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Research Article

Combinatorial Interpretation of General Eulerian Numbers

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Since the 1950s, mathematicians have successfully interpreted the traditional Eulerian numbers and q-Eulerian numbers combinatorially. In this paper, the authors give a combinatorial interpretation to the general Eulerian numbers defined on general arithmetic progressions $\{a, a+d, a+2d, \ldots\}$.

1. Introduction

Definition 1. Given a positive integer n, define Ω_n as the set of all permutations of $[n] = \{1, 2, 3, \ldots, n\}$. For a permutation $\pi = p_1 p_2 p_3 \ldots p_n \in \Omega_n$, i is called an ascent of π if $p_i < p_{i+1}$; i is called a weak exceedance of π if $p_i \geq i$.

It is well known that a traditional Eulerian number $A_{n,k}$ is the number of permutations $\pi \in \Omega_n$ that have k weak exceedances [1, page 215]. And $A_{n,k}$ satisfies the recurrence: $A_{n,1} = 1$, $(n \geq 1)$, $A_{n,k} = 0$ $(k > n)$,

$$A_{n,k} = kA_{n-1,k} + (n+1-k) A_{n-1,k-1} \quad (1 \leq k \leq n) \tag{1}$$

Besides the recursive formula (1), $A_{n,k}$ can be calculated directly by the following analytic formula [2, page 8]:

$$A_{n,k} = \sum_{i=0}^{k-1} (-1)^i (k-i)^n \binom{n+1}{i} \quad (1 \leq k \leq n). \tag{2}$$

Definition 2. Given a permutation $\pi = p_1 p_2 p_3 \cdots p_n \in \Omega_n$, define functions

$$\operatorname{maj} \pi = \sum_{p_j > p_{j+1}} j,$$
$$a(n,k,i) = \#\{\pi \mid \operatorname{maj} \pi = i \ \& \ \pi \text{ has } k \text{ ascents}\}. \tag{3}$$

Since the 1950s, Carlitz [3, 4] and his successors have generalized Euler's results to q-sequences $\{1, q, q^2, q^3, \ldots\}$. Under Carlitz's definition, the q-Eulerian numbers $A_{n,k}(q)$ are given by

$$A_{n,k}(q) = q^{(m-k+1)(m-k)/2} \sum_{i=0}^{k(n-k-1)} a(n, n-k, i) q^i, \tag{4}$$

where functions $a(n,k,i)$ are as defined in Definition 2.

In [5], instead of studying q-sequences, the authors have generalized Eulerian numbers to any general arithmetic progression

$$\{a, \ a+d, \ a+2d, \ a+3d, \ldots\}. \tag{5}$$

Under the new definition, and given an arithmetic progression as defined in (5), the general Eulerian numbers $A_{n,k}(a,d)$ can be calculated directly by the following equation [5, Lemma 2.6]:

$$A_{n,k}(a,d) = \sum_{i=0}^{k} (-1)^i [(k+1-i)d - a]^n \binom{n+1}{i}. \tag{6}$$

Interested readers can find more results about the general Eulerian numbers and even general Eulerian polynomials in [5].

2. Combinatorial Interpretation of General Eulerian Numbers

The following concepts and properties will be heavily used in this section.

Definition 3. Let $W_{n,k}$ be the set of n-permutations with k weak exceedances. Then $|W_{n,k}| = A_{n,k}$. Furthermore, given a permutation $\pi = p_1p_2p_3 \ldots p_n$, let $Q_n(\pi) = i$, where $p_i = n$.

Given a permutation $\pi \in \Omega_n$, it is known that π can be written as a one-line form like $\pi = p_1p_2p_3 \ldots p_n$, or π can be written in a disjoint union of distinct cycles. For π written in a cycle form, we can use a *standard representation* by writing (a) each cycle starting with its largest element and (b) the cycles in increasing order of their largest element. Moreover, given a permutation π written in a standard representation cycle form, define a function f as $f(\pi)$ to be the permutation obtained from π by erasing the parentheses. Then f is known as the *fundamental bijection* from Ω_n to itself [6, page 30]. Indeed, the inverse map f^{-1} of the fundamental bijection function f is also famous in illustrating the relation between the ascents and weak exceedances as follows [2, page 98].

Proposition 4. *The function f^{-1} gives a bijection between the set of permutations on $[n]$ with k ascents and the set $W_{n,k+1}$.*

Example 5. The standard representation of permutation $\pi = 5243716$ is $(2)(43)(7615) \in \Omega_7$, and $f(\pi) = 2437615$; $Q_7(\pi) = 5$; $\pi = 5243716$ has 3 ascents, while $f^{-1}(\pi) = (5243)(716) = 6453271 \in W_{7,4}$ has $3 + 1 = 4$ weak excedances because $p_1 = 6 > 1$, $p_2 = 4 > 2$, $p_3 = 5 > 3$, and $p_6 = 7 > 6$.

Now suppose we want to construct a sequence consisting of k vertical bars and the first n positive integers. Then the k vertical bars divide these n numbers into $k+1$ compartments. In each compartment, there is either no number or all the numbers are listed in a decreasing order. The following definition is analogous to the definition of [2, page 8].

Definition 6. A bar in the above construction is called extraneous if either

(a) it is immediately followed by another bar; or

(b) each of the rest compartment is either empty or consists of integers in a decreasing order if this bar is removed.

Example 7. Suppose $n = 7$, $k = 4$; then in the following arrangement

$$32\,|1|\,|7654| \tag{7}$$

the 1st, 2nd, and 4th bars are extraneous.

Now we are ready to give combinatorial interpretations to the general Eulerian numbers $A_{n,k}(a,d)$. First note that (6) implies that $A_{n,k}(a,d)$ is a homogeneous polynomial of degree n with respect to a and d. Indeed,

$$\begin{aligned}
&A_{n,k}(a,d)\\
&= \sum_{i=0}^{k}(-1)^i[(k+1-i)d-a]^n\binom{n+1}{i}\\
&= \sum_{i=0}^{k}(-1)^i\;[(k+1-i)(d-a)+(k-i)a]^n\binom{n+1}{i}\\
&= \sum_{j=0}^{n}\left[\sum_{i=0}^{k}(-1)^i(k+1-i)^{n-j}(k-i)^j\binom{n+1}{i}\right]\\
&\quad\times\binom{n}{j}(d-a)^{n-j}a^j\\
&= \sum_{j=0}^{n}c_{n,k}(j)\binom{n}{j}(d-a)^{n-j}a^j,
\end{aligned} \tag{8}$$

where

$$c_{n,k}(j) = \sum_{i=0}^{k}(-1)^i(k+1-i)^{n-j}(k-i)^j\binom{n+1}{i}, \quad 0 \le j \le n. \tag{9}$$

The following theorem gives combinatorial interpretations to the coefficients $c_{n,k}(j)$, $0 \le j \le n$.

Theorem 8. *Let the general Eulerian numbers $A_{n,k}(a,d)$ be written as in* (8). *Then*

$$\begin{aligned}
c_{n,k}(j) &= \#\{\pi \in W_{n,k+1},\ j < Q_n(\pi) \le n\}\\
&\quad + \#\{\pi \in W_{n,k},\ 1 \le Q_n(\pi) \le j\}.
\end{aligned} \tag{10}$$

Proof. We can check the result in (10) for two special values $j = 0$ and $j = n$ quickly. By (2),

when $j = 0$, $c_{n,k}(0) = \sum_{i=0}^{k}(-1)^i(k+1-i)^n\binom{n+1}{i} = A_{n,k+1}$;

when $j = n$, $c_{n,k}(n) = \sum_{i=0}^{k}(-1)^i(k-i)^n\binom{n+1}{i} = A_{n,k}$. Therefore, (10) is true for $j = 0$ and $j = n$.

Generally, for $1 \le j \le n-1$, we write down k bars with $k+1$ compartments in between. Place each element of $[n]$ in a compartment. If none of the k bars is extraneous, then the arrangement corresponds to a permutation with k ascents. Let B be the set of arrangements with at most one extraneous bar at the end and none of integers $\{1,2,\ldots,j\}$ locating in the last compartment. We will show that $c_{n,k}(j) = |B|$.

To achieve that goal, we use the Principle of Inclusion and Exclusion. There are $(k+1)^{n-j}k^j$ ways to put n numbers into $k+1$ compartments with elements $\{1,2,\ldots,j\}$ avoiding the last compartments.

Let B_i be the number of arrangements with the following features:

(1) none of $\{1,2,\ldots,j\}$ sits in the last compartment;

(2) each arrangement in B_i has at least i extraneous bars.

(3) in each arrangement in B_i, any two extraneous bars are not located right next to each other.

Then the Principle of Inclusion and Exclusion shows that

$$|B| = (k+1)^{n-j}k^j - B_1 + B_2 + \cdots + (-1)^k B_k. \tag{11}$$

Now we consider the value of B_i, where $1 \le i \le k$. Suppose that we have $k+1-i$ compartments with $k-i$ bars in between. There are $(k+1-i)^{n-j}(k-i)^j$ ways to insert n numbers into these $k+1-i$ compartments with first j integers avoiding the last compartment and list integers in each component in a decreasing order. Then insert i separating extraneous bars into $n+1$ positions. So we get

$$B_i = (k+1-i)^{n-j}(k-i)^j \binom{n+1}{i}. \tag{12}$$

Plug formula (12) into (11); we have $c_{n,k}(j) = |B|$.

Given an arrangement $\pi \in B$, if we remove the bars, then we obtain a permutation $\pi \in \Omega_n$. So without confusion, we just use the same notation π to represent both an arrangement in set B and a permutation on $[n]$. Now for each $\pi \in B$, π either

(case 1) has no extraneous bar and none of $\{1, 2, \ldots, j\}$ locates in the last compartment or

(case 2) has only one extraneous bar at the end.

If π is in case 1, then π has k ascents since each bar is non-extraneous. And the last compartment of π is nonempty. Therefore the last cycle of $f^{-1}(\pi)$ has to be $(n \ldots p_g)$. In other words, $Q_n(f^{-1}(\pi)) = p_g > j$ since none of $\{1, 2, \ldots, j\}$ locates in the last compartment. And by Proposition 4, $f^{-1}(\pi) \in W_{n,k+1}$.

If π is in case 2, then π has $k-1$ ascents since only the last bar is extraneous. Note that in this case, the arrangement with no elements of $\{1, 2, \ldots, j\}$ in the compartment second to the last or the last nonempty compartment has been removed by the Principle of Inclusion and Exclusion. Equivalently, at least one number of $\{1, 2, \ldots, j\}$ has to be in the compartment second to the last. So the last cycle of $f^{-1}(\pi)$ has to be $(n \ldots p_l)$, and $Q_n(f^{-1}(\pi)) = p_l \le j$. Also by Proposition 4, $f^{-1}(\pi) \in W_{n,k}$.

Combing all the results above, statement (10) is correct. □

The next Theorem describes some interesting properties of the coefficients $c_{n,k}$.

Theorem 9. *Let the coefficients $c_{n,k}$ be as described in Theorem 8. Then,*

(1) $\sum_{k=0}^{n} c_{n,k}(j) = n!$, *for any* $0 \le j \le n$;

(2) $c_{n,k}(j) = c_{n,n-k}(n-j)$, *for all* $0 \le j, k \le n$.

Before we can prove Theorem 9, we need the following lemma which is also interesting by itself.

Lemma 10. *Given a positive integer n, then*

$$\begin{aligned}\#\{\pi \in W_{n,k} \,\&\, Q_n(\pi) = j\} \\ = \#\{\pi \in W_{n,n+1-k} \,\&\, Q_n(\pi) = n+1-j\}\end{aligned} \tag{13}$$

for any $1 \le k, j \le n$.

Proof. First of all, given a positive integer n, we define a function $g : \Omega_n \to \Omega_n$ as follows:

$$\begin{gathered}\text{for } \pi = p_1 p_2 \ldots p_n \in \Omega_n, \\ g(\pi) = (n+1-p_1)(n+1-p_2)\ldots(n+1-p_n).\end{gathered} \tag{14}$$

For instance, for $\pi = 53214 \in \Omega_5$, $g(\pi) = 13452$. g is obviously a bijection of Ω_n to itself.

Now for some fixed $1 \le k, j \le n$, suppose $S_{k,j} = \{\pi \in W_{n,k} \,\&\, Q_n(\pi) = j\}$, and $T_{k,j} = \{\pi \in W_{n,n+1-k} \,\&\, Q_n(\pi) = n+1-j\}$. For any $\pi \in S_{k,j}$, we write π in the standard representation cycle form. So $\pi = (p_u \ldots) \ldots (n \ldots j)$ and $f(\pi) = p_u \ldots n \ldots j$ has $k-1$ ascents by Proposition 4. Now we compose $f(\pi)$ with the bijection function g as just defined. Then $g(f(\pi)) = n+1-p_u \ldots 1 \ldots n+1-j$ has $n-k$ ascents, which implies that $f^{-1}(g(f(\pi)))$ has $n+1-k$ weak excedances. So $f^{-1}(g(f(\pi))) \in W_{n,n+1-k}$. Note that the last cycle of $f^{-1}(g(f(\pi)))$ has to be $(n \ldots n+1-j)$. Therefore, $f^{-1}(g(f(\pi))) \in T_{k,j}$. Since both f and g are bijection functions, $f^{-1}gf$ gives a bijection between $S_{k,j}$ and $T_{k,j}$. □

Now we are ready to prove Theorem 9.

Proof of Theorem 9. For part 1, by Theorem 8,

$$\begin{aligned}\sum_{k=0}^{n} c_{n,k}(j) &= \sum_{k=0}^{n} \#\{\pi \in W_{n,k+1},\, j < Q_n(\pi) \le n\} \\ &\quad + \sum_{k=0}^{n} \#\{\pi \in W_{n,k},\, 1 \le Q_n(\pi) \le j\} \\ &= \sum_{k=0}^{n} \#\{\pi \in W_{n,k}\} = |\Omega_n| = n!.\end{aligned} \tag{15}$$

For part 2, also by Theorem 8,

$$\begin{aligned}c_{n,k}(j) &= \sum_{i=j+1}^{n} \#\{\pi \in W_{n,k+1},\, Q_n(\pi) = i\} \\ &\quad + \sum_{m=1}^{j} \#\{\pi \in W_{n,k},\, Q_n(\pi) = m\} \\ &= \sum_{i=j+1}^{n} \#\{\pi \in W_{n,n-k},\, Q_n(\pi) = n+1-i\} \\ &\quad + \sum_{m=1}^{j} \#\{\pi \in W_{n,n+1-k}, \\ &\qquad Q_n(\pi) = n+1-m\} \quad \text{by Lemma 10} \\ &= \#\{\pi \in W_{n,k},\, 1 \le Q_n(\pi) \le n-j\} \\ &\quad + \#\{\pi \in W_{n,n+1-k},\, n-j < Q_n(\pi) \le n\} \\ &= c_{n,n-k}(n-j).\end{aligned} \tag{16}$$

□

Remark 11. Using the analytic formula of $c_{n,k}(j)$ as in (9), part 2 of Theorem 9 implies the following identity:

$$\sum_{i=0}^{k}(-1)^i(k+1-i)^{n-j}(k-i)^j\binom{n+1}{i} = \sum_{l=0}^{n-k}(-1)^l(n+1-k-l)^j(n-k-l)^{n-j}\binom{n+1}{l}, \quad (17)$$

where n is a positive integer, and $0 \le j,\ k \le n$.

3. Another Combinatorial Interpretation of $c_{n,k}(1)$ and $c_{n,k}(n-1)$

In pursuing the combinatorial meanings of the coefficients $c_{n,k}$, the authors have found some other interesting properties about permutations. The results in this section will reveal close connections between the traditional Eulerian numbers $A_{n,k}$ and $c_{n,k}(j)$, where $j = 1$ or $j = n-1$.

One fundamental concept of permutation combinatorics is *inversion*. A pair (p_i, p_j) is called an *inversion* of the permutation $\pi = p_1p_2\dots p_n$ if $i < j$ and $p_i > p_j$ [6, page 36]. The following definition provides the main concepts of this section.

Definition 12. For a fixed positive integer n, let $AW_{n,k} = \{\pi = p_1p_2p_3\cdots p_n \mid \pi \in W_{n,k}$ and $p_1 < p_n\}$ (or (p_1, p_n) is not an inversion) and $BW_{n,k} = W_{n,k} \setminus AW_{n,k}$ (or (p_1, p_n) is an inversion).

It is obvious that $|AW_{n,k}| + |BW_{n,k}| = A_{n,k}$. The following theorem interprets coefficients $c_{n,k}(1)$ and $c_{n,k}(n-1)$ in terms of $AW_{n,k}$ and $BW_{n,k}$.

Theorem 13. *Let the coefficients $c_{n,k}$ of the general Eulerian numbers be written as in (9). $AW_{n,k}$ and $BW_{n,k}$ are as defined in Definition 12. Then*

(1) $c_{n,k}(1) = 2|AW_{n,k+1}|$,

(2) $c_{n,k}(n-1) = 2|BW_{n,k}|$.

Proof. For part (1), by Theorem 8, $c_{n,k}(1) = |S_1| + |S_2|$, where $S_1 = \{\pi = p_1p_2\dots p_n \mid \pi \in W_{n,k+1}\ \&\ p_1 \neq n\}$, $S_2 = \{\pi = p_1p_2\cdots p_n \mid \pi \in W_{n,k}\ \&\ p_1 = n\}$. Given a permutation $\pi = p_1p_2\dots p_n \in S_1$ and $p_n \neq n$, then both $p_1p_2\dots p_n$ and $p_np_2\dots p_1$ belong to S_1, so one of them has to be in $AW_{n,k+1}$. If $\pi = p_1p_2\dots p_n \in S_1$ and $p_n = n$, then $\pi \in AW_{n,k+1}$, but $p_np_2\dots p_1 \in S_2$. Therefore, $(1/2)c_{n,k}(1) = |AW_{n,k+1}|$.

Part (2) can be proved using exactly the same method. So we leave it to the readers as an exercise. □

$|AW_{n,k}|$ and $|BW_{n,k}|$ are interesting combinatorial concepts by themselves. Note that generally speaking, $|AW_{n,k}| \neq |BW_{n,k}|$. Indeed, $|AW_{n,k}| = |BW_{n,n+1-k}|$.

Theorem 14. *For any positive integer $n \ge 2$, the sets $AW_{n,k}$ and $BW_{n,k}$ are defined in Definition 12. Then $|AW_{n,k}| = |BW_{n,n+1-k}|$ for $1 \le k \le n$.*

Proof. It is an obvious result of part 2 of Theorems 9 and 13. □

Our last result of this paper is the following theorem which reveals that both $|AW_{n,k}|$ and $|BW_{n,k}|$ take exactly the same recursive formula as the traditional Eulerian numbers $A_{n,k}$ as shown in (1).

Theorem 15. *For a fixed positive integer n, let $AW_{n,k}$ and $BW_{n,k}$ be as defined in Definition 12; then*

$$k\left|AW_{n-1,k}\right| + (n+1-k)\left|AW_{n-1,k-1}\right| = \left|AW_{n,k}\right|, \quad (18)$$

$$k\left|BW_{n-1,k}\right| + (n+1-k)\left|BW_{n-1,k-1}\right| = \left|BW_{n,k}\right|. \quad (19)$$

Proof. A computational proof can be obtained straightforward by using (9) and Theorem 13. But here we provide a proof in a flavor of combinatorics.

Idea of the Proof. For (18), given a permutation $A_1 = p_1p_2p_3\cdots p_{n-1} \in AW_{n-1,k}$, for each position i with $p_i \ge i$, we insert n into a certain place of A_1, such that the new permutation A_1' is in $AW_{n,k}$. There are k such positions, so we can get k new permutations in $AW_{n,k}$. Similarly, if $A_2 = p_1p_2p_3\cdots p_{n-1} \in AW_{n-1,k-1}$, for each position i with $p_i < i$, and the position at the end of A_2, we insert n into a specific position of A_2 and the resulting new permutation A_2' is in $AW_{n,k}$. There are $n+1-k$ such positions, so we can get $n+1-k$ new permutations in $AW_{n,k}$. We will show that all the permutations obtained from the above constructions are distinct, and they have exhausted all the permutations in $AW_{n,k}$

For any fixed $A' = \pi_1\pi_2\pi_3\dots\pi_n \in AW_{n,k}$, then $\pi_1 < \pi_n$. We classify A' into the following disjoint cases:

Case a. Consider that $\pi_i = n$ with $i < n$. So $A' = \pi_1\pi_2\dots\pi_{i-1}n\pi_{i+1}\dots\pi_{n-1}\pi_n$.

(a1) $\pi_1 < \pi_{n-1}$, and $\pi_n \ge i$;

(a2) $\pi_1 < \pi_{n-1}$, and $\pi_n < i$;

(a3) $\pi_1 > \pi_{n-1}$, $\pi_n < n-1$, and $\pi_n \ge i$;

(a4) $\pi_1 > \pi_{n-1}$, $\pi_n < n-1$, and $\pi_n < i$;

(a5) $\pi_1 > \pi_{n-1}$, and $\pi_n = n-1$.

Case b. Consider that $\pi_n = n$. So $\pi_i = n-1$ for some $i < n$ and $A' = \pi_1\pi_2\dots\pi_{i-1}n-1\dots\pi_{n-1}n$:

(b1) $\pi_1 < \pi_{n-1}$;

(b2) $\pi_{n-1} < \pi_1 < n-1$, and $\pi_{n-1} \ge i$;

(b3) $\pi_{n-1} < \pi_1 < n-1$, and $\pi_{n-1} < i$;

(b4) $\pi_1 = n-1$.

Based on the classifications listed above, we can construct a map $f : \{AW_{n-1,k}, AW_{n-1,k-1}\} \to AW_{n,k}$ by applying the idea of the proof we have illustrated at the beginning of the proof. To save space, the map f is demonstrated in Table 1. From Table 1 we can see that in each case, the positions of inserting n are all different. So all the images obtained in a certain case are different. Since all the cases are disjoint, all the images $A' \in AW_{n,k}$ are distinct.

Table 1: The map $f : \{AW_{n-1,k}, AW_{n-1,k-1}\} \rightarrow AW_{n,k}$.

$A = p_1 p_2 \dots p_{n-1}$	Position i	Condition	$A' \in AW_{n,k}$
$A \in AW_{n-1,k}$	$1 < i \leq n-1$ and $p_i \geq i$	$p_i > p_1$	$A' = p_1 p_2 \dots p_{i-1} n p_{i+1} \dots p_{n-1} p_i$ With $p_1 < p_{n-1}, p_1 < p_i$ Case (a1)
		$p_i < p_1$ and $p_{n-1} < n-1$	$A' = p_1 p_2 \dots p_{i-1} n p_{i+1} \dots p_{n-2} p_i p_{n-1}$ With $p_i < p_1 < p_{n-1} < n-1$ Case (a3)
		$p_i < p_1$ and $p_{n-1} = n-1$	$A' = p_1 p_2 \dots p_{i-1} n-1 p_{i+1} \dots p_{n-2} p_i n$ With $p_i < p_1, p_i \geq i$ Case (b2)
	$i = 1$	$p_j = n-1$ and $j < n-1$	$A' = p_{n-1} p_2 \dots p_{j-1} n p_{j+1} \dots p_{n-2} p_1 n-1$ With $p_1 < p_{n-1}$ Case (a5)
		$p_{n-1} = n-1$	$A' = n-1 p_2 \dots p_{n-2} p_1 n$ Case (b4)
$A \in AW_{n-1,k-1}$	$1 < i \leq n-1$ and $p_i < i$	$p_i > p_1$	$A' = p_1 p_2 \dots p_{i-1} n p_{i+1} \dots p_{n-1} p_i$ With $p_1 < p_{n-1}, p_1 < p_i$ Case (a2)
		$p_i < p_1$ and $p_{n-1} < n-1$	$A' = p_1 p_2 \dots p_{i-1} n p_{i+1} \dots p_{n-2} p_i p_{n-1}$ With $p_i < p_1 < p_{n-1} < n-1$ Case (a4)
		$p_i < p_1$ and $p_{n-1} = n-1$	$A' = p_1 p_2 \dots p_{i-1} n-1 p_{i+1} \dots p_{n-2} p_i n$ With $p_i < p_1, p_i < i$ Case (b3)
	$i = n$		$A' = p_1 p_2 \dots p_{n-1} n$ Case (b1)

Table 2: The map $g : \{BW_{n-1,k}, BW_{n-1,k-1}\} \rightarrow BW_{n,k}$.

$B = p_1 p_2 \dots p_{n-1}$	Position i	Condition	$B' \in BW_{n,k}$
$B \in WB_{n-1,k}$	$1 < i < n-1$ and $p_i \geq i$	$p_1 > p_i$	$B' = p_1 \dots p_{i-1} n p_{i+1} \dots p_{n-1} p_i$ With $p_1 > p_{n-1}$ and $p_i \geq i$ Case (c1)
		$p_1 < p_i < n-1$	$B' = p_1 \dots p_{i-1} n p_{i+1} \dots p_i p_{n-1}$ With $p_1 < p_i < n-1$, $p_i \geq i$ Case (c3)
		$p_1 < p_i = n-1$	$B' = n p_2 \dots p_{i-1} n-1 p_{i+1} \dots p_1 p_{n-1}$ With $p_i = n-1$, $p_1 > p_{n-1}$ Case (d2)
	$i = 1$	$p_1 \geq 1$	$B' = n p_2 \dots p_{n-1} p_1$ With $p_{n-1} < p_1$ Case (d1)
$B \in WB_{n-1,k-1}$	$1 < i < n-1$ and $p_i < i$	$p_1 > p_i$	$B' = p_1 \dots p_{i-1} n p_{i+1} \dots p_{n-1} p_i$ With $p_1 > p_{n-1}$ and $p_i < i$ Case (c2)
		$p_1 < p_i$	$B' = p_1 \dots p_{i-1} n p_{i+1} \dots p_i p_{n-1}$ With $p_1 < p_i < n-1$, $p_i < i$ Case (c4)
	$i = n-1$ $p_i < i$	$p_1 > p_i = p_{n-1}$	$B' = p_1 \dots p_{n-2} n p_{n-1}$ Case (c6)
	$1 \leq i < n-1$ and $p_i = n-1$	$p_i = n-1$	$B' = p_1 \dots p_{i-1} n p_{i+1} \dots p_{n-2} n-1 p_{n-1}$ Case (c5)

Similarly, for each $B' = \pi_1\pi_2\pi_3 \ldots \pi_n \in BW_{n,k}$, then $\pi_1 > \pi_n$. We classify B' into the following disjoint cases.

Case c. Consider that $\pi_i = n$ with $1 < i \leq n-1$. So $B' = \pi_1\pi_2 \ldots \pi_{i-1} n \pi_{i+1} \ldots \pi_{n-1}\pi_n$:

(c1) $\pi_1 > \pi_{n-1}$, and $\pi_n \geq i$;

(c2) $\pi_1 > \pi_{n-1}$, and $\pi_n < i$;

(c3) $\pi_1 < \pi_{n-1} < n-1$, $\pi_{n-1} \geq i$;

(c4) $\pi_1 < \pi_{n-1} < n-1$, $\pi_{n-1} < i$;

(c5) $\pi_{n-1} = n-1$;

(c6) $\pi_{n-1} = n$.

Case d. Consider that $\pi_1 = n$. So $B' = n\pi_2 \ldots \pi_{n-2}\pi_{n-1}$:

(d1) $\pi_{n-2} < \pi_{n-1}$;

(d2) $\pi_{n-2} > \pi_{n-1}$.

To prove (19), we use a similar idea of proof as shown above. If $B_1 = p_1p_2p_3 \cdots p_{n-1} \in BW_{n-1,k}$, for each position i with $p_i \geq i$, we insert n into a certain place of B_1 to get $B_1' \in AW_{n,k}$. If $B_2 = p_1p_2p_3 \cdots p_{n-1} \in BW_{n-1,k-1}$, for each position i with $p_i < i$, and the position i where $p_i = n-1$, we insert n into a specific position of B_2 to obtain $B_2' \in AW_{n,k}$. Such a map $g : \{BW_{n-1,k}, BW_{n-1,k-1}\} \rightarrow BW_{n,k}$ is illustrated in Table 2. And the distinct images under g exhaust all the permutations in $BW_{n,k}$. □

Here is a concrete example for the constructions illustrated in Table 2.

Example 16. Suppose $n = 4$, $k = 2$. We want to obtain $BW_{4,2} = \{3142, 3412, 3421, 4132, 4213, 4312, 4321\}$ from $BW_{3,2} = \{321, 231\}$ and $BW_{3,1} = \{312\}$. For $321 \in BW_{3,2}$, $p_1 = 3 \geq 1$, then it corresponds to $B' = 4213$ which is case (d1) in Table 2; $p_2 = 2 \geq 2$, then it corresponds to $B' = 3412$ which is case (c1) in Table 2. Similarly, we can construct $\{4312, 4321\}$ from $231 \in BW_{3,2}$ and $\{3421, 3142, 4132\}$ from $312 \in BW_{3,1}$ using Table 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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