Incomplete Tribonacci-Lucas numbers and polynomials

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Abstract

In this paper, we define Tribonacci-Lucas polynomials and present Tribonacci-Lucas numbers and polynomials as a binomial sum. Then, we introduce incomplete Tribonacci-Lucas numbers and polynomials. In addition we derive recurrence relations, some properties and generating functions of these numbers and polynomials. Also, we find the generating function of incomplete Tribonacci polynomials which is given as the open problem in [12].

Keywords: Incomplete Tribonacci-Lucas numbers, Incomplete Tribonacci-Lucas polynomials, Binomial sums, Generating functions.

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1 Introduction

Recently, Fibonacci and Lucas numbers have investigated very largely and authors tried to developed and give some directions to mathematical calculations using these type of special numbers [2,9,16,17]. One of these directions goes through to the *Tribonacci* and the *Tribonacci-Lucas numbers.* In fact Tribonacci numbers have been firstly defined by M. Feinberg in 1963 and then some important properties over this numbers have been created by [5,8,10,13,19]. On the other hand, Tribonacci-Lucas numbers have been introduced and investigated by author in [4]. In addition, there exists another mathematical term, namely to be *incomplete*, on Fibonacci, Lucas and Tribonacci numbers. As a brief background, the incomplete Fibonacci, Lucas and Tribonacci numbers were introduced by authours [6,11,12,18], and further the generating functions of these numbers were presented by authours. Moreover, in [14,15], it is defined and examined recurrence relations of the incomplete Fibonacci, Lucas p-numbers and p-polynomials . We may also refer [3,18] for further studies about to have incompleteness of special numbers.

For $n \geq 2$, it is known that while the Tribonacci sequence $\{T_n\}_{n\in\mathbb{N}}$ is defined by

$$T_{n+1} = T_n + T_{n-1} + T_{n-2} \quad (T_0 = 0,\ T_1 = T_2 = 1), \tag{1}$$

the Tribonacci-Lucas sequence $\{K_n\}_{n\in\mathbb{N}}$ is defined by

$$K_{n+1} = K_n + K_{n-1} + K_{n-2} \ \ (K_0 = 3,\ K_1 = 1,\ K_2 = 3). \tag{2}$$

There is also well known that each of the Tribonacci and Tribonaccci-Lucas numbers is actually a linear combination of α^n, β^n and γ^n. In other words,

$$\left.\begin{array}{c} T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \\ \text{and} \\ K_n = \alpha^n + \beta^n + \gamma^n, \end{array}\right\} \tag{3}$$

where α, β and γ are roots of the characteristic equations of (1) and (2) such that

$$\begin{aligned} \alpha &= \frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3},\ \beta = \frac{1+w\sqrt[3]{19+3\sqrt{33}}+w^2\sqrt[3]{19-3\sqrt{33}}}{3}, \\ \gamma &= \frac{1+w^2\sqrt[3]{19+3\sqrt{33}}+w\sqrt[3]{19-3\sqrt{33}}}{3}, \quad w = \frac{-1+i\sqrt{3}}{2}. \end{aligned}$$

Meanwhile we note that equations in (3) are called the Binet formulas for Tribonacci and Tribonacci-Lucas numbers, respectively.

Moreover, authors defined a large class of polynomials by Fibonacci and Tribonacci numbers [1, 7]. Such polynomials are called Fibonacci polynomials and Tribonacci polynomials [9], respectively. In 1973, Hoggatt and Bicknell [7] introduced Tribonacci polynomials. The Tribonacci polynomials $T_n(x)$ are defined by the recurrence relation

$$T_{n+3}(x) = x^2 T_{n+2}(x) + x T_{n+1}(x) + T_n(x),$$

where $T_0(x) = 0,\ T_1(x) = 1,\ T_2(x) = x^2$.

On the other hand, in [12], incomplete Tribonacci polynomials are defined by

$$T_n^{(s)}(x) = \sum_{i=0}^{s}\sum_{j=0}^{i}\binom{i}{j}\binom{n-i-j-1}{i}x^{2n-3(i+j)-2}, \tag{4}$$

where $0 \leq s \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. In here, for $x = 1$, it is obtained incomplete Tribonacci numbers. The recurrence relation of these polynomials is

$$\begin{aligned} T_{n+3}^{(s)}(x) = \quad & x^2 T_{n+2}^{(s)}(x) + x T_{n+1}^{(s)}(x) + T_n^{(s)}(x) - x\sum_{j=0}^{s}\binom{s}{j}\binom{n-s-j}{s}x^{2n-3(s+j)} \\ & -\sum_{j=0}^{s}\binom{s}{j}\binom{n-s-j-1}{s}x^{2n-3(s+j)-2} \end{aligned} \tag{5}$$

Also the generating function of incomplete Tribonacci numbers is

$$Q_s\left(z\right)=\frac{T_{2s+1}+z\left(T_{2s+2}-T_{2s+1}\right)+z^2\left(T_{2s+3}-T_{2s+2}-T_{2s+1}-2\right)-g\left(z\right)}{1-z-z^2-z^3}, \tag{6}$$

where $g\left(z\right)=\left(z^2+z^3\right)\frac{(1+z)^s}{(1-z)^{s+1}}$ and T_n is n-th Tribonacci number.

In the light of the above paragraph, the main goal of this paper is to improve the Tribonacci-Lucas numbers with a different viewpoint . In order to do that we first define Tribonacci-Lucas polynomials and then by presenting Tribonacci-Lucas numbers and polynomials as a binomial sum, we define the incomplete Tribonacci-Lucas numbers and polynomials.

After that we find the generating function of incomplete Tribonacci polynomials which is given as the open problem in [12]. Also, we obtain some properties and generating functions of incomplete Tribonacci-Lucas numbers and polynomials.

2 Tribonacci-Lucas polynomials and pascal-like triangle

In the following table, we give the pascal-like triangle of Tribonacci-Lucas numbers and each element of this table is defined in similar way as in the tribonacci triangle.

$n\backslash i$	0	1	2	3	4	5	$\cdots$
0	3						
1	1	2					
2	1	6	2				
3	1	8	10	2			
4	1	10	24	14	2		
5	1	12	42	48	18	2	
$\vdots$				$\vdots$			

Table1. Tribonacci-Lucas triangle

Let $B\left(n,i\right)$ be the element in the n-th row and i-th column of the Tribonacci-Lucas triangle. By using the triangle, we have

$$B\left(n+1,i\right)=B\left(n,i\right)+B\left(n,i-1\right)+B\left(n-1,i-1\right), \tag{7}$$

where $B\left(n,0\right)=1,\ B\left(n,n\right)=2$ for $n\in\mathbb{Z}^+$.

By using the Table 1, we have the Tribonacci-Lucas numbers as binomial sum

$$K_n=\sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor}B\left(n-i,i\right).$$

In here, the sum of elements on the rising diagonal lines in the Tribonacci-Lucas triangle is the Tribonacci-Lucas number K_n. Furthermore, we write

$$K_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{i} \frac{n}{n-i-j} \binom{i}{j} \binom{n-i-j}{i}, \quad (n > i + j) \tag{8}$$

since these coefficients hold the relation

$$\begin{cases} B(n,i) = \sum\limits_{j=0}^{i} \frac{n+i}{n-j} \binom{i}{j} \binom{n-j}{i}, & (n > i) \\ B(n,i) = 2, & (n = i) \end{cases}.$$

Now, we introduce the Tribonacci-Lucas polynomial

$$K_{n+3}(x) = x^2 K_{n+2}(x) + x K_{n+1}(x) + K_n(x),$$

where $K_0(x) = 3,\ K_1(x) = x^2,\ K_2(x) = x^4 + 2x$. Note that $K_n(1) = K_n$, $n \in \mathbb{N}$. It is given a few Tribonacci-Lucas polynomials in the following:

$K_0(x) = 3,$ $\quad K_4(x) = x^8 + 4x^5 + 6x^2,$
$K_1(x) = x^2,$ $\quad K_5(x) = x^{10} + 5x^7 + 10x^4 + 5x,$
$K_2(x) = x^4 + 2x,$ $\quad K_6(x) = x^{12} + 6x^9 + 15x^6 + 14x^3 + 3,$
$K_3(x) = x^6 + 3x^3 + 3,$ $\quad K_7(x) = x^{14} + 7x^{11} + 21x^8 + 28x^5 + 14x^2.$

In similarly with the Tribonacci-Lucas triangle, we define the Tribonacci-Lucas polynomials triangle:

$n\backslash i$	0	1	2	3	4	5	$\cdots$
0	3						
1	x^2	$2x$					
2	x^4	$3x^3+3$	$2x^2$				
3	x^6	$4x^5+4x^2$	$5x^4+5x$	$2x^3$			
4	x^8	$5x^7+5x^4$	$9x^6+12x^3+3$	$7x^5+7x^2$	$2x^4$		
5	x^{10}	$6x^9+6x^6$	$14x^8+21x^5+7x^2$	$16x^7+24x^4+8x$	$9x^6+9x^3$	$2x^5$	
$\vdots$				$\vdots$			

Table2. Tribonacci-Lucas polynomials triangle

Let $B(n,i)(x)$ be the element in the n-th row and i-th column of the Tribonacci-Lucas polynomials triangle. By using the triangle, we have

$$B(n+1,i)(x) = x^2 B(n,i)(x) + x B(n,i-1)(x) + B(n-1,i-1)(x), \tag{9}$$

where $B(n,0)(x) = x^{2n},\ B(n,n)(x) = 2x^n$ for $n \in \mathbb{Z}^+$.

By using the Table 2, we have

$$K_n(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} B(n-i,i)(x).$$

In here, the sum of elements on the rising diagonal lines in the Table 2 is the Tribonacci-Lucas polynomials $K_n(x)$ .Furthermore, we write

$$K_n(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{i} \frac{n}{n-i-j} \binom{i}{j} \binom{n-i-j}{i} x^{2n-3i-3j}, \qquad (n > i+j) \tag{10}$$

since these coefficients satisfy the relation

$$\begin{cases} B(n,i)(x) = \sum\limits_{j=0}^{i} \frac{n+i}{n-j} \binom{i}{j} \binom{n-j}{i} x^{2n-i-3j}, & (n > i) \\ B(n,i)(x) = 2x^n, & (n = i) \end{cases}.$$

Thus, by considering equations (8) and (10), we introduce the incomplete Tribonacci-Lucas numbers and incomplete Tribonacci-Lucas polynomials.

Now, we get new recurrence relations, some properties and generating functions of incomplete Tribonacci-Lucas numbers and polynomials.

3 The incomplete Tribonacci-Lucas polynomials and Tribonacci-Lucas numbers

Definition 1 *The incomplete Tribonacci-Lucas polynomials* $K_n^{(s)}(x)$ *are defined by*

$$\begin{aligned} K_n^{(s)}(x) &= \sum_{i=0}^{s} B(n-i,i)(x) \qquad (11) \\ &= \sum_{i=0}^{s} \sum_{j=0}^{i} \frac{n}{n-i-j} \binom{i}{j} \binom{n-i-j}{i} x^{2n-3(i+j)}, \end{aligned}$$

where $0 \leq s \leq \lfloor \frac{n}{2} \rfloor$ *for* $n \in \mathbb{Z}^+$.

In Definition 1, for $x = 1$, we define the incomplete Tribonacci-Lucas numbers, that is, $K_n^{(s)}(1) = K_n(s)$.

To reveal the importance of this subject, we can express the relationships as in the following:

- $K_n^{(\lfloor \frac{n}{2} \rfloor)}(x) = K_n(x)$ (the relationship between incomplete Tribonacci-Lucas polynomials and Tribonacci-Lucas polynomials),

- $K_n\left(\left\lfloor \frac{n}{2} \right\rfloor\right) = K_n$ (the relationship between incomplete Tribonacci-Lucas numbers and Tribonacci-Lucas numbers).

From Definition 1, we have a few incomplete Tribonacci-Lucas polynomials as

$n \backslash s$	0	1	2	3	$\cdots$
1	x^2				
2	x^4	x^4+2x			
3	x^6	x^6+3x^3+3			
4	x^8	$x^8+4x^5+4x^2$	$x^8+4x^5+6x^2$		
5	x^{10}	$x^{10}+5x^7+5x^4$	$x^{10}+5x^7+10x^4+5x$		
6	x^{12}	$x^{12}+6x^9+6x^6$	$x^{12}+6x^9+15x^6+12x^3+3$	$x^{12}+6x^9+15x^6+14x^3+3$	
$\vdots$			$\vdots$		

Table3. Incomplete Tribonacci-Lucas polynomials

By taking account of Table 3, we can write

$$K_n^{(0)}(x) = x^{2n}, \tag{12}$$

$$K_n^{(1)}(x) = x^{2n} + nx^{2n-3} + nx^{2n-6}, \quad n \geq 3 \tag{13}$$

$$K_n^{\left(\left\lfloor \frac{n}{2} \right\rfloor\right)}(x) = K_n(x), \tag{14}$$

$$K_n^{\left(\left\lfloor \frac{n-2}{2} \right\rfloor\right)}(x) = \begin{cases} K_n(x) - 2x^{\frac{n}{2}}, & n \geq 2, \ even \\ K_n(x) - \left(nx^{\frac{n+3}{2}} + nx^{\frac{n-3}{2}}\right), & n \geq 2, \ odd \end{cases}. \tag{15}$$

Proposition 2 *For $0 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $n \in \mathbb{Z}^+$, we have the following recurrence relations;*

i) *The homogeneous recurrence relation of the incomplete Tribonacci-Lucas polynomials $K_n^{(s)}(x)$ is*

$$K_{n+3}^{(s+1)}(x) = x^2 K_{n+2}^{(s+1)}(x) + xK_{n+1}^{(s)}(x) + K_n^{(s)}(x). \tag{16}$$

ii) *The non-homogeneous recurrence relation of the incomplete Tribonacci-Lucas polynomials $K_n^{(s)}(x)$ is*

$$\begin{aligned} K_{n+3}^{(s)}(x) &= x^2 K_{n+2}^{(s)}(x) + xK_{n+1}^{(s)}(x) + K_n^{(s)}(x) \\ &\quad -xB(n+1-s,s)(x) - B(n-s,s)(x). \end{aligned} \tag{17}$$

Proof.

i) From Definition 1, let us label $x^2K_{n+2}^{(s+1)}(x)+xK_{n+1}^{(s)}(x)+K_n^{(s)}(x)$ by RHS.
Actually we write

$$\begin{aligned} RHS &= x^2\sum_{i=0}^{s+1}B(n+2-i,i)(x)+x\sum_{i=0}^{s}B(n+1-i,i)(x)+\sum_{i=0}^{s}B(n-i,i)(x) \\ &= x^2\sum_{i=0}^{s+1}B(n+2-i,i)(x)+x\sum_{i=1}^{s+1}B(n+2-i,i-1)(x)+\sum_{i=1}^{s+1}B(n+1-i,i-1)(x) \\ &= \sum_{i=0}^{s+1}\left(x^2B(n+2-i,i)(x)+xB(n+2-i,i-1)(x)+B(n+1-i,i-1)(x)\right) \\ &\quad -xB(n+2,-1)(x)-B(n+1,-1)(x). \end{aligned}$$

Then, by considering $\binom{n}{-1}=0$ and the equation (9), we finally have

$$RHS=\sum_{i=0}^{s+1}B(n+3-i,i)(x)=K_{n+3}^{(s+1)}(x)$$

as required.

ii) By considering the equations (9), (16) and Definition 1, we have

$$\begin{aligned} \sum_{i=0}^{s+1}B(n+3-i,i)(x) &= x^2\sum_{i=0}^{s+1}B(n+2-i,i)(x)+x\sum_{i=0}^{s}B(n+1-i,i)(x)+\sum_{i=0}^{s}B(n-i,i)(x) \\ \sum_{i=0}^{s}B(n+3-i,i)(x) &= x^2\sum_{i=0}^{s}B(n+2-i,i)(x)+x\sum_{i=0}^{s}B(n+1-i,i)(x)+\sum_{i=0}^{s}B(n-i,i)(x) \\ &\quad -B(n+2-s,s+1)+x^2B(n+1-s,s+1) \\ K_{n+3}^{(s)}(x) &= x^2K_{n+2}^{(s)}(x)+xK_{n+1}^{(s)}(x)+K_n^{(s)}(x)-xB(n+1-s,s)-B(n-s,s). \end{aligned}$$

■

By using Table 3, for $x=1$, we have incomplete Tribonacci-Lucas numbers in the following Table 4:

$n\backslash s$	0	1	2	3	$\cdots$
1	1				
2	1	3			
3	1	7			
4	1	9	11		
5	1	11	21		
6	1	13	37	39	
$\vdots$			$\vdots$		

Table4. Incomplete Tribonacci-Lucas numbers

Corollary 3 *For* $0 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor$ *and* $n \in \mathbb{Z}^{+}$, *we have the following recurrence relations;*

i) *The homogeneous recurrence relation of the incomplete Tribonacci-Lucas numbers* $K_n(s)$ *is*

$$K_{n+3}(s+1) = K_{n+2}(s+1) + K_{n+1}(s) + K_n(s). \tag{18}$$

ii) *The non-homogeneous recurrence relation of the incomplete Tribonacci-Lucas numbers* $K_n(s)$ *is*

$$\begin{aligned} K_{n+3}(s) &= K_{n+2}(s) + K_{n+1}(s) + K_n(s) \\ &\quad - B(n+1-s, s) - B(n-s, s). \end{aligned} \tag{19}$$

Proposition 4 *The relation between of incomplete Tribonacci polynomials* $T_n^{(s)}(x)$ *and incomplete Tribonacci-Lucas polynomials* $K_n^{(s)}(x)$ *is*

$$K_n^{(s)}(x) = T_{n+1}^{(s)}(x) + xT_{n-1}^{(s-1)}(x) + 2T_{n-2}^{(s-1)}(x),$$

where $1 \leq s \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ *and* $n > 2$.

Proof. Proof of its can easily do by using Definition 1 and the equation (4). ∎

Corollary 5 *The relation between of incomplete Tribonacci numbers* $T_n(s)$ *and incomplete Tribonacci-Lucas numbers* $K_n(s)$ *is*

$$K_n(s) = T_{n+1}(s) + T_{n-1}(s-1) + 2T_{n-2}(s-1),$$

where $1 \leq s \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ *and* $n > 2$.

Theorem 6 *For* $n,\ h \geq 1$ *and* $0 \leq s \leq \left\lfloor \frac{n}{2} \right\rfloor$, *the sum of incomplete Tribonacci-Lucas numbers is*

$$\sum_{i=0}^{h-1} K_{n+i}(s) = \frac{1}{2}\left(K_{n+h+2}(s+1) - K_{n+2}(s+1) + K_n(s) - K_{n+h}(s)\right). \tag{20}$$

Proof. Let us use the principle of mathematical induction on h to prove (20). While, for $h = 1$, it is easy to see that

$$K_n(s) = \frac{1}{2}\left(K_{n+3}(s+1) - K_{n+2}(s+1) + K_n(s) - K_{n+1}(s)\right).$$

As the usual next step of inductions, let us assume that it is true for all positive integers h. That is,

$$\sum_{i=0}^{h-1} K_{n+i}(s) = \frac{1}{2}\left(K_{n+h+2}(s+1) - K_{n+2}(s+1) + K_n(s) - K_{n+h}(s)\right).$$

Therefore, we have to show that it is true for $h+1$. In other words, we need to check

$$\sum_{i=0}^{h} K_{n+i}(s) = \frac{1}{2}\left(K_{n+h+3}(s+1) - K_{n+2}(s+1) + K_n(s) - K_{n+h+1}(s)\right).$$

Hence, we can write

$$\begin{aligned}\sum_{i=0}^{h} K_{n+i}(s) &= \sum_{i=0}^{h-1} K_{n+i}(s) + K_{n+h}(s) \\ &= \frac{1}{2}\left(K_{n+h+2}(s+1) - K_{n+2}(s+1) + K_n(s) - K_{n+h}(s)\right) + K_{n+h}(s) \\ &= \frac{1}{2}\left(K_{n+h+2}(s+1) - K_{n+2}(s+1) + K_n(s) + K_{n+h}(s)\right) \\ &= \frac{1}{2}\left(K_{n+h+3}(s+1) - K_{n+2}(s+1) + K_n(s) - K_{n+h+1}(s)\right).\end{aligned}$$

■

The following proposition give the sum of incomplete Tribonacci-Lucas poynomials, that is, sum of the n-th row of the Table 3.

Proposition 7 *For $l = \left\lfloor \frac{n}{2} \right\rfloor$, we have the equality*

$$\sum_{s=0}^{l} K_n^{(s)}(x) = (l+1) K_n(x) - \sum_{i=0}^{l}\sum_{j=0}^{i} \frac{in}{n-i-j}\binom{i}{j}\binom{n-i-j}{i} x^{2n-3(i+j)}. \tag{21}$$

Proof. From Definition 1, we have

$$K_n^{(s)}(x) = \sum_{i=0}^{s} B(n-i,i)(x).$$

Then, we can write

$$\begin{aligned}\sum_{s=0}^{l} K_n^{(s)}(x) &= \sum_{s=0}^{l}\sum_{i=0}^{s} B(n-i,i)(x) \\ &= (l+1) B(n,0)(x) + lB(n-1,1)(x) + \cdots + B(n-l,l)(x) \\ &= \sum_{i=0}^{l} (l+1-i) B(n-i,i)(x) \\ &= \sum_{i=0}^{l} (l+1) B(n-i,i)(x) - \sum_{i=0}^{l} iB(n-i,i)(x) \\ &= (l+1) K_n^{(l)}(x) - \sum_{i=0}^{l}\sum_{j=0}^{i} \frac{in}{n-i-j}\binom{i}{j}\binom{n-i-j}{i} x^{2n-3(i+j)}.\end{aligned}$$

Since l is $\left\lfloor \frac{n}{2} \right\rfloor$, we obtain

$$\sum_{s=0}^{l} K_n^{(s)}(x) = (l+1) K_n(x) - \sum_{i=0}^{l} \sum_{j=0}^{i} \frac{in}{n-i-j} \binom{i}{j} \binom{n-i-j}{i} x^{2n-3(i+j)}.$$

■

The following corollary shows the sum of the n-th row of the Table 4. It is obtained from (21) with $x = 1$.

Corollary 8 *For $l = \left\lfloor \frac{n}{2} \right\rfloor$, we have the following equality;*

$$\sum_{s=0}^{l} K_n(s) = (l+1) K_n - \sum_{i=0}^{l} \sum_{j=0}^{i} \frac{in}{n-i-j} \binom{i}{j} \binom{n-i-j}{i}. \tag{22}$$

4 Generating functions of the incomplete Tribonacci and Tribonacci-Lucas polynomials and numbers

Lemma 9 *Let $\{S_n\}_{n=0}^{\infty}$ be a complex sequence satisfying the non-homogeneous third-order recurrence relation. Then we have*

$$S_n = aS_{n-1} + bS_{n-2} + cS_{n-3} + r_n,$$

where $a, b, c \in \mathbb{C}$, $n \geq 3$ and $r_n : \mathbb{N} \to \mathbb{C}$ is a sequence. Hence the generating function $U(x)$ of S_n is given by

$$U(x) = \frac{S_0 - r_0 + x(S_1 - aS_0 - r_1) + x^2(S_2 - aS_1 - bS_0 - r_2) + G(x)}{1 - ax - bx^2 - cx^3}, \tag{23}$$

where $G(x)$ denotes the generating function of r_n.

Proof. Let $U(x)$ and $G(x)$ be two generating functions for complex sequences S_n and r_n, respectively, where

$$U(x) = S_0 + S_1 x + S_2 x^2 + S_3 x^3 + \ldots + S_n x^n + \ldots, \tag{24}$$

$$G(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3 + \ldots + r_n x^n + \ldots. \tag{25}$$

If $U(x)$ given in (24) multiply with ax, bx^2 and cx^3, respectively, then we get

$$\left.\begin{array}{r} axU(x) = aS_0 x + aS_1 x^2 + aS_2 x^3 + aS_3 x^4 + \ldots + aS_n x^{n+1} + \ldots \\ bx^2 U(x) = bS_0 x^2 + bS_1 x^3 + bS_2 x^4 + bS_3 x^5 + \ldots + bS_n x^{n+2} + \ldots \\ cx^3 U(x) = cS_0 x^3 + cS_1 x^4 + cS_2 x^5 + cS_3 x^6 + \ldots + cS_n x^{n+3} + \ldots \end{array}\right\}. \tag{26}$$

Consequently, by subtracting (25) and (26) from (24), it is obtained the equation

$$U(x) = \frac{S_0 - r_0 + x(S_1 - aS_0 - r_1) + x^2(S_2 - aS_1 - bS_0 - r_2) + G(x)}{1 - ax - bx^2 - cx^3}$$

which completes the proof of the Lemma. ■

Now, we examine the problem which is given for incomplete Tribonacci polynomials in [15] .

Theorem 10 *The generating function of the incomplete Tribonacci polynomials* $T_n^{(s)}(x)$ *is given by*

$$Q_s(x,z)=\sum_{i=0}^{\infty}T_i^{(s)}(x)\,z^i=z^{2s+1}U_s(x,z),$$

where $U_s(x,z)=\frac{T_{2s+1}(x)+z\left(T_{2s+2}(x)-x^2T_{2s+1}(x)\right)+z^2\left(T_{2s}(x)-2x^{s+1}\right)-\left(xz^2+z^3\right)\frac{(x+z)^s}{(1-x^2z)^{s+1}}}{(1-x^2z-xz^2-z^3)}$.

Proof. Let s be fixed positive integer. By using the equations (4) and (5), we have

$$\begin{aligned}
T_n^{(s)}(x) &= 0 \qquad (0\leq n<2s+1),\\
T_{2s+1}^{(s)}(x) &= T_{2s+1}(x),\\
T_{2s+2}^{(s)}(x) &= T_{2s+2}(x),\\
T_{2s+3}^{(s)}(x) &= T_{2s+3}(x)-x^{s+1},
\end{aligned}$$

and

$$\begin{aligned}
T_n^{(s)}(x) &= x^2T_{n-1}^{(s)}(x)+xT_{n-2}^{(s)}(x)+T_{n-3}^{(s)}(x)-\sum_{j=0}^{s}\binom{s}{j}\binom{n-3-s-j}{s}x^{2n-5-3(s+j)} \qquad (27)\\
&\quad-\sum_{j=0}^{s}\binom{s}{j}\binom{n-4-s-j}{s}x^{2n-8-3(s+j)},
\end{aligned}$$

where $n\geq 4+2s$. Also, we replace $S_0,S_1,...,S_n$ by $T_{2s+1}^{(s)}(x)$, $T_{2s+2}^{(s)}(x)$, ..., $T_{n+2s+1}^{(s)}(x)$, respectively. Assume that $r_0=r_1=0,\ r_2=x^{s+1}$ and

$$r_n=\sum_{j=0}^{s}\binom{s}{j}\binom{n-2+s-j}{s}x^{2n-3+s-3j}+\sum_{j=0}^{s}\binom{s}{j}\binom{n-3+s-j}{s}x^{2n-6+s-3j}.$$

Furthermore, by considering [20, page 127], the generating function $G(x,z)$ of the $\{r_n\}$ is

$$G(x,z)=\left(xz^2+z^3\right)\frac{(x+z)^s}{(1-x^2z)^{s+1}}.$$

Therefore, by using Lemma 9, the generating function $U_s(x,z)$ of the sequence S_n is

$$\begin{aligned}
U_s(x,z)\left(1-x^2z-xz^2-z^3\right)+G(x,z) &= T_{2s+1}(x)+z\left(T_{2s+2}(x)-x^2T_{2s+1}(x)\right)\\
&\quad+z^2\left(T_{2s}(x)-2x^{s+1}\right).
\end{aligned}$$

Eventually, we conclude that $Q_s(x,z) = z^{2s+1}U_s(x,z)$ as required. ■

In the above theorem, if we take $x=1$, it is obtained the generating function of the Tribonacci numbers in [12].

Corollary 11 *The generating function of the Tribonacci numbers $T_n(s)$ is given by*

$$Q_s(z) = \sum_{i=0}^{\infty} T_i(s) z^i = z^{2s+1}U_s(z),$$

where $U_s(z) = \frac{T_{2s+1}+z(T_{2s+2}-T_{2s+1})+z^2(T_{2s}-2)-\left(z^2+z^3\right)\frac{(1+z)^s}{(1-z)^{s+1}}}{(1-z-z^2-z^3)}$.

Theorem 12 *The generating function of the incomplete Tribonacci-Lucas polynomials $K_n^{(s)}(x)$ is given by*

$$W_s(x,z) = \sum_{n=0}^{\infty} K_n^{(s)}(x) z^n = z^{-1}Q_s(x,z) + \left(xz+2z^2\right)Q_{s-1}(x,z),$$

where $s>1$ and $Q_s(x,z)$ is the generating function of incomplete Tribonacci polynomials.

Proof. Let $W_s(x,z)$ be generating function of the incomplete Tribonacci-Lucas polynomials, that is $W_s(x,z) = \sum_{n=0}^{\infty} K_n^{(s)}(x) z^n$.

By using Proposition 4, Theorem 10 and the property of sum , we definitely have

$$\begin{aligned}
\sum_{n=0}^{\infty} K_n^{(s)}(x) z^n &= \sum_{n=0}^{\infty} \left(T_{n+1}^{(s)}(x) + xT_{n-1}^{(s-1)}(x) + 2T_{n-2}^{(s-1)}(x)\right) z^n \\
&= z^{-1}Q_s(x,z) + \left(xz+2z^2\right)Q_{s-1}(x,z).
\end{aligned}$$

■

For $x=1$ in Theorem 12, we can present the generating function of incomplete Tribonacci-Lucas numbers.

Corollary 13 *The generating function of the incomplete Tribonacci-Lucas numbers $K_n(s)$ is given by*

$$W_s(z) = \sum_{n=0}^{\infty} K_n(s) z^n = z^{-1}Q_s(z) + \left(z+2z^2\right)Q_{s-1}(z),$$

where $s>1$ and $Q_s(z)$ is the generating function of incomplete Tribonacci numbers.

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