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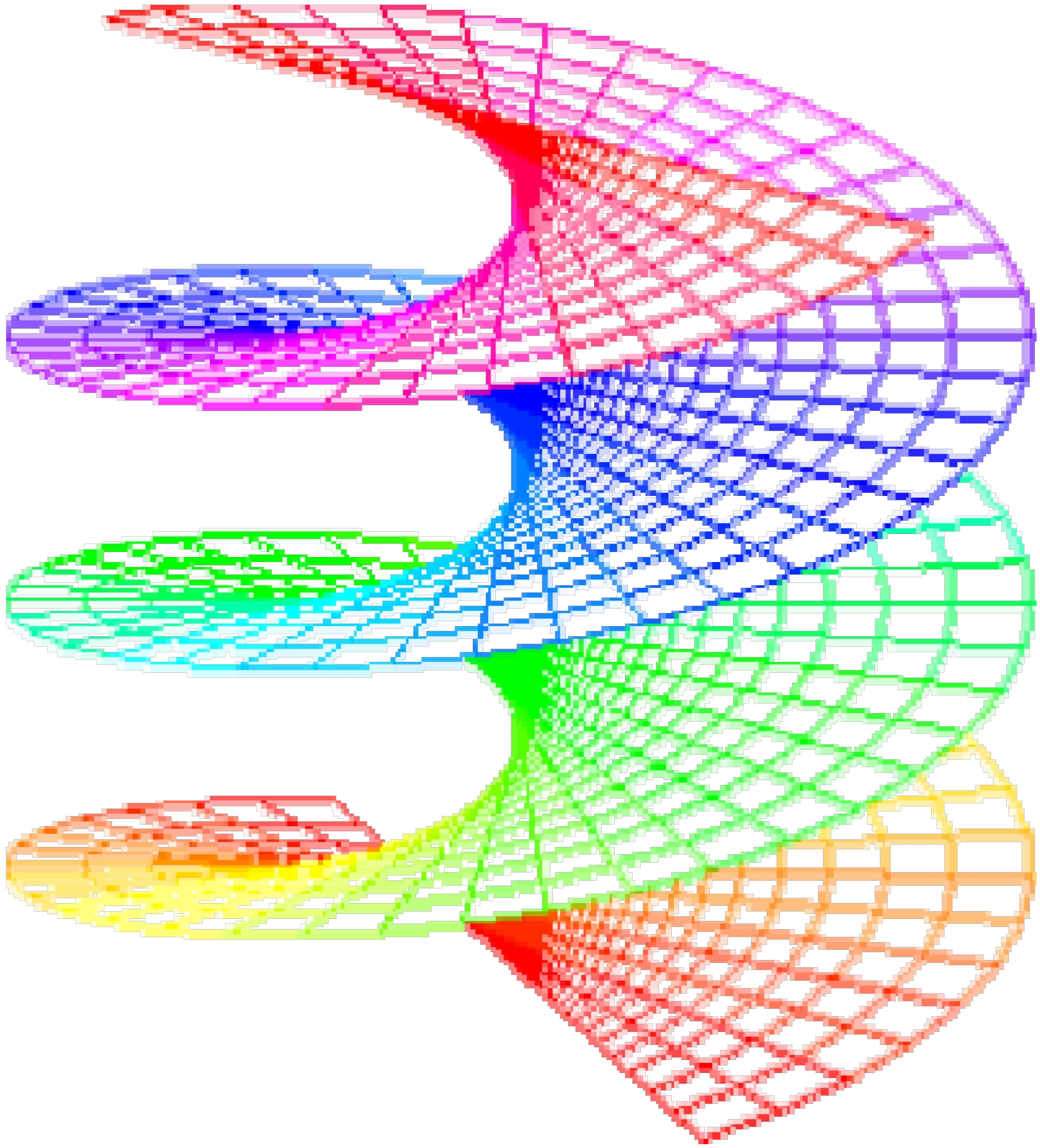

Department of Mathematics
Northwest University
Xi'an, Shaanxi, P. R. China

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Some identities involving Bernoulli numbers and Euler numbers

Yi Yuan

Research Center for Basic Science, Xi'an Jiaotong University
Xi'an, Shaanxi, P.R.China

Abstract The main purpose of this paper is using the elementary method to obtain some interesting identities involving the Bernoulli numbers and the Euler numbers.

Keywords The Bernoulli and the Euler numbers; Identity; Elementary method.

§1. Introduction

Let z be any complex number with $|z| < 2\pi$. The Bernoulli numbers B_n and the Euler numbers E_{2n} $(n = 0, 1, 2, \cdots)$ are defined by the following generated functions (See [1], [2] and [3]):

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < \frac{\pi}{2} \tag{1}$$

and

$$\frac{1}{\cos z} = \sum_{n=0}^{\infty} E_{2n} \frac{z^{2n}}{(2n)!}. \tag{2}$$

For example, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = -\frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, $B_{10} = -\frac{5}{66}$, $\cdots$, $B_{2n+1} = 0$ for $n \geq 1$, and

$$\sum_{k=0}^{r} \frac{2^{2k} B_{2k}}{(2k)!(2r+1-2k)!} = \frac{1}{(2r)!}$$

holds for any integer $r \geq 1$ (See exercise 16 for chapter 12 of [4]). $E_0 = 1$, $E_2 = 1$, $E_4 = 5$, $E_6 = 61$, $E_8 = 11385$, $E_{10} = 150521$, $\cdots$, and

$$\sum_{s=0}^{n} (-1)^s \binom{2n}{2s} E_{2s} = 0, \quad n \geq 1.$$

The Bernoulli numbers and the Euler numbers have extensive applications in combinational mathematics and analytic number theory. So there are many scholars have investigated their arithmetical properties. For example, G.Voronoi first proved a very useful congruence for Bernoulli numbers, one of its Corollaries (See [5] Proposition 15.2.3 and its Corollary) is that for any prime $p \equiv 3 \pmod 4$ with $p > 3$, we have

$$2\left(2 - \left(\frac{2}{p}\right)\right) B_m \equiv -\sum_{j=1}^{m-1} \left(\frac{j}{p}\right) (\bmod p),$$

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where (x/p) denotes the Legendre symbol and $m = (p+1)/2$. Liu Guodong [6] obtained some identities involving the Bernoulli numbers. That is, for any integers $n \geq 1$ and $k \geq 0$,

$$(a) \qquad \sum_{j=0}^{n} \binom{2n+1}{2j} \frac{2-2^{2j}}{(2k+1)^{2j}} B_{2j} = \frac{(2n+1)2^{2n}}{(2k+1)^{2n+1}} \sum_{s=0}^{k} s^{2n};$$

$$(b) \qquad \sum_{j=0}^{n} \binom{2n+1}{2j} \frac{2-2^{2j}}{(2k+2)^{2j}} B_{2j} = \frac{2n+1}{2^{2n}(k+1)^{2n+1}} \sum_{s=0}^{k} (2s+1)^{2n}.$$

For the Euler numbers, Zhang Wenpeng [3] obtained an important congruence, i.e.,

$$E_{p-1} = \begin{cases} 0 \pmod p, & p \equiv 1 \pmod 4, \\ -2 \pmod p, & p \equiv 3 \pmod 4. \end{cases}$$

where p be a prime.

Liu Guodong [7] proved that for any positive integers n and k,

$$E_{2n} \equiv (-1)^{n+k} 2^{2n+1} \sum_{i=1}^{k} (-1)^i i^{2n} \pmod{(2k+1)^2}.$$

Other results involving the Bernoulli numbers and the Euler numbers can also be found in [8], [9] and [10]. This paper as a note of [6] and [7], we use the elementary method to obtain some other identities for the Bernoulli numbers and the Euler numbers. That is, we shall prove the following:

Theorem 1. For any positive integers n and k, we have the identity

$$\sum_{t=0}^{n} \binom{2n+2}{2t} \left(2-2^{2t}\right) \frac{B_{2t}}{(2k)^{2t}} = \frac{4(n+1)}{(2k)^{2n+2}} \sum_{m=1}^{k} (2m-1)^{2n+1}.$$

Theorem 2. For any positive integers n and k, we have

$$E_{2n} - (2k)^{2n} \sum_{t=0}^{n} (-1)^{n+k-t} \binom{2n}{2t} \frac{E_{2t}}{(2k)^{2t}} = 2 \sum_{m=0}^{k-1} (-1)^{m+n} (2m+1)^{2n}.$$

From Theorem 2 we may immediately deduce the following:

Corollary 1. For any odd prime p, we have the congruence

$$E_{\frac{p^2-1}{4}} \equiv \begin{cases} (-1)^{\frac{p^2-1}{8}}\, 2 \pmod p, & p \equiv 3 \pmod 4; \\ (-1)^{\frac{p^2-1}{8}} \frac{4\sqrt{p}}{\pi} L(1, \chi_2\chi_4) \pmod p, & p \equiv 1 \pmod 4, \end{cases}$$

where χ_2 denotes the Legendre symbol modulo p, χ_4 denotes the non-principal character mod 4, and $L(1, \chi_2\chi_4)$ denotes the Dirichlet L-function corresponding to character $\chi_2\chi_4 \mod 4p$.

This Corollary is interesting, because it shows us some relations between the Euler numbers and the Dirichlet L-function. From Corollary 1 we can also get the following:

Corollary 2. For any prime p with $p \equiv 3 \pmod 4$, we have the congruence

$$E_{\frac{p^2-1}{4}} \equiv 2 \left(\frac{2}{p}\right) \equiv \begin{cases} 2 \pmod p, & \text{if } p \equiv 7 \pmod 8; \\ -2 \pmod p, & \text{if } p \equiv 3 \pmod 8. \end{cases}$$

§2. Some Lemmas

To complete the proof of Theorems, we need the following three simple lemmas. First we have

Lemma 1. For any integer $n \geq 1$, we have have the identities

$$(A) \qquad 2\sum_{m=1}^{n} \sin(2m-1)x = \frac{1-\cos 2nx}{\sin x};$$

$$(B) \qquad 2\sum_{m=0}^{n-1} (-1)^m \cos(2m+1)x = \frac{1-(-1)^n \cos 2nx}{\cos x}.$$

Proof. In fact, this Lemma is the different forms of the exercise 3.2.9 of [11], where is

$$\sum_{m=1}^{n} \frac{\sin(2m-1)x}{\sin x} = \left(\frac{\sin nx}{\sin x}\right)^2.$$

Note that $2\sin^2 nx = 1 - \cos 2nx$, from the above we can deduce the formula (A) of Lemma 1.

If we substitute x by $\pi/2 - y$ in (A), we may immediately get formula (B).

Lemma 2. For any real number x with $0 < |x| < \pi$, we have the identity

$$\frac{1}{\sin x} = \sum_{n=0}^{\infty} (-1)^n \left(2 - 2^{2n}\right) \frac{B_{2n}}{(2n)!} x^{2n-1}.$$

Proof. (See reference [12]).

Lemma 3. Let p be an odd prime, χ be an even primitive character $\bmod p$. Then we have

$$\sum_{n \leq p/4} \chi(n) = \frac{G(\chi)}{\pi} L(1, \overline{\chi}\chi_4),$$

where $G(\chi) = \sum_{n=1}^{p-1} \chi(n)\, e^{\frac{2\pi i n}{p}}$ is the Gauss sums, χ_4 denotes the non-principal character $\bmod 4$, and $L(1, \overline{\chi}\chi_4)$ denotes the Dirichlet L-function corresponding to character $\overline{\chi}\chi_4 \bmod 4p$.

Proof. (See Theorem 3.7 of [13]).

§3. Proof of the theorems

In this section, we shall complete the proof of Theorems. First we prove Theorem 1. Note that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{and} \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

from Lemma 2 and (A) of Lemma 1 we have

$$\begin{aligned} & 2\sum_{m=1}^{n}\sum_{s=0}^{\infty}(-1)^s\frac{(2m-1)^{2s+1}}{(2s+1)!}x^{2s+1} \\ = & \left(\sum_{s=0}^{\infty}(-1)^s\left(2-2^{2s}\right)\frac{B_{2s}}{(2s)!}x^{2s-1}\right)\left(1-\sum_{s=0}^{\infty}(-1)^s\frac{(2n)^{2s}}{(2s)!}x^{2s}\right) \\ = & \left(\sum_{s=0}^{\infty}(-1)^s\left(2-2^{2s}\right)\frac{B_{2s}}{(2s)!}x^{2s-1}\right)\left(\sum_{s=0}^{\infty}(-1)^s\frac{(2n)^{2s+2}}{(2s+2)!}x^{2s+2}\right) \\ = & \sum_{s=0}^{\infty}(-1)^s\left(\sum_{t=0}^{s}\left(2-2^{2t}\right)\frac{B_{2t}}{(2t)!}\frac{(2n)^{2s-2t+2}}{(2s-2t+2)!}\right)x^{2s+1}. \end{aligned} \tag{3}$$

Comparing the coefficient of x^{2k+1} on both side of (3), we get

$$2\sum_{m=1}^{n}\frac{(2m-1)^{2k+1}}{(2k+1)!}=\sum_{t=0}^{k}\left(2-2^{2t}\right)\frac{B_{2t}}{(2t)!}\frac{(2n)^{2k-2t+2}}{(2k-2t+2)!}$$

or

$$\sum_{t=0}^{k}\binom{2k+2}{2t}\left(2-2^{2t}\right)\frac{B_{2t}}{(2n)^{2t}}=\frac{4(k+1)}{(2n)^{2k+2}}\sum_{m=1}^{n}(2m-1)^{2k+1}.$$

This proves Theorem 1.

Now we prove Theorem 2. From (2) and (B) of Lemma 1 we have

$$\begin{aligned} & 2\sum_{m=0}^{n-1}(-1)^m\sum_{s=0}^{\infty}(-1)^s\frac{(2m+1)^{2s}}{(2s)!}x^{2s} \\ = & \left(\sum_{s=0}^{\infty}E_{2s}\frac{x^{2s}}{(2s)!}\right)\left(1-(-1)^n\sum_{s=0}^{\infty}(-1)^s\frac{(2n)^{2s}}{(2s)!}x^{2s}\right) \\ = & \sum_{s=0}^{\infty}E_{2s}\frac{x^{2s}}{(2s)!}-(-1)^n\sum_{s=0}^{\infty}\sum_{t=0}^{s}\frac{E_{2t}}{(2t)!}(-1)^{s-t}\frac{(2n)^{2s-2t}}{(2s-2t)!}x^{2s}. \end{aligned} \tag{4}$$

Comparing the coefficient of x^{2k} on both side of (4), we may immediately deduce

$$2\sum_{m=0}^{n-1}(-1)^{m+k}\frac{(2m+1)^{2k}}{(2k)!}=\frac{E_{2k}}{(2k)!}-\sum_{t=0}^{k}(-1)^{n+k-t}\frac{E_{2t}}{(2t)!}\frac{(2n)^{2k-2t}}{(2k-2t)!}$$

or

$$2\sum_{m=0}^{n-1}(-1)^{m+k}(2m+1)^{2k}=E_{2k}-(2n)^{2k}\sum_{t=0}^{k}(-1)^{n+k-t}\binom{2k}{2t}\frac{E_{2t}}{(2n)^{2t}}.$$

This completes the proof of Theorem 2.

To prove Corollary 1, taking $k=p$ and $n=(p^2-1)/8$ in Theorem 2 we may get

$$2E_{2n}+(2p)^{2n}\sum_{t=0}^{n-1}(-1)^{n-t}\binom{2n}{2t}\frac{E_{2t}}{(2p)^{2t}}=2\sum_{m=0}^{p-1}(-1)^{m+n}(2m+1)^{2n}$$

or

$$E_{\frac{p^2-1}{4}} \equiv (-1)^{\frac{p^2-1}{8}} \sum_{m=0}^{p-1} (-1)^m (2m+1)^{\frac{p^2-1}{4}} \pmod p. \tag{5}$$

For any integer a with $(a,p)=1$, from the Euler's criterion (See Theorem 9.2 of [4]) we know that

$$a^{\frac{p-1}{2}} \equiv \left(\frac{a}{p}\right) \pmod p,$$

where $(a/p)=\chi_2(a)$ is the Legendre symbol modulo p.

By this formula we may get

$$a^{\frac{p^2-1}{4}} \equiv \left(\frac{a}{p}\right)^{\frac{p+1}{2}} \equiv \begin{cases} 1 \pmod p, & \text{if } p\equiv 3 \pmod 4; \\ \left(\frac{a}{p}\right) \pmod p, & \text{if } p \equiv 1 \pmod 4. \end{cases} \tag{6}$$

If $p\equiv 3 \pmod 4$, note that $\left(\frac{0}{p}\right)=0$, from (5) and (6) we can get

$$\begin{aligned} E_{\frac{p^2-1}{4}} &\equiv (-1)^{\frac{p^2-1}{8}} \sum_{m=0}^{p-1} (-1)^m (2m+1)^{\frac{p^2-1}{4}} \\ &\equiv (-1)^{\frac{p^2-1}{8}} \sum_{m=0}^{p-1} (-1)^m \left(\frac{2m+1}{p}\right)^2 \pmod p \\ &\equiv (-1)^{\frac{p^2-1}{8}} 2 \pmod p. \end{aligned}$$

If $p\equiv 1 \pmod 4$, note that $\left(\frac{-1}{p}\right)=1$ (an even character mod p), $G(\chi_2)=\sqrt{p}$ and $\sum_{m=1}^{p-1}\left(\frac{m}{p}\right)=0$, from (5), (6) and Lemma 3 we may obtain

$$\begin{aligned} E_{\frac{p^2-1}{4}} &\equiv (-1)^{\frac{p^2-1}{8}} \sum_{m=0}^{p-1} (-1)^m (2m+1)^{\frac{p^2-1}{4}} \\ &\equiv (-1)^{\frac{p^2-1}{8}} \sum_{m=0}^{p-1} (-1)^m \left(\frac{2m+1}{p}\right) \pmod p \\ &\equiv (-1)^{\frac{p^2-1}{8}} \left[2\sum_{m=0}^{(p-1)/2} \left(\frac{4m+1}{p}\right) - \sum_{m=0}^{p-1} \left(\frac{2m+1}{p}\right) \right] \pmod p \\ &\equiv (-1)^{\frac{p^2-1}{8}}\ 2 \sum_{m=0}^{(p-1)/2} \left(\frac{m+\overline{4}}{p}\right) \pmod p \\ &\equiv (-1)^{\frac{p^2-1}{8}}\ 2 \sum_{m=\frac{1-p}{4}}^{(p-1)/4} \left(\frac{m}{p}\right) \pmod p \\ &\equiv (-1)^{\frac{p^2-1}{8}}\ 4 \sum_{m=1}^{(p-1)/4} \left(\frac{m}{p}\right) \pmod p \\ &\equiv (-1)^{\frac{p^2-1}{8}} \frac{4\sqrt{p}}{\pi} L(1,\chi_2\chi_4) \pmod p, \end{aligned}$$

where $\overline{a}$ denotes the solution of the congruence $ax \equiv 1 \pmod{p}$ and $\overline{4} = \frac{1-p}{4}$.

This completes the proof of Corollary 1.

Note. Using the exercise 3.2.7 and 3.2.8 of [11], we can also deduce the other identities and congruences involving the Bernoulli numbers and the Euler numbers.

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