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Solutions of polynomial Pell's equation

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A B S T R A C T

Let $D = F^2 + 2G$ be a monic quartic polynomial in $\mathcal{Z}[x]$, where $\deg G < \deg F$. Then for $F/G \in \mathcal{Q}[x]$, a necessary and sufficient condition for the solution of the polynomial Pell's equation $X^2 - DY^2 = 1$ in $\mathcal{Z}[x]$ has been shown. Also, the polynomial Pell's equation $X^2 - DY^2 = 1$ has nontrivial solutions $X, Y \in \mathcal{Q}[x]$ if and only if the values of period of the continued fraction of $\sqrt{D}$ are 2, 4, 6, 8, 10, 14, 18, and 22 has been shown. In this paper, for the period of the continued fraction of $\sqrt{D}$ is 4, we show that the polynomial Pell's equation has no nontrivial solutions $X, Y \in \mathcal{Z}[x]$.

Let D be a monic quartic polynomial with integer coefficients. We consider the polynomial Pell's equation

$$X^2 - DY^2 = 1 \tag{1}$$

where solutions X, Y are polynomials with integer coefficients. Solving Pell's equation in $\mathcal{Z}[x]$ has been studied by Mollin [2–6], Nathanson [7], Ramasamy [8], Webb and Yokota [9,10,12]. The authors [9] gave a necessary and sufficient condition for which the polynomial Pell's equation has a nontrivial solution in $\mathcal{Z}[x]$ in the case $D = F^2 + 2G$, $F, G \in \mathcal{Q}[x]$, and $F/G \in \mathcal{Q}[x]$. This gives a partial answer to the open problem which asks to determine the polynomial D for which Eq. (1) has nontrivial solutions in $\mathcal{Z}[x]$.

Given $D = F^2 + 2G$ with $\deg G < \deg F$, it is known [1,12] that $X^2 - DY^2 = 1$ is solvable in $\mathcal{Q}[x]$ if and only if the period of the continued fraction of $\sqrt{D}$ is one of the followings: 2, 4, 6, 8, 10, 14, 18, or 22. We recall that the period of the continued fraction of $\sqrt{D}$ is 2 if and only if $F/G \in \mathcal{Q}[x]$. So to answer the open problem for a monic quartic polynomial, we only need to consider the case where $D = F^2 + 2G$ with $F/G \notin \mathcal{Q}[x]$, and the period of the continued fraction of $\sqrt{D}$ is one of 4, 6, 8, 10, 14, 18, or 22.

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In [4], Mollin has shown that for $D_k(X) = (B_k - 1)^2 A_k^2 X^2 (A_k^2 X + 2)^2 + 2(B_k - 1)^2 X(A_k^2 X + 2) + 2(B_k - 1)X + C$, the period of continued fraction of $\sqrt{D_K}$ cannot be 4. Mollin [5] also has shown that for $d = (ba^k + \frac{a-1}{2b})^2 + 2a^k$, where a, b, k are natural numbers with $a \equiv 1 \pmod{2b}$, the length of continued fraction expansion of $\sqrt{d}$ is given by $4k + 2$.

With these evidence, we believe that the polynomial Pell's equation (1) has no nontrivial solution in $\mathcal{Z}[x]$ except for the case $F/G \in \mathcal{Q}[x]$.

In this paper, we give a partial answer to the open problem by showing the following:

Theorem 1. *Let D be a monic quartic polynomial in $\mathcal{Z}[x]$. Suppose that the period of the continued fraction of $\sqrt{D}$ is 4. Then the polynomial Pell's equation $X^2 - DY^2 = 1$ has no nontrivial solutions $X, Y \in \mathcal{Z}[x]$.*

Let $D = x^4 + ax^3 + bx^2 + cx + d \in \mathcal{Z}[x]$. Then we can rewrite D as

$$D = \left(x^2 + \frac{a}{2}x + \frac{4b - a^2}{8}\right)^2 + \frac{8c - a(4b - a^2)}{8}x + \frac{64d - (4b - a^2)^2}{64}.$$

For $8c - a(4b - a^2) = 0$, we can write D as $D = F^2 + 2G$, where $F/G \in \mathcal{Q}[x]$. Then as we have shown in [9], $\sqrt{D} = \langle F, \overline{F/G, 2F}\rangle$, and the period of the continued fraction of $\sqrt{D}$ is 2. Thus we assume $8c - a(4b - a^2) \neq 0$. Applying the linear translation $\tau : x \longrightarrow x - \frac{8}{8c - a(4b - a^2)} \cdot \frac{64d - (4b - a^2)^2}{64}$, we obtain

$$D^* = \left(x^2 + Ax + B\right)^2 + Cx,$$

where

$$A = \frac{2a(8c - a(4b - a^2)) - (64d - (4b - a^2)^2)}{4(8c - a(4b - a^2))}, \tag{2}$$

$$B = \frac{8(4b - a^2)(8c - a(4b - a^2))^2 + (64d - (4b - a^2)^2)^2}{64(8c - a(4b - a^2))^2} - \frac{4a(8c - a(4b - a^2))(64d - (4b - a^2)^2)}{64(8c - a(4b - a^2))^2}, \tag{3}$$

$$C = \frac{8c - a(4b - a^2)}{8}. \tag{4}$$

We note that by taking the linear translation τ, the period of the continued fraction of $\sqrt{D^*}$ and the period of the continued fraction of $\sqrt{D}$ are the same. Similarly, the leading coefficients of the numerator and the denominator of the third convergents P_3^*/Q_3^* are the same as the leading coefficients of the numerator and the denominator of the third convergents P_3/Q_3.

For $4b - a^2 = 0$ and $d = 0$, we have $B = 0$ which implies that $(x^2 + Ax)/Cx \in \mathcal{Q}[x]$. Thus, the period of the continued fraction of $\sqrt{D^*}$ is 2. So, we assume either $4b - a^2 \neq 0$ or $d \neq 0$.

Now by Lemma 1 below, the minimal solution of $X^2 - D^*Y^2 = 1$ is given by $P_3^* + Q_3^*\sqrt{D^*}$. Since every solution W of $X^2 - DY^2 = 1$ is generated by the minimal solution, we have $W = (P_3^* + Q_3^*\sqrt{D^*})^n = X_{n-1}^* + Y_{n-1}^*\sqrt{D}$. Similarly, every solution U of $X^2 - DY^2 = 1$ is given by $U = (P_3 + Q_3\sqrt{D})^n = X_{n-1} + Y_{n-1}\sqrt{D}$. We note that X_{n-1}^* and X_{n-1} can be expressed in the following way:

$$X_{n-1}^* = \sum_j \binom{n}{2j} \left(P_3^*\right)^{n-2j} \left(Q_3^*\right)^{2j} \left(D^*\right)^j,$$

$$X_{n-1} = \sum_j \binom{n}{2j} (P_3)^{n-2j} (Q_3)^{2j} (D)^j.$$

Then since D^* and D are monic, the leading coefficients of X^*_{n-1} and X_{n-1} are the same. Thus to show that there is no nontrivial solution in $\mathcal{Z}[x]$ for the polynomial Pell's equation $X^2 - DY^2 = 1$, it is enough to show that the leading coefficient of X^*_{n-1} is not in $\mathcal{Z}$.

Therefore, to prove Theorem 1, it is enough to show

Theorem 2. *Let D^* be defined above. Suppose that the period of the continued fraction of $\sqrt{D^*}$ is 4. Then the leading coefficient of X^*_{n-1} is not in $\mathcal{Z}[x]$.*

1. Background

As in [11], $\mathcal{K} = \mathcal{Q}((x^{-1}))$ is the field of formal Laurent series in x^{-1} over $\mathcal{Q}$. Then $\alpha \in \mathcal{K}$ implies that

$$\alpha = \sum_{j=t}^{\infty} a_j x^{-j}, \quad \text{where } a_j \in \mathcal{Q},\ a_t \neq 0,\ \operatorname{sgn}\alpha = a_t.$$

We define the non-archimedian absolute value by

$$|\alpha| = e^{-t}.$$

So, $|F/G| = e^{\deg F - \deg G}$ for $F, G \in \mathcal{Q}[x]$. We use the symbol $\lfloor \alpha \rfloor$ to mean the integer part of α:

$$\lfloor \alpha \rfloor = \sum_{j=t}^{0} a_j x^{-j} = a_t x^{-t} + \cdots + a_0 \in \mathcal{Q}[x].$$

For $D \in \mathcal{Z}[x]$, a continued fraction for $\sqrt{D}$ is obtained by putting $\alpha_0 = \sqrt{D}$ and, recursively for $n \geqslant 0$, putting

$$F_n = \lfloor \alpha_n \rfloor \quad \text{and} \quad \alpha_{n+1} = 1/(\alpha_n - F_n).$$

We define $M_0 = F$, $L_0 = 2G$, $L_{-1} = 1$. Then

$$\sqrt{D} = \sqrt{F^2 + 2G} = F + \frac{1}{\frac{\sqrt{F^2+L_0}+F}{2G}} = M_0 + \frac{1}{\frac{\sqrt{M_0^2+L_0}+M_0}{L_0}}.$$

Let $F_1 = \lfloor \frac{\sqrt{M_0^2+L_0}+M_0}{L_0} \rfloor$. Then $F_1 = \lfloor \frac{2M_0}{L_0} \rfloor$. Now write $2M_0 = F_1 L_0 + \varepsilon_0$, $\deg \varepsilon_0 < \deg L_0$.

Since $\frac{\sqrt{M_0^2+L_0}+M_0}{L_0} = F_1 + \frac{M_0^2+L_0-(L_0F_1-M_0)^2}{(\sqrt{M_0^2+L_0}+F_1L_0-M_0)L_0}$, we let $L_1 = \frac{M_0^2+L_0-(F_1L_0-M_0)^2}{L_0}$. Then $L_1 = 1 - F_1(F_1L_0 - 2M_0)$, $M_1 = M_0 - \varepsilon_0 = F_1L_0 - M_0$, and $D = M_1^2 + L_0L_1$.

Continue this, we have for $n \geqslant 1$,

$$\begin{aligned} M_n &= F_n L_{n-1} - M_{n-1}, \\ L_n &= L_{n-2} - F_n(F_n L_{n-1} - 2M_{n-1}), \\ F_{n+1} &= 2\left\lfloor \frac{M_n}{L_n} \right\rfloor, \\ D &= M_{n-1}^2 + L_{n-2}L_{n-1}. \end{aligned}$$

We write convergents to $\sqrt{D}$ as $P_n/Q_n = \langle F_0, F_1, \ldots, F_n\rangle$, where

$$\begin{pmatrix} P_n & Q_n \\ P_{n-1} & Q_{n-1} \end{pmatrix} = \begin{pmatrix} F_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_{n-1} & Q_{n-1} \\ P_{n-2} & Q_{n-2} \end{pmatrix} \quad \text{for } n \geqslant 0$$

and

$$\begin{pmatrix} P_{-1} & Q_{-1} \\ P_{-2} & Q_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $P_{n-1}^2 - DQ_{n-1}^2 = (-1)^n L_{n-1}$. Similarly, we write convergents to $\sqrt{D^*}$ as $P_n^*/Q_n^* = \langle F_0^*, F_1^*, \ldots, F_n^*\rangle$.

We will call $W = U + V\sqrt{D}$ a rational solution of (1) if $U^2 - DV^2 = 1$ and $U, V \in \mathcal{Q}[x]$. We define

$$T = \{U + V\sqrt{D}\colon\ U^2 - DV^2 = 1,\ \operatorname{sgn} U > 0,\ \operatorname{sgn} V > 0,\ \text{where } U, V \in \mathcal{Q}[x]\},$$

and T_0 to be the subset of T such that $U, V \in \mathcal{Z}[x]$.

Among all rational solutions in T, we say $P + Q\sqrt{D}$ is a minimal solution if and only if

$$|P + Q\sqrt{D}| \leqslant |U + V\sqrt{D}| \quad \text{for all } U + V\sqrt{D} \in T.$$

Then by Lemma 3 in [9], the minimal solution is unique, and every rational solution $W \in T$ can be expressed as $W = W_0^n$ for some $n \geqslant 1$, where W_0 is the minimal solution.

Let $\nu_2(m/n) = i - j$, where $(m, n) = 1, 2^i \| m, 2^j \| n$. For $A = x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$, denote the coefficient a_j of x^j in A by $[x^j]A$.

2. Lemmas

Here and in the sequel, we denote $D^* = (x^2 + AX + B)^2 + Cx$.

Lemma 1. *Suppose that the period of the continued fraction of $\sqrt{D^*}$ is 4. Then the minimal solution for the polynomial Pell's equation $X^2 - D^*Y^2 = 1$ is given by $P_3^* + Q_3^*\sqrt{D^*}$.*

Proof. Suppose that the minimal solution of $X^2 - D^*Y^2 = 1$ is given by $U^* + V^*\sqrt{D^*}$. Then by Lemma 2 in [9], $U^* = \lambda P_k^*$, $V^* = \lambda Q_k^*$ for some $\lambda \in \mathcal{Q}$ and $k \geqslant 0$. Thus

$$(U^*)^2 - D^*(V^*)^2 = \lambda^2\left((P_k^*)^2 - D^*(Q_k^*)^2\right) = \lambda^2(-1)^{k+1}L_k^*.$$

Now by direct calculation of L_k^*, we obtain

$$\begin{aligned}
L_1^* &= \frac{4B}{C}x + \frac{4AB + C}{C},\\
L_2^* &= \frac{C^2(4AB + C)}{16B^3}x + \frac{C^2(16B^3 - 4ABC - C^2)}{64B^4},\\
L_3^* &= \frac{64B^4(16B^3 - 4ABC - C^2)}{C^2(4AB + C)^3}x - \frac{512B^6(-8A^2B^2 + 8B^3 - 6ABC - C^2)}{C^2(4AB + C)^4}.
\end{aligned}$$

Since for $B = 0$, we know the period of the continued fraction of $\sqrt{D^*}$ is 2. Thus we assume $B \neq 0$, which implies that $L_1^* \notin \mathcal{Q}$. We note that $(P_3^*)^2 - D^*(Q_3^*)^2 = L_3^* = 1$ implies that $C \neq 0$ and

$4AB + C \neq 0$. Thus $L_2^* \notin \mathcal{Q}$. Then we have $|U^* + V^*\sqrt{D^*}| \geqslant |P_3^* + Q_3^*\sqrt{D^*}|$. Thus, $P_3^* + Q_3^*\sqrt{D^*}$ is the minimal solution. This proves the lemma. □

We now note that the period of $\sqrt{D^*}$ is 4 implies that $L_3^* = 1$, which in turn implies

$$16B^3 - 4ABC - C^2 = 0. \tag{5}$$

Lemma 2. *Suppose that the period of the continued fraction of $\sqrt{D^*}$ is 4 and the minimal solution is $P_3^* + Q_3^*\sqrt{D^*}$. Then $[x^5]P_3^* = \frac{2}{BC} = [x^3]Q_3^*$.*

Proof. By expanding $\sqrt{D^*}$ using the continued fraction, we obtain

$$\sqrt{D^*} = \left\langle x^2 + Ax + B, \overline{\frac{2(x+A)}{C}, \frac{4BCx - C^2}{8B^2}, \frac{2(x+A)}{C}, 2\left(x^2 + Ax + B\right)}\right\rangle.$$

Then

$$\begin{aligned}
P_3^* &= \frac{2(x+A)}{C} P_2^* + P_1^* \\
&= \frac{2(x+A)}{C}\left(\frac{4BCx - C^2}{8B^2} P_1^* + P_0^*\right) + P_1^* \\
&= \frac{2(x+A)}{C}\left(\frac{4BCx - C^2}{8B^2}\right)\left(\frac{2(x+A)}{C} P_0^* + 1\right) + \frac{2(x+A)}{C} P_0^* + P_1^* \\
&= \frac{2(x+A)}{C}\left(\frac{4BCx - C^2}{8B^2}\right)\left(\frac{2(x+A)}{C}\left(x^2 + Ax + B\right) + 1\right) + \frac{2(x+A)}{C} P_0^* + P_1^*, \\
Q_3^* &= \frac{2(x+A)}{C}\left(\frac{4BCx - C^2}{8B^2}\right)\left(\frac{2(x+A)}{C}\right) + \frac{2(x+A)}{C} + Q_1^*.
\end{aligned}$$

Therefore,

$$\left[x^5\right]P_3^* = \frac{2}{C}\frac{4BC}{8B^2}\frac{2}{C} = \frac{2}{BC} = \left[x^3\right]Q_3^*. \qquad \square$$

3. Main theorem

Proof of Theorem 2. Suppose contrary that the leading coefficient of X_{n-1}^* is in $\mathcal{Z}$. Then since

$$X_{n-1}^* = \sum_j \binom{n}{2j}\left(P_3^*\right)^{n-2j}\left(Q_3^*\right)^{2j}\left(D^*\right)^j$$

has the leading coefficient

$$\sum_j \binom{n}{2j}\left(\frac{2}{BC}\right)^n = 2^{n-1}\left(\frac{2}{BC}\right)^n,$$

we must have $\frac{2}{BC} \in \mathcal{Z}$. Let $\frac{2}{BC} = 2^l m$, where $l \geqslant 0$, $m \in \mathcal{Z}$. Then

$$BC = \frac{1}{2^{l-1}m}, \quad l \geqslant 0,\ m \in \mathcal{Z}.$$

Let $B = \frac{1}{2^s\alpha}$, $C = \frac{1}{2^t\beta}$, where $\alpha, \beta \in \mathcal{Q}$ and $\nu_2(\alpha) = \nu_2(\beta) = 0$. Then since $BC = \frac{1}{2^{l-1}m}$, we have $\alpha\beta = m$ and $s + t = l - 1$.

Putting $C = \frac{1}{2^t\beta}$ into Eq. (4), we have $\frac{2^{3-t}}{\beta} = 8c - a(4b - a^2) \in \mathcal{Z}$, which implies that $\beta = \frac{1}{k}$, $k \in \mathcal{Z}$, which in turn implies that $\alpha = km \in \mathcal{Z}$ and

$$k2^{3-t} = 8c - a\big(4b - a^2\big). \tag{6}$$

This shows that $t \leqslant 3$.

Now by replacing $C = \frac{k}{2^t}$, $B = \frac{1}{2^s\alpha}$ in Eq. (5), we have

$$A = \frac{16B^3 - C^2}{4BC} = \left(\frac{2^{-2s+t-2}}{k\alpha^2} - 2^{s-t-2}k\alpha\right).$$

Also by replacing $8c - a(4b - a^2)$ of Eq. (2) by $k2^{3-t}$, we obtain

$$\begin{aligned} A &= \frac{2a(8c - a(4b - a^2)) - (64d - (4b - a^2)^2)}{4(8c - a(4b - a^2))} \\ &= \frac{k2^{4-t}a - (64d - (4b - a^2)^2)}{k2^{5-t}}. \end{aligned}$$

Equating these two equations, we have

$$\frac{2^{-2s+3}}{\alpha^2} - k^2 2^{s-2t+3}\alpha = k2^{4-t}a - \big(64d - \big(4b - a^2\big)^2\big). \tag{7}$$

We note that since $t \leqslant 3$, the right-hand side of Eq. (9) is in $\mathcal{Z}$, which implies that $\frac{1}{\alpha^2} \in \mathcal{Z}$. But since $\alpha = km \in \mathcal{Z}$, we must have $\alpha^2 = 1$ and $k^2 = 1$.

This shows that

$$2^{-2s+3} - 2^{s-2t+3}\alpha = 2^{4-t}ak - \big(64d - \big(4b - a^2\big)^2\big). \tag{8}$$

Suppose first that a is odd. Then by Eq. (8), $64d - (4b - a^2)^2$ is even. But this is impossible, since $64d - (4b - a^2)^2$ is odd for a odd.

So we assume that a is even. Then by letting $a = 2a'$ in Eq. (6), we have $k2^{3-t} = 8(c - a'(b - a'^2)) \in \mathcal{Z}$, which implies that $t \leqslant 0$. By letting $a = 2a'$ in Eq. (8), we have

$$2^{-2s+3} - 2^{s-2t+3}\alpha = 2^{4-t}ak - 2^4\big(4d - \big(b - a'^2\big)^2\big). \tag{9}$$

Since $t \leqslant 0$, dividing both sides of Eq. (9) by 2^4, we have

$$2^{-2s-1} - 2^{s-2t-1}\alpha = 2^{-t}ak - \big(4d - \big(b - a'^2\big)^2\big) \in \mathcal{Z}, \tag{10}$$

which in turn implies that either $-2s - 1 = s - 2t - 1$ or $-2s - 1 \geqslant 0$, $s - 2t - 1 \geqslant 0$. The first case implies that $3s = 2t$ and the second case implies that $s \leqslant -1$.

We first treat the first case. Since $t \leqslant 0$ and $3s = 2t$, we have $s \leqslant 0$. Thus, in this case, we have $A = 0$, $B, C \in \mathcal{Z}$ which contradicts $\frac{2}{BC} \in \mathcal{Z}$.

Thus, we are left with the second case. Since $t+s = l-1$ and $t \leqslant 0$, $l \geqslant 0$, we have $s \geqslant -1-t \geqslant -1$. Therefore $s = -1$. Then

$$B = \pm 2, \qquad C = \pm\frac{1}{2^t}, \qquad A = \pm\left(2^t - 2^{-t-3}\right)$$

and

$$4d - \left(b - a'^2\right)^2 = \pm 2^{-t} a - 2 + 2^{-2t-2}.$$

Then since $8c - a(4b - a^2) = \pm 2^{3-t}$, we have

$$\tau^{-1}\colon \quad x + \left(\pm 2^{-2} a - 2^{t-1} + 2^{-t-4}\right).$$

Now we calculate D for $A = 2^t - 2^{-t-3}$, $B = 2$, $C = \frac{1}{2^t}$, and $u = x + 2^{-2}a - 2^{t-1} + 2^{-t-4}$.

$$\begin{aligned} D &= \left(u^2 + \left(2^t - 2^{-t-3}\right)u + 2\right)^2 + \frac{u}{2^t} \\ &= x^4 + ax^3 + \left(\frac{33+3a^2}{8} - \frac{2^{-2t}}{128} - \frac{3}{2^{1-2t}}\right)x^2 + \cdots. \end{aligned}$$

We claim that the coefficient of x^2 is not in $\mathcal{Z}$. For $t = 0$,

$$\frac{33+3a^2}{8} - \frac{2^{-2t}}{128} - \frac{3}{2^{1-2t}} = \frac{16(33+3a^2) - 1 - 192}{128}$$

and the numerator is odd. This shows that the coefficient of x^2 is not in $\mathcal{Z}$. For $t = -1$,

$$\frac{33+3a^2}{8} - \frac{2^{-2t}}{128} - \frac{3}{2^{1-2t}} = \frac{4(33+3a^2) - 1 - 12}{32}$$

and the numerator is odd. This shows that the coefficient of x^2 is not in $\mathcal{Z}$. For $t \leqslant -2$,

$$\frac{33+3a^2}{8} - \frac{2^{-2t}}{128} - \frac{3}{2^{1-2t}} = \frac{2^{-2-2t}(33+3a^2) - 2^{-6-4t} - 3}{2^{1-2t}}$$

and the numerator is odd. This shows that the coefficient of x^2 is not in $\mathcal{Z}$. Therefore, D is not in $\mathcal{Z}[x]$, which is impossible.

For $A = -(2^t - 2^{-t-3})$, $B = -2$, $C = -\frac{1}{2^t}$, and $u = x - 2^{-2}a - 2^{t-1} + 2^{-t-4}$, we have

$$\begin{aligned} D &= \left(u^2 - \left(2^t - 2^{-t-3}\right)u - 2\right)^2 - \frac{u}{2^t} \\ &= x^4 + \left(2^{-1-t} - 2^{2+t} - a\right)x^3 + \cdots. \end{aligned}$$

For $t = 0$, the coefficient of x^3 is not in $\mathcal{Z}$. For $t \leqslant -3$,

$$2^{-1-t} - 2^{2+t} - a = \frac{2^{-3-2t} - 1 - 2^{-2-t}a}{2^{-2-t}}$$

and the numerator is odd and the denominator is even. Thus the coefficient of x^3 is not in $\mathcal{Z}$. Now we are left with the case $t = -1$ and $t = -2$. For $t = -1$, we look at the coefficient of x^2. Since $t = -1$, we have $A = -(\frac{1}{2} - \frac{1}{4}) = -\frac{1}{4}$, $B = -2$, $C = -\frac{1}{2}$, and $u = x - \frac{a}{4} - \frac{1}{8}$. Then

$$D = \left(u^2 - \frac{u}{4} - 2\right)^2 - \frac{u}{2}$$
$$= x^4 - (a+1)x^3 + \frac{2(3a^2 - a^3) - 117 + 24a + 9a^2}{32}x^2 + \cdots.$$

Thus, the coefficient of x^2 is not in $\mathcal{Z}$. Finally for $t = -2$. Note that in this case, we have $A = \frac{1}{4}$, $B = -2$, $C = -\frac{1}{4}$, and $u = x - \frac{a}{4} + \frac{1}{8}$. Then

$$D = \left(u^2 + \frac{u}{4} - 2\right)^2 - \frac{u}{2}$$
$$= x^4 + (1-a)x^3 + \frac{2(3a^2 - a^3) - 117 - 24a + 9a^2}{32}x^2 + \cdots.$$

Thus, the coefficient of x^2 is not in $\mathcal{Z}$. Therefore, the leading coefficient of X^*_{n-1} is not an integer. □

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