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On the reciprocal sums of higher-order sequences

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Abstract

Let $\{u_n\}$ be a higher-order recursive sequence. Several identities are obtained for the infinite sums and finite sums of the reciprocals of higher-order recursive sequences.

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Keywords: infinite sums; finite sums; reciprocal; higher-order recurrences

1 Introduction

The so-called Fibonacci zeta function and Lucas zeta function defined by

$$\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s} \quad \text{and} \quad \zeta_L(s) = \sum_{n=1}^{\infty} \frac{1}{L_n^s},$$

where the F_n and L_n denote the Fibonacci numbers and Lucas numbers, have been considered in several different ways. Navas [1] discussed the analytic continuation of these series. Elsner *et al.* [2] obtained that for any positive distinct integer s_1, s_2, s_3, the numbers $\zeta_F(2s_1)$, $\zeta_F(2s_2)$, and $\zeta_F(2s_3)$ are algebraically independent if and only if at least one of s_1, s_2, s_3 is even.

Ohtsuka and Nakamura [3] studied the partial infinite sums of reciprocal Fibonacci numbers and proved the following conclusions:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-2} - 1 & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1 & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-1}F_n & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

Where $\lfloor \cdot \rfloor$ denotes the floor function.

Further, Wu and Zhang [4, 5] generalized these identities to the Fibonacci polynomials and Lucas polynomials. Similar properties were also investigated in [6–8]. Related properties of the Fibonacci polynomials and Lucas polynomials can be found in [9–12].

Recently, some authors considered the nearest integer of the sums of reciprocal Fibonacci numbers and other famous sequences and obtained several new interesting identities, see [13] and [14]. Kilic and Arikan [15] defined a kth-order linear recursive sequence

$\{u_n\}$ for any positive integer $p \geq q$ and $n > k$ as follows:

$$u_n = pu_{n-1} + qu_{n-2} + u_{n-3} + \cdots + u_{n-k},$$

and they proved that there exists a positive integer n_0 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1} \quad (n \geq n_0),$$

where $\| \cdot \|$ denotes the nearest integer. (Clearly, $\|x\| = \lfloor x + \frac{1}{2} \rfloor$.)

In this paper, we unify the above results by proving some theorems that include all the results, [3–8] and [13–15], as special cases. We consider the following type of higher-order recurrence sequences. For any positive integer $a_1, a_2, \ldots, a_m$, we define mth-order linear recursive sequences $\{u_n\}$ for $n > m$ as follows:

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_{m-1} u_{n-m+1} + a_m u_{n-m}, \tag{1}$$

with initial values $u_i \in \mathbb{N}$ for $0 \leq i < m$ and at least one of them is not zero. If $m = 2$, $a_1 = a_2 = 1$, then $u_n = F_n$ are the Fibonacci numbers. If $m = 2$, $a_1 = 2$, $a_2 = 1$, then $u_n = P_n$ are the Pell numbers. Our main results are the following.

Theorem 1 *Let $\{u_n\}$ be an mth-order sequence defined by* (1) *with the restriction $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$. For any positive real number $\beta > 2$, there exists a positive integer n_1 such that*

$$\left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1} \quad (n \geq n_1).$$

Taking $\beta \to +\infty$, from Theorem 1 we may immediately deduce the following.

Corollary 1 *Let $\{u_n\}$ be an mth-order sequence defined by* (1) *with the restriction $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$. Then there exists a positive integer n_2 such that*

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1} \quad (n \geq n_2).$$

For a positive real number $1 < \beta \leq 2$, whether there exits an identity for

$$\left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{1}{u_k} \right)^{-1} \right\|$$

is an interesting open problem.

2 Several lemmas

To complete the proof of our theorem, we need the following.

Lemma 1 *Let $a_1, a_2, \ldots, a_m$ be positive integers with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ and $m \in \mathbb{N}$ with $m \geq 2$. Then, for the polynomial*

$$f(x) = x^m - a_1 x^{m-1} - a_2 x^{m-2} - \cdots - a_{m-1} x - a_m,$$

we have

(I) *Polynomial $f(x)$ has exactly one positive real zero α with $a_1 < \alpha < a_1 + 1$.*
(II) *Other $m-1$ zeros of $f(x)$ lie within the unit circle in the complex plane.*

Proof For any positive integer $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ and $m \geq 2$, we have

$$\begin{aligned} f(a_1) &= a_1^m - a_1^m - a_2 a_1^{m-2} - \cdots - a_{m-1} a_1 - a_m \\ &= -a_2 a_1^{m-2} - \cdots - a_{m-1} a_1 - a_m < 0, \end{aligned}$$

and

$$\begin{aligned} f(a_1 + 1) &= (a_1 + 1)^m - a_1 (a_1 + 1)^{m-1} - a_2 (a_1 + 1)^{m-2} - \cdots - a_m \\ &> (a_1 + 1)^m - a_1 \bigl((a_1 + 1)^{m-1} + (a_1 + 1)^{m-2} + \cdots + 1\bigr) \\ &= (a_1 + 1)^m - a_1 \cdot \frac{(a_1 + 1)^m - 1}{a_1} = 1 > 0. \end{aligned}$$

Thus there exits a positive real zero α of $f(x)$ with $a_1 < \alpha < a_1 + 1$. According to Descarte's rule of signs, $f(x) = 0$ has at most one positive real root. So, $f(x)$ has exactly one positive real zero α with $a_1 < \alpha < a_1 + 1$. This completes the proof of (I) in Lemma 1.

Observe that from (I) in Lemma 1 we have

$$\text{if } x \in \mathbb{R} \text{ such that } x > \alpha, \text{ then } f(x) > 0, \tag{2}$$

$$\text{if } x \in \mathbb{R} \text{ such that } 0 < x < \alpha, \text{ then } f(x) < 0. \tag{3}$$

Let

$$\begin{aligned} g(x) &= (x - 1) f(x) \\ &= x^{m+1} - (a_1 + 1) x^m + (a_1 - a_2) x^{m-1} + (a_2 - a_3) x^{m-2} + \cdots + (a_{m-1} - a_m) x + a_m. \end{aligned}$$

Since $f(x)$ has exactly one positive real zero α, $g(x)$ has two positive real zeros α and 1. Observe that

$$\text{if } x \in \mathbb{R} \text{ such that } x > \alpha, \text{ then } g(x) > 0, \tag{4}$$

$$\text{if } x \in \mathbb{R} \text{ such that } 1 < x < \alpha, \text{ then } g(x) < 0. \tag{5}$$

To complete the proof of (II) in Lemma 1, it is sufficient to show that there is no zero on and outside of the unit circle. □

Claim 1 *$f(x)$ has no complex zero z_1 with $|z_1| > \alpha$.*

Proof Assume that there exits such a zero. So, we have

$$f(z_1) = z_1^m - a_1 z_1^{m-1} - a_2 z_1^{m-2} - \cdots - a_{m-1} z_1 - a_m = 0,$$

then we obtain

$$\begin{aligned} &\left|z_1^m\right| \le a_1\left|z_1^{m-1}\right| + a_2\left|z_1^{m-2}\right| + \cdots + a_{m-1}|z_1| + a_m, \\ &f\bigl(|z_1|\bigr) = \left|z_1^m\right| - a_1\left|z_1^{m-1}\right| - a_2\left|z_1^{m-2}\right| - \cdots - a_{m-1}|z_1| - a_m \le 0. \end{aligned}$$

This contradicts with (2). □

Claim 2 *$f(x)$ has no complex zero z_2 with $1 < |z_2| < \alpha$.*

Proof Assume that there exits such a zero. Since $f(z_2) = 0$,

$$g(z_2) = z_2^{m+1} - (a_1 + 1)z_2^m + (a_1 - a_2)z_2^{m-1} + \cdots + (a_{m-1} - a_m)z_2 + a_m = 0,$$

then we obtain

$$(a_1 + 1)|z_2|^m \le |z_2|^{m+1} + (a_1 - a_2)|z_2|^{m-1} + \cdots + (a_{m-1} - a_m)|z_2| + a_m.$$

So, we have $g(|z_2|) \ge 0$, which contradicts with (5). □

Claim 3 *On the circle $|z_3| = \alpha$ and $|z_3| = 1$, $f(x)$ has the unique zero α.*

Proof If $f(z_3) = 0$, then

$$g(z_3) = z_3^{m+1} - (a_1 + 1)z_3^m + (a_1 - a_2)z_3^{m-1} + \cdots + (a_{m-1} - a_m)z_3 + a_m = 0,$$

then we obtain

$$(a_1 + 1)|z_3|^m \le |z_3|^{m+1} + (a_1 - a_2)|z_3|^{m-1} + \cdots + (a_{m-1} - a_m)|z_3| + a_m. \tag{6}$$

If $z_3 = \alpha$ or $z_3 = 1$, then $g(z_3) = 0$, so (6) must be an equality. Therefore, $z_3^{m+1}, (a_1 - a_2)z_3^{m-1}, (a_2 - a_3)z_3^{m-2}, \ldots, (a_{m-1} - a_m)z_3$ and a_m all lie on the same ray issuing from the origin. Since $(a_1 - a_2), (a_2 - a_3), \ldots, a_m$, are all the elements of $\mathbb{R}^+$, $z_3^{m+1}, z_3^{m-1}, z_3^{m-2}, \ldots, z_3$ must be the elements of $\mathbb{R}^+$. Therefore we obtain $f(z_3) \in \mathbb{R}^+$. On the circle $|z_3| = \alpha$ and $|z_3| = 1$, there are two conditions $z_3 = 1$ or $z_3 = \alpha$. Since $f(1) \neq 0$, α is the unique zero of $f(x)$, Claim 3 holds.

From the three claims, (II) in Lemma 1 is proven. □

Lemma 2 *Let $m \ge 2$ and let $\{u_n\}_{n\ge 0}$ be an integer sequence satisfying the recurrence formula* (1). *Then the closed formula of u_n is given by*

$$u_n = c\alpha^n + \mathcal{O}\bigl(d^{-n}\bigr) \quad (n \to \infty),$$

where $c > 0$, $d > 1$, and $a_1 < \alpha < a_1 + 1$ is the positive real zero of $f(x)$.

Proof Let $\alpha, \alpha_1, \dots, \alpha_t$ be the distinct roots of $f(x) = 0$, where $f(x) = 0$ is the characteristic equation of the recurrence formula (1). From Lemma 1 we know that α is the simple root of $f(x) = 0$, then let r_j, for $j = 1, 2, \dots, t$, denote the multiplicity of the root α_j. From the properties of mth-order linear recursive sequences, u_n can be expressed as follows:

$$u_n = c\alpha^n + \sum_{i=1}^{t} P_i(n)\alpha_i^n, \tag{7}$$

where

$$P_i(n) \in \mathbb{R}[n], \quad \deg P_i(n) = r_i - 1, r_1 + r_2 + \cdots + r_t = m - 1, \quad \text{and} \quad c \in \mathbb{R}.$$

For example, for positive integers $1 \le u, v, w \le t$, if α_u is the simple root of $f(x)$, then $P_u(n) = g_1$, where $g_1 \in \mathbb{R}$, and $\deg P_u(n) = 0$; if α_v is the double root of $f(x)$, then $P_v(n) = g_2 n + g_3$, where $g_2, g_3 \in \mathbb{R}$, and $\deg P_v(n) = 1$; if α_w is the multiple root of $f(x)$ with the multiplicity r_w, then $P_w(n) = b_1 n^{r_w - 1} + b_2 n^{r_w - 2} + \cdots + b_{r_w - 1} n + b_{r_w}$, where $b_1, b_2, \dots, b_{r_w} \in \mathbb{R}$, and $\deg P_w(n) = r_w - 1$.

From Lemma 1 we have $|\alpha_i| < 1$ for $1 \le i \le t$. Since each term of tail in (7) goes to 0 as $n \to \infty$, we can find the constant $M \in \mathbb{R}$ and $d \in \mathbb{R}$ with $d > 1$ for $n > n_0$ such that

$$\left| \sum_{i=1}^{t} P_i(n)\alpha_i^n \right| \le \sum_{i=1}^{t} \left| P_i(n)\alpha_i^n \right| \le M d^{-n},$$

which completes the proof (note that if all the roots of $f(x)$ are distinct, we can choose $d^{-1} = \max\{|\alpha_1|, |\alpha_2|, \dots, |\alpha_{m-1}|\}$ and $M = m - 1$). □

3 Proof of Theorem 1

In this section, we shall complete the proof of Theorem 1. From the geometric series as $\epsilon \to 0$, we have

$$\frac{1}{1 \pm \epsilon} = 1 \mp \epsilon + \mathcal{O}(\epsilon^2) = 1 + \mathcal{O}(\epsilon).$$

Using Lemma 2, we have

$$\begin{aligned} \frac{1}{u_k} &= \frac{1}{c\alpha^k + \mathcal{O}(d^{-k})} = \frac{1}{c\alpha^k (1 + \mathcal{O}((\alpha d)^{-k}))} \\ &= \frac{1}{c\alpha^k} \left(1 + \mathcal{O}\left((\alpha d)^{-k}\right)\right) \\ &= \frac{1}{c\alpha^k} + \mathcal{O}\left(\left(\alpha^2 d\right)^{-k}\right). \end{aligned} \tag{8}$$

Thus

$$\begin{aligned} \sum_{k=n}^{\lfloor \beta n \rfloor} \frac{1}{u_k} &= \frac{1}{c} \sum_{k=n}^{\lfloor \beta n \rfloor} \frac{1}{\alpha^k} + \mathcal{O}\left(\sum_{k=n}^{\lfloor \beta n \rfloor} \left(\alpha^2 d\right)^{-k} \right) \\ &= \frac{\alpha}{c(\alpha - 1)} \cdot \alpha^{-n} - \frac{1}{c(\alpha - 1)} \cdot \alpha^{-\lfloor \beta n \rfloor} + \mathcal{O}\left(\alpha^{-2n} d^{-n}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha}{c(\alpha-1)} \cdot \alpha^{-n} + \mathcal{O}\bigl(\alpha^{-2n}\alpha^{-\lfloor \beta n\rfloor+2n}\bigr) + \mathcal{O}\bigl(\alpha^{-2n}d^{-n}\bigr) \\
&= \frac{\alpha}{c(\alpha-1)}\alpha^{-n} + \mathcal{O}\bigl(\alpha^{-2n}h\bigr),
\end{aligned}
$$

where $h = \max\{\alpha^{-\lfloor \beta n\rfloor+2n}, d^{-n}\}$.

Taking reciprocal, we get

$$
\begin{aligned}
\left(\sum_{k=n}^{\lfloor \beta n\rfloor} \frac{1}{u_k}\right)^{-1} &= \frac{1}{\frac{\alpha}{c(\alpha-1)}\alpha^{-n}(1+\mathcal{O}(\alpha^{-n}h))} \\
&= \frac{\alpha-1}{\alpha}c\alpha^{n}\bigl(1+\mathcal{O}\bigl(\alpha^{-n}h\bigr)\bigr) \\
&= \frac{\alpha-1}{\alpha}c\alpha^{n} + \mathcal{O}(h) \\
&= u_n - u_{n-1} + \mathcal{O}(h).
\end{aligned}
$$

Since $h = \max\{\alpha^{-\lfloor \beta n\rfloor+2n}, d^{-n}\} < 1$, there exists $n \geq n_1$ sufficient large so that the modulus of the last error term becomes less than 1/2, which completes the proof.

Proof of Corollary 1 From identity (8), we have

$$
\frac{1}{u_k} = \frac{1}{c\alpha^k} + \mathcal{O}\bigl(\bigl(\alpha^2 d\bigr)^{-k}\bigr).
$$

Thus

$$
\sum_{k=n}^{\infty} \frac{1}{u_k} = \frac{1}{c}\sum_{k=n}^{\infty}\frac{1}{\alpha^k} + \mathcal{O}\left(\sum_{k=n}^{\infty}\bigl(\alpha^2 d\bigr)^{-k}\right) = \frac{\alpha}{c(\alpha-1)}\alpha^{-n} + \mathcal{O}\bigl(\bigl(\alpha^2 d\bigr)^{-n}\bigr).
$$

Taking reciprocal, we get

$$
\begin{aligned}
\left(\sum_{k=n}^{\infty} \frac{1}{u_k}\right)^{-1} &= \frac{1}{\frac{\alpha}{c(\alpha-1)}\alpha^{-n}(1+\mathcal{O}((\alpha d)^{-n}))} \\
&= \frac{\alpha-1}{\alpha}c\alpha^{n}\bigl(1+\mathcal{O}\bigl((\alpha d)^{-n}\bigr)\bigr) \\
&= \frac{\alpha-1}{\alpha}c\alpha^{n} + \mathcal{O}\bigl(d^{-n}\bigr) \\
&= u_n - u_{n-1} + \mathcal{O}\bigl(d^{-n}\bigr).
\end{aligned}
$$

So, there exists $n \geq n_2$ sufficiently large so that the modulus of the last error term becomes less than 1/2, which completes the proof. □

4 Related results

The following results are obtained similarly.

Theorem 2 *Let* $\{u_n\}$ *be an mth-order sequence defined by* (1) *with the restriction* $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$. *Let p and q be positive integers with* $0 \leq q < p$. *For any real number* $\beta > 2$, *there exist positive integers n_3, n_4 and n_5 depending on $a_1, a_2, \dots,$ and a_m such that*

$$\text{(a)}\quad \left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{(-1)^k}{u_k} \right)^{-1} \right\| = (-1)^n (u_n + u_{n-1}) \quad (n \ge n_3),$$

$$\text{(b)}\quad \left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{1}{u_{pk+q}} \right)^{-1} \right\| = u_{pn+q} - u_{pn-p+q} \quad (n \ge n_4),$$

$$\text{(c)}\quad \left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{(-1)^k}{u_{pk+q}} \right)^{-1} \right\| = (-1)^n (u_{pn+q} + u_{pn-p+q}) \quad (n \ge n_5).$$

For $\beta \to +\infty$, we deduce the following identity of infinite sum as a special case of Theorem 2.

Corollary 2 *Let $\{u_n\}$ be an mth-order sequence defined by (1) with the restriction $a_1 \ge a_2 \ge \cdots \ge a_m \ge 1$. Let p and q be positive integers with $0 \le q < p$. Then there exist positive integers n_6, n_7 and n_8 depending on $a_1, a_2, \dots,$ and a_m such that*

$$\text{(e)}\quad \left\| \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{u_k} \right)^{-1} \right\| = (-1)^n (u_n + u_{n-1}) \quad (n \ge n_6),$$

$$\text{(f)}\quad \left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_{pk+q}} \right)^{-1} \right\| = u_{pn+q} - u_{pn-p+q} \quad (n \ge n_7),$$

$$\text{(g)}\quad \left\| \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{u_{pk+q}} \right)^{-1} \right\| = (-1)^n (u_{pn+q} + u_{pn-p+q}) \quad (n \ge n_8).$$

Proof We shall prove only (c) in Theorem 2 and other identities are proved similarly. From Lemma 2 we have

$$\frac{(-1)^k}{u_{pk+q}} = \frac{(-1)^k}{c\alpha^{pk+q} + \mathcal{O}(d^{-pk-q})} = \frac{(-1)^k}{c\alpha^{pk+q}} \bigl(1 + \mathcal{O}\bigl((\alpha d)^{-pk-q}\bigr)\bigr).$$

Thus

$$\begin{aligned}
\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{(-1)^k}{u_{pk+q}} &= \frac{(-1)^n \alpha^p}{c\alpha^{pn+q}(\alpha^p + 1)} + \frac{(-1)^n \alpha^p}{c\alpha^{p\lfloor \beta n \rfloor + q}(\alpha^p + 1)} + \mathcal{O}\bigl(\bigl(\alpha^2 d\bigr)^{-pn-q}\bigr) \\
&= \frac{(-1)^n \alpha^p}{c\alpha^{pn+q}(\alpha^p + 1)} + \mathcal{O}\bigl(\alpha^{-p\lfloor \beta n \rfloor - q}\bigr) + \mathcal{O}\bigl(\alpha^{-2pn-2q} d^{-pn-q}\bigr) \\
&= \frac{(-1)^n \alpha^p}{c\alpha^{pn+q}(\alpha^p + 1)} + \mathcal{O}\bigl(\alpha^{-2pn} \alpha^{-p\lfloor \beta n \rfloor + 2pn}\bigr) + \mathcal{O}\bigl(\alpha^{-2pn} d^{-pn}\bigr) \\
&= \frac{(-1)^n \alpha^p}{c\alpha^{pn+q}(\alpha^p + 1)} + \mathcal{O}\bigl(\alpha^{-2pn} h^p\bigr),
\end{aligned}$$

where $h = \max\{\alpha^{-\lfloor \beta n \rfloor + 2n}, d^{-n}\}$.

Taking reciprocal, we get

$$\begin{aligned}\left(\sum_{k=n}^{\lfloor\beta n\rfloor}\frac{(-1)^k}{u_{pk+q}}\right)^{-1} &= (-1)^n\left(c\alpha^{pn+q}+c\alpha^{pn-p+q}\right)\left(1+\mathcal{O}\left(\alpha^{-pn}h^p\right)\right)\\ &= (-1)^n(u_{pn+q}+u_{pn-p+q})+\mathcal{O}\left(h^p\right).\end{aligned}$$

Since $h=\max\{\alpha^{-\lfloor\beta n\rfloor+2n},d^{-n}\}<1$, there exists $n\geq n_5$ sufficiently large so that the modulus of the last error term becomes less than 1/2, which completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ZW obtained the theorems and completed the proof. HZ corrected and improved the final version. Both authors read and approved the final manuscript.

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