_2, \ldots, w_r$. Let the integers $m_1, m_2, \ldots, m_r$ be determined such that f has a zero of order $2m_j$ at m_j when $m_j > 0$, and f has a pole of order $-2m_j$ at m_j when $m_j < 0$. Let $h(z) = f(z)/f'(z)$ for all $z \in D \setminus \{w_1, w_2, \ldots, w_r\}$. Then h is a meromorphic function on D which is holomorphic throughout $D \setminus \{w_1, w_2, \ldots, w_r\}$. Moreover this function h has simple poles at $w_1, w_2, \ldots, w_r$, and the residue of h at the pole w_j is $2m_j$ for $j = 1, 2, \ldots, r$. It follows from Cauchy's Residue Theorem (Theorem 6.16) that

$$\int_\sigma \frac{f'(\zeta)\, d\zeta}{f(\zeta)} = 4\pi i \sum_{j=1}^{r} m_j n(\sigma, w_j)$$

for all piecewise continuously differentiable closed paths σ in D (see also Theorem 7.12). Therefore

$$\frac{1}{4\pi i} \int_\sigma \frac{f'(\zeta)\, d\zeta}{f(\zeta)} \in \mathbb{Z}$$

for all closed paths σ in D that do not pass through any zero or pole of f. It follows from that that if z_0 and z are elements of $D \setminus \{w_1, w_2, \ldots, w_r\}$, and if γ_1 and γ_2 are closed paths in D from z_0 to z that do not pass through any any zero or pole of f, then

$$\frac{1}{4\pi i} \left(\int_{\gamma_1} \frac{f'(\zeta)\, d\zeta}{f(\zeta)} - \int_{\gamma_2} \frac{f'(\zeta)\, d\zeta}{f(\zeta)} \right) \in \mathbb{Z},$$

and therefore

$$\exp\left(c_0 + \frac{1}{2} \int_{\gamma_1} \frac{f'(\zeta)\, d\zeta}{f(\zeta)} \right) = \exp\left(c_0 + \frac{1}{2} \int_{\gamma_2} \frac{f'(\zeta)\, d\zeta}{f(\zeta)} \right)$$

for all complex numbers c_0. It follows from this that there is a well-defined continuous function $g \colon D \setminus \{w_1, w_2, \ldots, w_r\} \to \mathbb{C}$ on $D \setminus \{w_1, w_2, \ldots, w_r\}$ whose value $g(z)$ at any element z of $D \setminus \{w_1, w_2, \ldots, w_r\}$ satisfies

$$g(z) = \exp\left(c_0 + \frac{1}{2} \int_{\gamma} \frac{f'(\zeta)\, d\zeta}{f(\zeta)} \right)$$

for all continuous piecewise continuously differentiable paths γ from z_0 to z, where c_0 is chosen so that $\exp(2c_0) = f(z_0)$. Then $g(z_0)^2 = f(z_0)$. We shall prove that $g(z)^2 = f(z)$ for all $z \in D \setminus \{w_1, w_2, \ldots, w_r\}$.

Let z_1 be an element of D that is not a zero or pole of f. Then there exists some positive real number δ such that

$$\{z \in \mathbb{C} : |z - z_1| < \delta\} \subset D \setminus \{w_1, w_2, \ldots, w_r\}.$$

But then

$$\begin{aligned} g(z) &= \exp\left(c_0 + \frac{1}{2}\int_{\gamma_1} \frac{f'(\zeta)\, d\zeta}{f(\zeta)} + \frac{1}{2}\int_{[z_1,z]} \frac{f'(\zeta)\, d\zeta}{f(\zeta)}\right) \\ &= g(z_1) \exp\left(\frac{1}{2}\int_{[z_1,z]} \frac{f'(\zeta)\, d\zeta}{f(\zeta)}\right) \end{aligned}$$

when $|z - z_1| < \delta$, where γ_1 is some piecewise continuously differentiable path from z_0 to z_1 that does not pass through any zero or pole of f. (Here $\displaystyle\int_{[z_1,z]} \frac{f'(\zeta)\, d\zeta}{f(\zeta)}$ denotes the path integral of f'/f taken along the line segment joining z_1 to z.) Now any holomorphic function defined on an open disk may be represented by a power series throughout that disk (Theorem 7.1). Therefore there exist complex numbers $a_0, a_1, a_2, \ldots$ such that

$$\frac{f'(z)}{f(z)} = \sum_{n=0}^{+\infty} a_n (z - z_1)^n$$

for all $z \in D$ satisfying $|z - z_1| < \delta$. Then

$$\int_{[z_1,z]} \frac{f'(\zeta)\, d\zeta}{f(\zeta)} = \sum_{n=0}^{+\infty} \frac{a_n}{n+1} (z - z_1)^{n+1}.$$

It follows that this integral defines a holomorphic function on the disk $\{z \in \mathbb{C} : |z - z_1| < \delta$. Moreover

$$\frac{d}{dz} \int_{[z_1,z]} \frac{f'(\zeta)\, d\zeta}{f(\zeta)} = \frac{f'(z)}{f(z)}$$

when $|z - z_1| < \delta$. It follows that

$$\begin{aligned} g'(z) &= g(z_1) \frac{d}{dz} \exp\left(\frac{1}{2}\int_{[z_1,z]} \frac{f'(\zeta)\, d\zeta}{f(\zeta)}\right) \\ &= \frac{1}{2} g(z_1) \exp\left(\frac{1}{2}\int_{[z_1,z]} \frac{f'(\zeta)\, d\zeta}{f(\zeta)}\right) \frac{d}{dz} \int_{[z_1,z]} \frac{f'(\zeta)\, d\zeta}{f(\zeta)} \\ &= \frac{g(z) f'(z)}{2f(z)}. \end{aligned}$$

Therefore

$$\frac{d}{dz}\frac{g(z)^2}{f(z)} = \frac{2g(z)g'(z)}{f(z)} - \frac{g^2(z)f'(z)}{f(z)^2} = \frac{g(z)}{f(z)^2}(2g'(z)f(z) - g(z)f'(z)) = 0.$$

It follows that $g(z)^2 = f(z)$ throughout $D \setminus \{w_1, w_2, \ldots, w_r\}$, and thus the function g is the required meromorphic function. ∎

9.6 Construction of Jacobi's Elliptic Functions

Let $\wp$ be the Weierstrass elliptic function associated with a two-dimensional lattice Λ generated by primitive periods $2\omega_1$ and $2\omega_2$, let $\omega_3 = -(\omega_1 + \omega_2)$, and let

$$e_1 = \wp(\omega_1), \quad e_2 = \wp(\omega_2), \quad e_3 = \wp(\omega_3)$$

Then

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

Now the function $\wp(z) - e_1$ has double poles located at the elements of the lattice Λ, and double zeros located at the elements of the set $\Lambda + \omega_1$. This function has no other zeros or poles in the complex plane. Therefore every zero of the function $\wp(z) - e_1$ is a double zero, and every pole is a double pole. The same is true of the functions $\wp(z) - e_2$ and $\wp(z) - e_3$. This function has no other zeros or poles in the complex plane. It therefore follows from Theorem 9.14 that there exist meromorphic functions S, C and D on the complex plane, defined such that

$$S(z)^2 = \frac{e_1 - e_2}{\wp(z) - e_2}, \quad C(z)^2 = \frac{\wp(z) - e_1}{\wp(z) - e_2}, \quad D(z)^2 = \frac{\wp(z) - e_3}{\wp(z) - e_2}.$$

We choose the signs of the functions C and D such that

$$C(0) = \lim_{z\to 0} C(z) = 1, \quad D(0) = \lim_{z\to 0} D(z) = 1.$$

Then

$$S(0) = \lim_{z\to 0} S(z) = 0,$$

and

$$S(z)^2 + C(z)^2 = 1, \quad \frac{e_3 - e_2}{e_1 - e_2}S(z)^2 + D(z)^2 = 1.$$

Now the function $z^{-1}S(z)$ tends to a non-zero limit at $z \to 0$, and $(z^{-1}S(z))^2$ is an even function of z. It follows that $z^{-1}S(z)$ is an even function of z, and therefore $S(z)$ is an odd function of z. The functions C and D are non-zero at zero, and their squares are even functions, and therefore C and D are even functions.

Lemma 9.15 *The function $S(z)$ satisfies*

$$S(z + 2\omega_1) = -S(z), \quad S(z + 2\omega_2) = S(z)$$

for all complex numbers z for which $S(z)$ is defined. Therefore this function is doubly-periodic, with primitive periods $4\omega_1$ and $2\omega_2$.

Proof The function S satisfies $S(z+2\omega_1)^2 = S(z)^2$ and $S(z+2\omega_2)^2 = S(z)^2$ and wherever $S(z)$ is defined, and therefore

$$S(z + 2\omega_1) = b_1 S(z), \quad S(z + 2\omega_2) = b_2 S(z),$$

wherever $S(z)$ is defined, where b_1 and b_2 are constants, independent of z, whose values are either $+1$ or -1. But $S(\omega_1)^2 = 1$ and $S(\omega_1) = b_1 S(-\omega_1) = -b_1 S(\omega_1)$ and therefore $b_1 = -1$. Thus

$$S(z + 2\omega_1) = -S(z)$$

wherever $S(z)$ is defined.

Let $f(z) = zS(z - \omega_2)$ for all complex numbers z for which $S(z - \omega_2)$ is defined. Then

$$\begin{aligned} (e_1 - e_2) f(-z)^2 &= z^2(\wp(-z - \omega_2) - e_2) = z^2(\wp(z + \omega_2) - e_2) \\ &= z^2(\wp(z - \omega_2) - e_2) = (e_1 - e_2) f(z)^2 \end{aligned}$$

Thus $f(z)^2$ is an even function of f. Moreover $\lim_{z\to 0} f(z) \neq 0$. It follows that f is itself and even function. But then $S(z - \omega_2)$ is an odd function of z, and therefore $S(2\omega_2) = S(0)$. It follows that $b_2 = 1$, and thus $S(z + 2\omega_2 = S(z)$, as required. ∎

In a similar fashion, one can verify that

$$C(z + 2\omega_1) = -C(z), \quad C(z + 2\omega_2) = -C(z),$$

$$D(z + 2\omega_1) = D(z), \quad D(z + 2\omega_2) = -D(z).$$

Moreover, using the differential equation satisfied by the Weierstrass elliptic function $\wp(z)$, it follows easily that

$$S'(z) = \sqrt{e_1 - e_2} C(z) D(z).$$

It then follows that

$$C'(z) = -\sqrt{e_1 - e_2} D(z) S(z), \quad D'(z) = -\sqrt{e_1 - e_2} k^2 D(z) D(z),$$

where

$$k = -\sqrt{\frac{e_3 - e_2}{e_1 - e_2}}.$$

Thus if one defines

$$\operatorname{sn} u = S((e_1 - e_2)^{-\frac{1}{2}} u), \quad \operatorname{cn} u = C((e_1 - e_2)^{-\frac{1}{2}} u), \quad \operatorname{dn} u = D((e_1 - e_2)^{-\frac{1}{2}} u),$$

then sn, cn and dn are elliptic functions which satisfy the identities

$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1, \quad k^2 \operatorname{sn}^2 u + \operatorname{dn}^2 u = 1,$$

$$\frac{d}{du} \operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u, \quad \frac{d}{du} \operatorname{cn} u = -\operatorname{dn} u \operatorname{sn} u, \quad \frac{d}{du} \operatorname{dn} u = -k^2 \operatorname{sn} u \operatorname{cn} u.$$

These functions are the elliptic functions of Jacobi.