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Research Article

On the Higher Power Sums of Reciprocal Higher-Order Sequences

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Let $\{u_n\}$ be a higher-order linear recursive sequence. In this paper, we use the properties of error estimation and the analytic method to study the reciprocal sums of higher power of higher-order sequences. Then we establish several new and interesting identities relating to the infinite and finite sums.

1. Introduction

The so-called Fibonacci zeta function and Lucas zeta function, defined by

$$\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s}, \qquad \zeta_L(s) = \sum_{n=1}^{\infty} \frac{1}{L_n^s}, \tag{1}$$

where the F_n and L_n denote the Fibonacci numbers and Lucas numbers, have been considered in several different ways; see [1, 2]. Ohtsuka and Nakamura [3] studied the partial infinite sums of reciprocal Fibonacci numbers and proved the following conclusions:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \text{ is even}, n \geq 2, \\ F_{n-2} - 1 & \text{if } n \text{ is odd}, n \geq 1, \end{cases}$$
$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1 & \text{if } n \text{ is even}, n \geq 2, \\ F_{n-1}F_n & \text{if } n \text{ is odd}, n \geq 1, \end{cases} \tag{2}$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

Further, Wu and Zhang [4, 5] generalized these identities to the Fibonacci polynomials and Lucas polynomials. Various properties of the Fibonacci polynomials and Lucas polynomials have been studied by many authors; see [6–13].

Recently, some authors considered the nearest integer of the sums of reciprocal Fibonacci numbers and other well-known sequences and obtained several meaningful results; see [14–16]. In particular, in [16], Kılıç and Arıkan studied a problem which is a little different from that of [3], namely, that of determining the nearest integer to $\left(\sum_{k=n}^{\infty}(1/v_k)\right)^{-1}$. Specifically, suppose that $\|x\| = \lfloor x + (1/2) \rfloor$ (the nearest integer function) and $\{v_n\}_{n\geq 0}$ is an integer sequence satisfying the recurrence formula

$$v_n = pv_{n-1} + qv_{n-2} + v_{n-3} + \cdots + v_{n-k}, \tag{3}$$

for any positive integer $p \geq q$ and $n > k$. Then we can conclude that there exists a positive integer n_0 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{v_k} \right)^{-1} \right\| = v_n - v_{n-1}, \tag{4}$$

for all $n > n_0$.

In [17], Wu and Zhang unified the above results by proving the following conclusion that includes all the results, [3–8, 15, 16], as special cases.

Proposition 1. *For any positive integer $n > m$, the mth-order linear recursive sequence $\{u_n\}$ is defined as follows:*

$$u_n = a_1u_{n-1} + a_2u_{n-2} + \cdots + a_{m-1}u_{n-m+1} + a_mu_{n-m}, \tag{5}$$

with initial values $u_i \in \mathbb{N}$ for $0 \leq i < m$ and at least one of them not being zero. For any positive real number $\beta > 2$ and any

positive integer $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$, there exists a positive integer n_1 such that

$$\left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1}, \quad (n \geq n_1). \tag{6}$$

In particular, taking $\beta \to +\infty$, there exists a positive integer n_2 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1}, \quad (n \geq n_2). \tag{7}$$

It seems difficult to deal with $\left(\sum_{k=n}^{\infty}(1/u_k^s)\right)^{-1}$ for all integers $s \geq 2$, because it is quite unclear *a priori* what the shape of the result might be. In [18], Xu and Wang applied the method of undetermined coefficients and constructed a number of delicate inequalities in order to study the infinite sum of the cubes of reciprocal Pell numbers and then obtained the following meaningful result.

Proposition 2. *For any positive integer $n \geq 1$, we have the identity*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k^3} \right)^{-1} \right\rfloor = \begin{cases} P_n^2 P_{n-1} + 3P_n P_{n-1}^2 + \left\lfloor -\frac{61}{82}P_n - \frac{91}{82}P_{n-1} \right\rfloor \\ \qquad \text{if } n \text{ is even}, n \geq 2, \\ P_n^2 P_{n-1} + 3P_n P_{n-1}^2 + \left\lfloor \frac{61}{82}P_n + \frac{91}{82}P_{n-1} \right\rfloor \\ \qquad \text{if } n \text{ is odd}, n \geq 1. \end{cases} \tag{8}$$

To find and prove this result is a substantial achievement since such a complex formula would not be clear beforehand that a result would even be possible. However, there is no research considering the higher power ($s > 2$) of reciprocal sums of some recursive sequences. The main purpose of this paper is using the properties of error estimation and the analytic method to study the higher power of the reciprocal sums of $\{u_n\}$ and obtain several new and interesting identities. The results are as follows.

Theorem 3. *Let $\{u_n\}$ be an mth-order sequence defined by* (5) *with the restrictions $a_1, a_2, \ldots, a_m \in \mathbb{N}$ and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 2$. For any real number $\beta > 2$ and positive integer $1 \leq s < \lfloor \log_{(\alpha/a_1)} \alpha d \rfloor$, where $\alpha, \alpha_1, \ldots, \alpha_{m-1}$ are the roots of the characteristic equation of u_n and $d^{-1} = \max\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_{m-1}|\}$, then there exists a positive integer n_3 such that*

$$\left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{a_1^{sk}}{u_k^s} \right)^{-1} - \left(\frac{u_n^s}{a_1^{sn}} - \frac{u_{n-1}^s}{a_1^{sn-s}} \right) \right\| = 0, \quad (n \geq n_3). \tag{9}$$

Taking $\beta \to +\infty$, from Theorem 3 we may immediately deduce the following.

Corollary 4. *Let $\{u_n\}$ be an mth-order sequence defined by* (5) *with the restrictions $a_1, a_2, \ldots, a_m \in \mathbb{N}$ and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 2$. For positive integer $1 \leq s < \lfloor \log_{(\alpha/a_1)} \alpha d \rfloor$, where $\alpha, \alpha_1, \ldots, \alpha_{m-1}$ are the roots of the characteristic equation of u_n and $d^{-1} = \max\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_{m-1}|\}$, then there exists a positive integer n_4 such that*

$$\left\| \left(\sum_{k=n}^{\infty} \frac{a_1^{sk}}{u_k^s} \right)^{-1} - \left(\frac{u_n^s}{a_1^{sn}} - \frac{u_{n-1}^s}{a_1^{sn-s}} \right) \right\| = 0, \quad (n \geq n_4). \tag{10}$$

For positive real number $1 < \beta \leq 2$, whether there exits an identity for

$$\left\| \left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{a_1^{sk}}{u_k^s} \right)^{-1} \right\| \tag{11}$$

is an interesting open problem.

2. Several Lemmas

To complete the proof of our theorem, we need two lemmas.

Lemma 5. *Let $a_1, a_2, \ldots, a_m \in \mathbb{N}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ and $m \in \mathbb{N}$ with $m \geq 2$. Then for the polynomial*

$$f(x) = x^m - a_1 x^{m-1} - a_2 x^{m-2} - \cdots - a_{m-1}x - a_m, \tag{12}$$

we have the following:

(I) polynomial $f(x)$ has exactly one positive real zero α with $a_1 < \alpha < a_1 + 1$;

(II) other $m-1$ zeros of $f(x)$ lie within the unit circle in the complex plane.

Proof. See Lemma 1 of [16]. □

Lemma 6. *Let $m \geq 2$ and $\{u_n\}_{n \geq 0}$ be an integer sequence satisfying the recurrence formula* (5). *Then for any positive integer s, we have*

$$u_n^s = c^s \alpha^{sn} + O\left(\alpha^{sn-n} d^{-n}\right) \quad (n \longrightarrow \infty), \tag{13}$$

where $c > 0$, $d > 1$, and $a_1 < \alpha < a_1 + 1$ is the positive real zero of $f(x)$.

Proof. From Lemma 2 of [16], the closed formula of u_n is given by

$$u_n = c\alpha^n + O\left(d^{-n}\right) \quad (n \longrightarrow \infty), \tag{14}$$

where $c > 0$, $d > 1$, $a_1 < \alpha < a_1 + 1$, and α is the positive real zero of $f(x)$. Now we prove Lemma 6 by mathematical induction. From formula (14), we have

$$u_n^2 = c^2\alpha^{2n} + O\left(d^{-2n}\right) + O\left(\alpha^n d^{-n}\right) = c^2\alpha^{2n} + O\left(\alpha^n d^{-n}\right). \tag{15}$$

That is, the lemma holds for $s = 2$. Suppose that for all integers $2 \leq s \leq k$ we have

$$u_n^s = c^s \alpha^{sn} + O\left(\alpha^{sn-n} d^{-n}\right) \quad (n \longrightarrow \infty). \tag{16}$$

Then for $s = k + 1$ we have

$$\begin{aligned} u_n^{k+1} &= \left(c^k\alpha^{kn} + O\left(\alpha^{kn-n}d^{-n}\right)\right)\cdot\left(c\alpha^n + O\left(d^{-n}\right)\right) \\ &= c^{k+1}\alpha^{kn+n} + O\left(\alpha^{kn}d^{-n}\right) \\ &\quad + O\left(\alpha^{kn}d^{-n}\right) + O\left(\alpha^{kn-n}d^{-2n}\right) \\ &= c^{k+1}\alpha^{(k+1)n} + O\left(\alpha^{kn}d^{-n}\right). \end{aligned} \tag{17}$$

That is, Lemma 6 also holds for $s = k + 1$. This completes the proof of Lemma 6 by mathematical induction. □

3. Proof of Theorem 3

In this section, we shall complete the proof of Theorem 3. From the geometric series as $\epsilon \to 0$, we have

$$\frac{1}{1 \pm \epsilon} = 1 \mp \epsilon + O\left(\epsilon^2\right) = 1 + O\left(\epsilon\right). \tag{18}$$

Using Lemma 6, we have

$$\begin{aligned} \frac{a_1^{sk}}{u_k^s} &= \frac{a_1^{sk}}{c^s\alpha^{sk} + O\left(\alpha^{sk-k}d^{-k}\right)} \\ &= \frac{a_1^{sk}}{c^s\alpha^{sk}\left(1 + O\left(\alpha^{-k}d^{-k}\right)\right)} = \frac{a_1^{sk}}{c^s\alpha^{sk}}\left(1 + O\left(\alpha^{-k}d^{-k}\right)\right) \\ &= \frac{a_1^{sk}}{c^s\alpha^{sk}} + O\left(\frac{a_1^{sk}}{\alpha^{sk+k}d^k}\right). \end{aligned} \tag{19}$$

Consequently,

$$\begin{aligned} \sum_{k=n}^{\lfloor\beta n\rfloor}\frac{a_1^{sk}}{u_k^s} &= \frac{1}{c^s}\sum_{k=n}^{\lfloor\beta n\rfloor}\left(\frac{a_1}{\alpha}\right)^{sk} + O\left(\sum_{k=n}^{\lfloor\beta n\rfloor}\frac{a_1^{sk}}{\alpha^{sk+k}d^k}\right) \\ &= \frac{\alpha^s}{c^s\left(\alpha^s - a_1^s\right)}\cdot\left(\frac{a_1}{\alpha}\right)^{sn} - \frac{1}{c^s\left(\alpha^s - a_1^s\right)}\cdot\left(\frac{a_1}{\alpha}\right)^{s\lfloor\beta n\rfloor} \\ &\quad + O\left(\frac{a_1^{sn}}{\alpha^{sn+n}d^n}\right) \\ &= \frac{\alpha^s}{c^s\left(\alpha^s - a_1^s\right)}\cdot\left(\frac{a_1}{\alpha}\right)^{sn} + O\left(\frac{a_1^{sn}}{\alpha^{sn}}\cdot\frac{a_1^{s\lfloor\beta n\rfloor - sn}}{\alpha^{s\lfloor\beta n\rfloor - sn}}\right) \\ &\quad + O\left(\frac{a_1^{sn}}{\alpha^{sn}}\cdot\frac{1}{\alpha^n d^n}\right) \\ &= \frac{\alpha^s}{c^s\left(\alpha^s - a_1^s\right)}\cdot\left(\frac{a_1}{\alpha}\right)^{sn} + O\left(\frac{a_1^{sn}}{\alpha^{sn}}\cdot h\right), \end{aligned} \tag{20}$$

where $h = \max\{(a_1^{s\lfloor\beta n\rfloor - sn}/\alpha^{s\lfloor\beta n\rfloor - sn}), (1/\alpha^n d^n)\}$.

Taking the reciprocal of this expression yields

$$\begin{aligned} &\left(\sum_{k=n}^{\lfloor\beta n\rfloor}\frac{a_1^{sk}}{u_k^s}\right)^{-1} \\ &= (1)\times\left(\frac{\alpha^s}{c^s\left(\alpha^s - a_1^s\right)}\cdot\left(\frac{a_1}{\alpha}\right)^{sn}\right. \\ &\qquad \left.\cdot\left(1 + O\left(\frac{a_1^{sn}h}{\alpha^{sn}}\cdot\frac{\alpha^{sn}}{a_1^{sn}}\right)\right)\right)^{-1} \\ &= \frac{c^s\left(\alpha^s - a_1^s\right)}{\alpha^s}\cdot\left(\frac{\alpha}{a_1}\right)^{sn}\cdot\left(1 + O\left(h\right)\right) \\ &= \frac{c^s\alpha^{sn}}{a_1^{sn}} - \frac{c^s\alpha^{sn-s}}{a_1^{sn-s}} + O\left(\frac{\alpha^{sn}}{a_1^{sn}}\cdot h\right) \\ &= \frac{u_n^s}{a_1^{sn}} - \frac{u_{n-1}^s}{a_1^{sn-s}} + O\left(\frac{\alpha^{sn}}{a_1^{sn}}\cdot h\right). \end{aligned} \tag{21}$$

Case 1. If $h = (a_1^{s\lfloor\beta n\rfloor - sn}/\alpha^{s\lfloor\beta n\rfloor - sn})$, then for any real number $\beta > 2$ and positive integer s we have

$$\frac{\alpha^{sn}}{a_1^{sn}}\cdot h = \frac{\alpha^{sn}}{a_1^{sn}}\cdot\frac{a_1^{s\lfloor\beta n\rfloor - sn}}{\alpha^{s\lfloor\beta n\rfloor - sn}} = \left(\frac{a_1}{\alpha}\right)^{s\lfloor\beta n\rfloor - 2sn} < 1. \tag{22}$$

Case 2. If $h = (1/\alpha^n d^n)$, for any positive integer $a_1 \geq 2$, $1 < (\alpha/a_1) < \alpha d$ holds. Then for any positive integer s with

$$1 \leq s < \left\lfloor \log_{(\alpha/a_1)}\alpha d\right\rfloor, \tag{23}$$

we have

$$\frac{\alpha^{sn}}{a_1^{sn}}\cdot h = \frac{\alpha^{sn-n}}{a_1^{sn}d^n} = \left(\frac{\alpha^{s-1}}{a_1^s d}\right)^n < 1. \tag{24}$$

In both cases, it follows that for any real number $\beta > 2$ and positive integer $1 \leq s < \lfloor\log_{(\alpha/a_1)}\alpha d\rfloor$ there exists $n \geq n_3$ sufficiently large so that the modulus of the last error term of identity (21) becomes less than 1/2. This completes the proof of Theorem 3.

Proof of Corollary 4. From identity (19), we have

$$\frac{a_1^{sk}}{u_k^s} = \frac{a_1^{sk}}{c^s\alpha^{sk}} + O\left(\frac{a_1^{sk}}{\alpha^{sk+k}d^k}\right). \tag{25}$$

Consequently,

$$\begin{aligned} \sum_{k=n}^{\infty}\frac{a_1^{sk}}{u_k^s} &= \frac{1}{c^s}\sum_{k=n}^{\infty}\left(\frac{a_1}{\alpha}\right)^{sk} + O\left(\sum_{k=n}^{\infty}\frac{a_1^{sk}}{\alpha^{sk+k}d^k}\right) \\ &= \frac{\alpha^s}{c^s\left(\alpha^s - a_1^s\right)}\cdot\left(\frac{a_1}{\alpha}\right)^{sn} + O\left(\frac{a_1^{sn}}{\alpha^{sn+n}d^n}\right). \end{aligned} \tag{26}$$

Taking the reciprocal of this expression yields

$$\begin{aligned}
&\left(\sum_{k=n}^{\infty}\frac{a_1^{sk}}{u_k^s}\right)^{-1}\\
&=\frac{1}{\left(\alpha^s/c^s\left(\alpha^s-a_1^s\right)\right)\cdot\left(a_1/\alpha\right)^{sn}\cdot\left(1+O\left(1/\alpha^n d^n\right)\right)}\\
&=\frac{c^s\left(\alpha^s-a_1^s\right)}{\alpha^s}\cdot\left(\frac{\alpha}{a_1}\right)^{sn}\cdot\left(1+O\left(\frac{1}{\alpha^n d^n}\right)\right)\\
&=\frac{c^s\alpha^{sn}}{a_1^{sn}}-\frac{c^s\alpha^{sn-s}}{a_1^{sn-s}}+O\left(\frac{\alpha^{sn-n}}{a_1^{sn}d^n}\right)\\
&=\frac{u_n^s}{a_1^{sn}}-\frac{u_{n-1}^s}{a_1^{sn-s}}+O\left(\frac{\alpha^{sn-n}}{a_1^{sn}d^n}\right).
\end{aligned}\tag{27}$$

For any positive integer s with

$$1\le s<\left\lfloor\log_{(\alpha/a_1)}\alpha d\right\rfloor,\tag{28}$$

we have

$$\frac{\alpha^{sn-n}}{a_1^{sn}d^n}=\left(\frac{\alpha^{s-1}}{a_1^s d}\right)^n<1.\tag{29}$$

So there exists $n\ge n_4$ sufficiently large so that the modulus of the last error term of identity (27) becomes less than $1/2$. This completes the proof of Corollary 4. □

4. Computation

We can determine the power s of different sequence u_n by MATHEMATICA as the following examples.

Example 7. Let u_n be the second-order linear recursive sequence (see Table 1).

Example 8. Let u_n be the third-order linear recursive sequence (see Table 2).

Example 9. Let u_n be the fifth-order linear recursive sequence (see Table 3).

Therefore, we may immediately deduce the following corollaries.

Corollary 10. *Let $P_n=2P_{n-1}+P_{n-2}$ be the Pell numbers. For any real number $\beta>2$ and positive integer $1\le s<9$, there exists a positive integer n_5 such that*

$$\left\|\left(\sum_{k=n}^{\lfloor\beta n\rfloor}\frac{2^{sk}}{P_k^s}\right)^{-1}-\left(\frac{P_n^s}{2^{sn}}-\frac{P_{n-1}^s}{2^{sn-s}}\right)\right\|=0,\quad\left(n\ge n_5\right).\tag{30}$$

Corollary 11. *Let $T_n=4T_{n-1}+3T_{n-2}+2T_{n-3}$ be the generalized Tribonacci numbers. For any real number $\beta>2$ and positive integer $1\le s<11$, there exists a positive integer n_6 such that*

$$\left\|\left(\sum_{k=n}^{\lfloor\beta n\rfloor}\frac{4^{sk}}{T_k^s}\right)^{-1}-\left(\frac{T_n^s}{4^{sn}}-\frac{T_{n-1}^s}{4^{sn-s}}\right)\right\|=0,\quad\left(n\ge n_6\right).\tag{31}$$

Table 1

u_n	α	d	$\log_{\alpha/a_1}\alpha d$
$u_n=2u_{n-1}+u_{n-2}$	2.4142	2.4142	9.3653
$u_n=3u_{n-1}+u_{n-2}$	3.3028	3.3028	24.8500
$u_n=4u_{n-1}+u_{n-2}$	4.2361	4.2361	50.3460
$u_n=5u_{n-1}+u_{n-2}$	5.1926	5.1926	87.1630

Table 2

u_n	α	d	$\log_{\alpha/a_1}\alpha d$
$u_n=2u_{n-1}+u_{n-2}+u_{n-3}$	2.5468	1.5959	5.8020
$u_n=3u_{n-1}+2u_{n-2}+u_{n-3}$	3.6274	1.9044	10.1772
$u_n=4u_{n-1}+3u_{n-2}+2u_{n-3}$	4.7246	1.5370	11.9086
$u_n=5u_{n-1}+4u_{n-2}+3u_{n-3}$	5.8074	1.2050	12.9970

Table 3

u_n	α	d	$\log_{\alpha/a_1}\alpha d$
$u_n=2u_{n-1}+u_{n-2}+u_{n-3}+u_{n-4}+u_{n-5}$	2.6083	1.1855	4.2511
$u_n=3u_{n-1}+3u_{n-2}+2u_{n-3}+2u_{n-4}+u_{n-5}$	3.9300	1.3189	6.0936
$u_n=4u_{n-1}+3u_{n-2}+u_{n-3}+u_{n-4}+u_{n-5}$	4.6959	1.4156	11.8100
$u_n=5u_{n-1}+3u_{n-2}+2u_{n-3}+u_{n-4}+u_{n-5}$	5.6055	1.4828	18.5257

Corollary 12. *Let $u_n=5u_{n-1}+3u_{n-2}+2u_{n-3}+u_{n-4}+u_{n-5}$ be a fifth-order sequence. For any real number $\beta>2$ and positive integer $1\le s<18$, there exists a positive integer n_7 such that*

$$\left\|\left(\sum_{k=n}^{\lfloor\beta n\rfloor}\frac{5^{sk}}{u_k^s}\right)^{-1}-\left(\frac{u_n^s}{5^{sn}}-\frac{u_{n-1}^s}{5^{sn-s}}\right)\right\|=0,\quad\left(n\ge n_7\right).\tag{32}$$

5. Related Results

The following results are obtained similarly.

Theorem 13. *Let $\{u_n\}$ be an mth-order sequence defined by (5) with the restrictions $a_1,a_2,\ldots,a_m\in\mathbb{N}$ and $a_1\ge a_2\ge\cdots\ge a_m\ge 2$. Let p and q be positive integers with $0\le q<p$. For any real number $\beta>2$ and positive integer $1\le s<\lfloor\log_{(\alpha/a_1)}\alpha^p d^p\rfloor$, where $\alpha,\alpha_1,\ldots,\alpha_{m-1}$ are the roots of the characteristic equation of u_n and $d^{-1}=\max\{|\alpha_1|,|\alpha_2|,\ldots,|\alpha_{m-1}|\}$, then there exist positive integers n_8, n_9, and n_{10} depending on $a_1,a_2,\ldots$, and a_m such that the following hold.*

(a) $\left\|\left(\sum_{k=n}^{\lfloor\beta n\rfloor}\left((-a_1)^{sk}/u_k^s\right)\right)^{-1}-(-1)^{sn}\left(u_n^s/a_1^{sn}+u_{n-1}^s/a_1^{sn-s}\right)\right\|=0,\ (n\ge n_8)$.

(b) $\left\|\left(\sum_{k=n}^{\lfloor\beta n\rfloor}\left(a_1^{spk+sq}/u_{pk+q}^s\right)\right)^{-1}-\left(u_{pn+q}^s/a_1^{spn+sq}-u_{pn-p+q}^s/a_1^{spn+sq-sp}\right)\right\|=0,\ (n\ge n_9)$.

(c) $\left\|\left(\sum_{k=n}^{\lfloor\beta n\rfloor}\left((-a_1)^{spk+sq}/u_{pk+q}^s\right)\right)^{-1}-(-1)^{spn+sq}\left(u_{pn+q}^s/a_1^{spn+sq}+u_{pn-p+q}^s/a_1^{spn+sq-sp}\right)\right\|=0,\ (n\ge n_{10})$.

For $\beta\to+\infty$, we deduce the following identity of infinite sum as special case of Theorem 13.

Corollary 14. *Let $\{u_n\}$ be an mth-order sequence defined by (5) with the restrictions $a_1, a_2, \ldots, a_m \in \mathbb{N}$ and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 2$. Let p and q be positive integers with $0 \leq q < p$. For any positive integer $1 \leq s < \lfloor \log_{(\alpha/a_1)} \alpha d \rfloor$, where $\alpha, \alpha_1, \ldots, \alpha_{m-1}$ are the roots of the characteristic equation of u_n and $d^{-1} = \max\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_{m-1}|\}$, then there exist positive integers n_{11}, n_{12}, and n_{13} depending on a_1, a_2, …, and a_m such that the following hold.*

(d) $\| (\sum_{k=n}^{\infty} ((-a_1)^{sk}/u_k^s))^{-1} - (-1)^{sn}(u_n^s/a_1^{sn} + u_{n-1}^s/a_1^{sn-s}) \| = 0, (n \geq n_{11})$.

(e) $\| (\sum_{k=n}^{\infty} (a_1^{spk+sq}/u_{pk+q}^s))^{-1} - (u_{pn+q}^s/a_1^{spn+sq} - u_{pn-p+q}^s/a_1^{spn+sq-sp}) \| = 0, (n \geq n_{12})$.

(f) $\| (\sum_{k=n}^{\infty} ((-a_1)^{spk+sq}/u_{pk+q}^s))^{-1} - (-1)^{spn+sq}(u_{pn+q}^s/a_1^{spn+sq} + u_{pn-p+q}^s/a_1^{spn+sq-sp}) \| = 0, (n \geq n_{13})$.

Proof. We shall prove only (c) in Theorem 13 and other identities are proved similarly. From identity (19), we have

$$\frac{(-a_1)^{spk+sq}}{u_{pk+q}^s} = \frac{(-a_1)^{spk+sq}}{c^s \alpha^{spk+sq}} \left(1 + O\left(\alpha^{-pk-q} d^{-pk-q}\right)\right). \quad (33)$$

Consequently,

$$\begin{aligned}
&\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{(-a_1)^{spk+sq}}{u_{spk+sq}^s} \\
&= \frac{1}{c^s} \sum_{k=n}^{\lfloor \beta n \rfloor} \left(\frac{-a_1}{\alpha}\right)^{spk+sq} + O\left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{a_1^{spk+sq}}{\alpha^{spk+sq+pk+q} d^{pk+q}}\right) \\
&= \frac{(-1)^{spn+sq} \cdot \alpha^s}{c^s (\alpha^s - a_1^s)} \cdot \left(\frac{a_1}{\alpha}\right)^{spn+sq} - \frac{(-1)^{spn+sq}}{c^s (\alpha^s - a_1^s)} \\
&\quad \cdot \left(\frac{a_1}{\alpha}\right)^{sp\lfloor \beta n \rfloor + sq} + O\left(\frac{a_1^{spn}}{\alpha^{spn+pn} d^{pn}}\right) \\
&= \frac{(-1)^{spn+sq} \cdot \alpha^s}{c^s (\alpha^s - a_1^s)} \cdot \left(\frac{a_1}{\alpha}\right)^{spn+sq} + O\left(\frac{a_1^{spn}}{\alpha^{spn}} \cdot \frac{a_1^{sp\lfloor \beta n \rfloor - spn}}{\alpha^{sp\lfloor \beta n \rfloor - spn}}\right) \\
&\quad + O\left(\frac{a_1^{spn}}{\alpha^{spn}} \cdot \frac{1}{\alpha^{pn} d^{pn}}\right) \\
&= \frac{(-1)^{spn+sq} \cdot \alpha^s}{c^s (\alpha^s - a_1^s)} \cdot \left(\frac{a_1}{\alpha}\right)^{spn+sq} + O\left(\frac{a_1^{spn}}{\alpha^{spn}} \cdot h^p\right),
\end{aligned} \quad (34)$$

where $h = \max\{(a_1^{s\lfloor \beta n \rfloor - sn}/\alpha^{s\lfloor \beta n \rfloor - sn}), (1/\alpha^n d^n)\}$.

Taking the reciprocal of this expression yields

$$\begin{aligned}
&\left(\sum_{k=n}^{\lfloor \beta n \rfloor} \frac{(-a_1)^{spk+sq}}{u_{pk+q}^s}\right)^{-1} \\
&= (-1)^{spn+sq} \cdot \frac{c^s (\alpha^s - a_1^s)}{\alpha^s} \cdot \left(\frac{\alpha}{a_1}\right)^{spn+sq} \cdot \left(1 + O\left(h^p\right)\right) \\
&= \frac{u_{pn+q}^s}{a_1^{spn+sq}} + \frac{u_{pn-p+q}^s}{a_1^{spn+sq-sp}} + O\left(\frac{\alpha^{spn}}{a_1^{spn}} \cdot h^p\right).
\end{aligned} \quad (35)$$

Case 1. If $h = (a_1^{s\lfloor \beta n \rfloor - sn})/(\alpha^{s\lfloor \beta n \rfloor - sn})$, then for any real number $\beta > 2$ and positive integer s we have

$$\frac{\alpha^{sn}}{a_1^{sn}} \cdot h^p = \frac{\alpha^{sn}}{a_1^{sn}} \cdot \frac{a_1^{sp\lfloor \beta n \rfloor - spn}}{\alpha^{sp\lfloor \beta n \rfloor - spn}} = \left(\frac{a_1}{\alpha}\right)^{sp\lfloor \beta n \rfloor - spn - sn} < 1. \quad (36)$$

Case 2. If $h = (1/\alpha^n d^n)$, for any positive integer $a_1 \geq 2$, $1 < (\alpha/a_1) < \alpha d$ holds. Then for any positive integer s with

$$1 \leq s < \left\lfloor \log_{(\alpha/a_1)} \alpha^p d^p \right\rfloor, \quad (37)$$

we have

$$\frac{\alpha^{sn}}{a_1^{sn}} \cdot h^p = \frac{\alpha^{sn-pn}}{a_1^{sn} d^{pn}} = \left(\frac{\alpha^{s-p}}{a_1^s d^p}\right)^n < 1. \quad (38)$$

In both cases, it follows that for any real number $\beta > 2$ and positive integer $1 \leq s < \lfloor \log_{(\alpha/a_1)} \alpha^p d^p \rfloor$, there exists $n \geq n_{10}$ sufficiently large so that the modulus of the last error term of identity (35) becomes less than $1/2$. This completes the proof of Theorem 13(c). □

Conflict of Interests

The authors declare that there is no conflict interests regarding the publication of this paper.

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