

# Apéry Numbers, Jacobi Sums, and Special Values of Generalized $p$ -adic Hypergeometric Functions

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We present some summation formulae for some special values of ratios of generalized  $p$ -adic hypergeometric functions in terms of the  $p$ -adic gamma function. By well-known methods such formulae yield expressions for roots of congruence zeta-functions in terms of Jacobi sums in the nonsingular case. We also express various formal-group congruences, including those involving Apéry numbers, in terms of  $p$ -adic hypergeometric functions at singular and nonsingular points of the associated differential equation. These congruences yield  $p$ -adic analytic formulae for unit roots of certain Hecke polynomials. © 1992 Academic Press, Inc.

## 1. INTRODUCTION

In this article we present some explicit formulae for values at  $x = 1$  and at  $x = -1$  of  $p$ -adic analytic continuations of ratios of generalized hypergeometric functions. For certain cases of  ${}_3F_2$  and  ${}_2F_1$  functions we evaluate the continuation in terms of the  $p$ -adic gamma function, in forms which yield products of Jacobi sums (via the Gross–Koblitz formula). We also express certain formal-group congruences satisfied by Apéry numbers (cf. [19, 17, 2]) and by binomial coefficients (cf. [4]) in terms of  $p$ -adic hypergeometric functions.

The  $p$ -adic analogue of Gauss' evaluation of  ${}_2F_1(x, \beta; \gamma; 1)$  has been treated by N. Koblitz [13] and J. Diamond [5]. In Section 3 we obtain similar  $p$ -adic analogues of the classical theorems of Kummer, Saalschütz, Dixon, and Watson via various combinatorial identities and the basic properties of the  $p$ -adic gamma function. While we do not give a complete answer to the question, "When may the ratio  $F(x)/F(x^p)$  be prolonged to  $x = 1$  or  $x = -1$ ?", we do give formulae for the value of the continuation in many cases where the value is a  $p$ -adic unit which is expressible in terms of Jacobi sums over the prime field  $\mathbb{F}_p$ .

The requirement that one should obtain Jacobi sums over  $\mathbb{F}_p$  amounts essentially to a restriction of the hypergeometric parameters to be rational numbers in  $(0, 1]$  with denominator dividing  $p - 1$ . This restriction has the advantage of simplifying certain considerations in the  $p$ -adic theory of hypergeometric functions. When the parameters do not lie in  $(1/(p - 1))\mathbb{Z}$  the condition necessary for a unit-valued continuation becomes more involved (cf. e.g., Eq. (3.2) below for the case of  ${}_2F_1(1)$  with  $\gamma = 1$ ; also hypothesis (ii) of Theorem 2.3), just as the condition for Jacobi sums over  $\mathbb{F}_{p^r}$  ( $r > 1$ ) to be  $p$ -adic units is more involved than for those over  $\mathbb{F}_p$ . However, in some cases the results in Section 3 hold in greater generality than the stated hypotheses.

Restricting our attention to  $\mathbb{F}_p$  also facilitates an interpretation of our results in the language of formal group laws. Corollary 2.2 below is essentially an expression of congruences of the Atkin–Swinnerton–Dyer type for binomial coefficients in terms of Jacobi sums. One may likewise view Theorems 3.1 and 3.2 below as expressions of formal-group congruences for products of binomial coefficients in terms of  $p$ -adic hypergeometric functions. In Theorem 4.1 we show how the formal-group congruences associated to Apéry numbers may also be viewed in this manner, and use this interpretation to express the  $p$ -adic unit roots of certain Hecke polynomials in terms of  $p$ -adic hypergeometric functions (Corollary 4.2).

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## 2. NOTATIONS AND PRELIMINARIES

Throughout this paper  $p$  will denote an odd prime,  $\mathbb{F}_p$  the finite field of  $p$  elements,  $\mathbb{Z}_p$  the ring of  $p$ -adic integers,  $\mathbb{Q}_p$  the field of  $p$ -adic numbers,  $\mathbb{C}_p$  the completion of an algebraic closure of  $\mathbb{Q}_p$ , and  $\mathfrak{O}$  the ring of integers of  $\mathbb{C}_p$ . We let  $\pi \in \mathfrak{O}$  be a fixed solution to  $\pi^{p-1} = -p$  and let  $\zeta$  be the unique  $p$ th root of unity in  $\mathfrak{O}$  such that  $\zeta \equiv 1 + \pi \pmod{\pi^2\mathfrak{O}}$ .

The  $p$ -adic gamma function  $\Gamma_p$  is defined by setting  $\Gamma_p(0) = 1$ , and for positive integers  $n$  by

$$\Gamma_p(n) = (-1)^n \prod_{\substack{0 < j < n \\ p \nmid j}} j. \tag{2.1}$$

If  $x, y \in \mathbb{Z}^+$  and  $x \equiv y \pmod{p'\mathbb{Z}}$  then  $\Gamma_p(x) \equiv \Gamma_p(y) \pmod{p'\mathbb{Z}}$ ; therefore the function has a unique extension to a continuous function  $\Gamma_p: \mathbb{Z}_p \rightarrow \mathbb{Z}_p^\times$ , which is Lipschitz with constant 1 [20, Corollary 3.3]. There are functional equations of translation and reflection for  $\Gamma_p$ :

$$\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x), & x \in \mathbb{Z}_p^\times; \\ -\Gamma_p(x), & x \in p\mathbb{Z}_p; \end{cases} \tag{2.2}$$

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{\hat{x}}, \quad x \in \mathbb{Z}_p, \tag{2.3}$$

where  $\hat{x}$  is defined by the conditions  $\hat{x} \in \{1, 2, \dots, p\}$  and  $\hat{x} \equiv x \pmod{p\mathbb{Z}_p}$ . In addition  $\Gamma_p$  satisfies the Gauss multiplication formula

$$\prod_{i=0}^{n-1} \Gamma_p\left(\frac{x+i}{n}\right) = \Gamma_p(x) \prod_{i=0}^{n-1} \Gamma_p\left(\frac{i}{n}\right) n^{1-\hat{x}} (n^{-(p-1)}(x-\hat{x})/p) \tag{2.4}$$

for all  $x \in \mathbb{Z}_p$  and  $n \in \mathbb{Z}^+$  with  $p \nmid n$  (cf. [4]), it being understood that the last factor is obtained from the binomial expansion when the exponent is not an integer.

The relation of the *p*-adic gamma function to Gauss sums and Jacobi sums is as follows. If  $\psi: \mathbb{F}_p \rightarrow \mathbb{Q}_p(\zeta)$  is a fixed nontrivial additive character and  $\chi: \mathbb{F}_p^\times \rightarrow \mathbb{Q}_p$  is a multiplicative character (extended to  $\mathbb{F}_p$  by defining  $\chi(0) = 0$ ), the Gauss sum  $g(\chi) \in \mathbb{Q}_p(\zeta)$  is defined by

$$g(\chi) = - \sum_{t \in \mathbb{F}_p} \psi(t) \chi(t). \tag{2.5}$$

For the remainder of this paper we fix the additive character  $\psi$  on  $\mathbb{F}_p$  defined by  $\psi(t) = \zeta^t$  for all  $t \in \mathbb{Z}$ . The Teichmüller character  $\omega$  on  $\mathbb{F}_p$  is the unique multiplicative character satisfying  $\omega(t) \equiv t \pmod{p\mathbb{Z}_p}$  for all  $t \in \mathbb{Z}$ . The Gross–Koblitz formula [10] for Gauss sums states that for  $0 < a < p - 1$  we have

$$g(\omega^{-a}) = \pi^a \Gamma_p\left(\frac{a}{p-1}\right). \tag{2.6}$$

If  $\chi_1, \chi_2: \mathbb{F}_p^\times \rightarrow \mathbb{Q}_p$  are multiplicative characters (again extended to  $\mathbb{F}_p$  by  $\chi_i(0) = 0$ ), the Jacobi sum  $J(\chi_1, \chi_2) \in \mathbb{Q}_p$  is defined by

$$J(\chi_1, \chi_2) = - \sum_{t \in \mathbb{F}_p} \chi_1(t) \chi_2(1-t). \tag{2.7}$$

We have the well-known relation

$$J(\chi_1, \chi_2) = \frac{g(\chi_1)g(\chi_2)}{g(\chi_1\chi_2)} \tag{2.8}$$

between the Gauss sums and Jacobi sums (cf. [4]). From the Gross–Koblitz formula we conclude that for  $a, b < a + b < p - 1$ ,

$$J(\omega^{-a}, \omega^{-b}) = \frac{\Gamma_p\left(\frac{a}{p-1}\right)\Gamma_p\left(\frac{b}{p-1}\right)}{\Gamma_p\left(\frac{a+b}{p-1}\right)}. \tag{2.9}$$

It follows that to obtain Jacobi sums over  $\mathbb{F}_p$ , we may take our arguments in  $\Gamma_p$  to be rational numbers in  $(0, 1)$  with denominators dividing  $p - 1$ .

We now give an elementary proposition expressing certain ratios of binomial coefficients in terms of  $\Gamma_p$ , which will be useful in the sequel.

**PROPOSITION 2.1.** *Suppose  $0 < a < b < p - 1$ , and for  $r \geq 0$  set  $n_r = b(p^r - 1)/(p - 1)$ ,  $m_r = a(p^r - 1)/(p - 1)$ . Then for  $r > 0$ ,*

$$\frac{\binom{n_r}{m_r}}{\binom{n_{r-1}}{m_{r-1}}} = -\frac{\Gamma_p(1 + n_r)}{\Gamma_p(1 + m_r) \Gamma_p(1 + n_r - m_r)} = \frac{\Gamma_p(-m_r) \Gamma_p(m_r - n_r)}{\Gamma_p(-n_r)}.$$

*Proof.* From the definition one computes

$$-\Gamma_p(1 + n_r) = (-1)^{n_r} p^{n_r - 1} \frac{n_r!}{n_{r-1}!},$$

and similarly for  $1 + m_r$  and  $1 + n_r - m_r$ , from which the first equality follows. The second equality is then obtained from the reflection formula for  $\Gamma_p$ , by noting that  $(\widehat{-m_r}) + (\widehat{m_r - n_r}) - (\widehat{-n_r}) = (p - a) + (p + a - b) - (p - b) = p$ , and  $(-1)^p = -1$ .

Note that by induction, this proposition implies that for each  $r \geq 0$ , the binomial coefficient  $\binom{n_r}{m_r}$  is a  $p$ -adic unit. Using the Gross–Koblitz formula and the fact that  $\Gamma_p$  is Lipschitz with constant 1, we also obtain the following corollary (cf. [20, Eq. (35); 4, Eq. (29)]):

**COROLLARY 2.2.** *With notation as in Proposition 2.1, for each  $r > 0$  we have*

$$\frac{\binom{n_r}{m_r}}{\binom{n_{r-1}}{m_{r-1}}} \equiv J(\omega^{-a}, \omega^{a-b}) \pmod{p^r \mathbb{Z}_p}.$$

In order to discuss the  $p$ -adic theory of hypergeometric functions, we first define a map  $a \mapsto a'$  on  $\mathbb{Q} \cap \mathbb{Z}_p$  by requiring that  $pa' - a = \mu_a \in \{0, 1, 2, \dots, p - 1\}$  (see [6, 7]). Note that  $\mu_a + \hat{a} = p$ . We write  $a^{(0)} = a$ , and  $a^{(i)} = a^{(i-1)'}$  for  $i > 0$ ; we also will write  $\mu_a^{(i)}$  for  $\mu_{a^{(i)}}$ . It is easy to verify that this map is well-defined and continuous, that  $a^{(i)} = 0$  for some  $i$  if and only if  $a$  is zero or a negative integer, and that  $a' = a$  if and only if  $a$  is a rational number in  $[0, 1]$  with denominator dividing  $p - 1$ . We also note

for future reference that if  $0 < a < p - 1$  and  $m_r = a(p' - 1)/(p - 1)$ , then  $(-m_r)' = -m_{r-1}$  and  $(1 + m_r)' = 1 + m_{r-1}$ .

The generalized hypergeometric function  ${}_kF_{k-1}(\alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_{k-1}; x)$  is defined for (complex or *p*-adic) parameters  $\alpha_i, \gamma_j$  by the series

$${}_kF_{k-1} \left( \begin{matrix} \alpha_1, \dots, \alpha_k \\ \gamma_1, \dots, \gamma_{k-1} \end{matrix} ; x \right) = \sum_{s=0}^{\infty} \frac{(\alpha_1)_s \cdots (\alpha_k)_s}{(\gamma_1)_s \cdots (\gamma_{k-1})_s s!} x^s, \tag{2.10}$$

for all values of  $x$  for which it converges. We require that no  $\gamma_j$  be zero or a negative integer, since in that case the series is undefined. In the case where some  $\alpha_i$  is zero or a negative integer, the series defines a polynomial. If the parameters satisfy

$$1 + \alpha_1 = \gamma_1 + \alpha_2 = \cdots = \gamma_{k-1} + \alpha_k, \tag{2.11}$$

the series is said to be *well-poised*, and if

$$\alpha_1 + \cdots + \alpha_k = \gamma_1 + \cdots + \gamma_{k-1} - 1, \tag{2.12}$$

the series is said to be *Saalschützian*.

The function (2.10) satisfies the linear algebraic differential equation

$$\left( E \prod_{j=1}^{k-1} (E - \gamma_j + 1) - x \prod_{i=1}^k (E + \alpha_i) \right) y = 0, \tag{2.13}$$

[1, p. 184] where  $E = x (d/dx)$ ; this equation is of the Fuchsian type with regular singularities at  $x = 0, 1$ , and  $\infty$ . Our formulae (Sections 3, 4) for the values at  $x = 1$  may thus be viewed as being associated to the singular fibres of families of varieties.

In [6, 7], B. Dwork has shown that for general  $\alpha_i, \gamma_j \in \mathbb{Q} \cap \mathbb{Z}_p$ , a certain ratio of the hypergeometric functions (2.10) can be extended as an analytic element (i.e., uniform limit of rational functions) to a domain larger than the disk of convergence ( $|x| < 1$ ) of the series. The particular statement of this result we will use is as follows:

**THEOREM 2.3 (Dwork).** *Suppose that  $\alpha_1, \dots, \alpha_k, \gamma_1, \dots, \gamma_{k-1} \in \mathbb{Q} \cap \mathbb{Z}_p$ , that none of the  $\gamma_j$  are zero or negative integers, that  $\gamma_j \neq 1$  for  $1 \leq j \leq q$  and  $\gamma_j = 1$  for  $q < j \leq k - 1$ . For  $i \geq 0$  set*

$$F^{(i)}(X) = \sum_{n=0}^{\infty} A^{(i)}(n) X^n = {}_kF_{k-1} \left( \begin{matrix} \alpha_1^{(i)}, \dots, \alpha_k^{(i)} \\ \gamma_1^{(i)}, \dots, \gamma_{k-1}^{(i)} \end{matrix} ; X \right) \in \mathbb{Q}_p[[X]],$$

and for  $i, s \geq 0$  set  $F_s^{(i)}(X) = \sum_{n=0}^{p^s-1} A^{(i)}(n) X^n$ . Suppose further that the parameters satisfy the conditions

- (i)  $|\gamma_j^{(i)}| = 1$  for all  $i \geq 0, j = 1, \dots, k - 1$ .

(ii) For each fixed  $i \geq 0$ , supposing the indices are rearranged so that  $\mu_{\alpha_1}^{(i)} \leq \dots \leq \mu_{\alpha_k}^{(i)}$  and  $\mu_{\gamma_1}^{(i)} \leq \dots \leq \mu_{\gamma_q}^{(i)}$ , we have  $\mu_{\gamma_j}^{(i)} > \mu_{\alpha_{r+1}}^{(i)}$  for  $j = 1, \dots, q$ .

Then

(i) For all  $i \geq 0$  we have  $F^{(i)}(X) \in \mathbb{Z}_p \llbracket X \rrbracket$ , and for  $r \geq s \geq 0$ ,

$$F_{r+1}^{(0)}(X) F_s^{(1)}(X^p) \equiv F_r^{(1)}(X^p) F_{s+1}^{(0)}(X) \pmod{p^{s+1} \mathbb{Z}_p \llbracket X \rrbracket}.$$

(ii) If  $\mathfrak{D} = \{x \in \mathbb{C}_p : |F_1^{(i)}(x)| = 1 \text{ for all } i \geq 0\}$ , then  $F^{(0)}(x)/F^{(1)}(x^p)$  is the restriction to  $|x| < 1$  of an analytic element  $\mathfrak{f}$  of support  $\mathfrak{D}$ , given by

$$\mathfrak{f}(x) = \lim_{r \rightarrow \infty} F_{r+1}^{(0)}(x)/F_r^{(1)}(x^p)$$

uniformly for  $x \in \mathfrak{D}$ .

*Proof.* When none of the  $\alpha_j$  are zero or negative integers, the result is given by Dwork in [6, Theorem 2, Theorem 3; 7, Lemma 2.2, Theorem 3.1]; the hypothesis (ii) above is a restatement of hypothesis (vi) of [7, p. 303] in the special case  $\beta = 0$ . But if any  $\alpha_j$  is zero or a negative integer, then (ii) is trivial and (i) may be obtained by approximating the  $\alpha_j$  which are non-positive integers by suitable  $p$ -adic numbers which are not non-positive integers, and using a continuity argument.

As in [13, 5], we also remark that, since the functions  $F_r^{(m)}(X)$  are rational functions of the  $\alpha_i^{(m)}, \gamma_j^{(m)}$ , if  $x_0$  lies in the domain  $\mathfrak{D}$  of support for the prolongment of the ratio associated to  ${}_k F_{k-1}(\alpha_1, \dots, \alpha_k; \gamma_1, \dots, \gamma_{k-1}; x)$  for all  $(\alpha_i, \gamma_j)$  lying in some subset  $S$  of  $(\mathbb{Q} \cap \mathbb{Z}_p)^{2k-1}$ , then because of the uniformity of the limit in (ii) above,  $\mathfrak{f}(x_0)$  is a continuous function of the parameters  $(\alpha_i, \gamma_j)$  on the set  $S$ .

In the remainder of this article we will use the symbol

$${}_k \mathfrak{F}_{k-1} \left( \begin{matrix} \alpha_1, \dots, \alpha_k \\ \gamma_1, \dots, \gamma_{k-1} \end{matrix} ; x \right) \tag{2.14}$$

to denote the analytic element  $\mathfrak{f}$  of support  $\mathfrak{D}$  which extends the ratio

$$\frac{{}_k F_{k-1} \left( \begin{matrix} \alpha_1, \dots, \alpha_k \\ \gamma_1, \dots, \gamma_{k-1} \end{matrix} ; x \right)}{{}_k F_{k-1} \left( \begin{matrix} \alpha'_1, \dots, \alpha'_k \\ \gamma'_1, \dots, \gamma'_{k-1} \end{matrix} ; x^p \right)}. \tag{2.15}$$

### 3. JACOBI SUMS AND $p$ -ADIC HYPERGEOMETRIC FUNCTIONS

In [13], Koblitz has proven that

$${}_2 \mathfrak{F}_1 \left( \begin{matrix} \alpha, \beta \\ 1 \end{matrix} ; 1 \right) = \frac{\Gamma_p(\alpha) \Gamma_p(\beta)}{\Gamma_p(\alpha + \beta)} \tag{3.1}$$

when  $\alpha, \beta \in \mathbb{Z}_p$  satisfy the conditions

$$a_i + b_i < p \quad \text{for } i \geq 0, \tag{3.2}$$

where  $-\alpha = a_0 + a_1p + a_2p^2 + \dots$ ,  $-\beta = b_0 + b_1p + b_2p^2 + \dots$  are the  $p$ -adic expansions of  $-\alpha, -\beta$ , and that there is a unit-valued continuation to  $x = 1$  (i.e.,  $1 \in \mathfrak{D}$ ) if and only if (3.2) holds. For  $a, b < a + b < p - 1$ , this yields the Jacobi sum (over  $\mathbb{F}_p$ )

$${}_2\mathfrak{F}_1 \left( \begin{matrix} a & b \\ p-1 & p-1 \\ 1 \end{matrix} ; 1 \right) = J(\omega^{-a}, \omega^{-b}). \tag{3.3}$$

Koblitz' method is to begin with the Vandermonde convolution formula

$$\binom{m+n}{n} = \sum_{k=0}^{\min(m,n)} \binom{m}{k} \binom{n}{k} = {}_2F_1 \left( \begin{matrix} -m, -n \\ 1 \end{matrix} ; 1 \right) \tag{3.4}$$

for appropriate  $m, n \in \mathbb{Z}^+$ , and take ratios and  $p$ -adic limits. This basic method is also applicable in other situations. We begin by proving a  $p$ -adic analogue of Kummer's theorem [15], which gives the value at  $x = -1$  of a well-poised  ${}_2F_1$  series. This classical formula demonstrates that a well-poised  ${}_2F_1(-1)$  may be expressed in terms of gamma functions even though  $x = -1$  is not a singular point of the differential equation (2.13) satisfied by the  ${}_2F_1$ . Since  $x = -1$  is indeed an ordinary point, the present theorem relates to the cohomology of curves.

**THEOREM 3.1.** *Suppose  $0 < 2a \leq b < p - 1$  and  $2(b - a) < p - 1$ . Then*

$${}_2\mathfrak{F}_1 \left( \begin{matrix} 2a & b \\ p-1 & p-1 \\ 1 + \frac{2a-b}{p-1} \end{matrix} ; -1 \right) = (-1)^a \frac{\Gamma_p \left( \frac{a}{p-1} \right) \Gamma_p \left( \frac{b-a}{p-1} \right)}{\Gamma_p \left( \frac{2a}{p-1} \right) \Gamma_p \left( \frac{b-2a}{p-1} \right)}.$$

*Proof.* If  $n \geq 2m \geq 0$ , then

$$\binom{n}{2m-k} \binom{n-2m+k}{k} = \binom{n}{2m} \binom{2m}{k} \tag{3.5}$$

[14, p. 3]. Substituting this in the combinatorial identity

$$(-1)^m \binom{n}{m} = \sum_{k=0}^{2m} (-1)^k \binom{n}{k} \binom{n}{2m-k} \tag{3.6}$$

[14, p. 14], yields

$$(-1)^m \binom{n}{m} = \binom{n}{2m} \sum_{k=0}^{2m} (-1)^k \frac{\binom{n}{k} \binom{2m}{k}}{\binom{n-2m+k}{k}}, \tag{3.7}$$

which we rewrite as

$$(-1)^m \frac{\binom{n}{m}}{\binom{n}{2m}} = {}_2F_1 \left( \begin{matrix} -2m, -n \\ 1+n-2m \end{matrix}; -1 \right). \tag{3.8}$$

For  $r \geq 0$  let  $n = n_r = b(p^r - 1)/(p - 1)$ ,  $m = m_r = a(p^r - 1)/(p - 1)$ ; then  $n_r \geq 2m_r$ . From Proposition 2.1 we see that  $\binom{n_r}{m_r}, \binom{n_r}{2m_r} \in \mathbb{Z}_p^\times$ ; therefore from (3.5) and (3.7) we find that

$${}_2F_1 \left( \begin{matrix} -2m_r, -n_r \\ 1+n_r-2m_r \end{matrix}; X \right) \in \mathbb{Z}_p[X] \tag{3.9}$$

and (3.8) implies that the value at  $x = -1$  is a  $p$ -adic unit, for each  $r \geq 0$ . Now as  $r \rightarrow \infty$ ,  $-2m_r \rightarrow \alpha = 2a/(p - 1)$ ,  $-n_r \rightarrow \beta = b/(p - 1)$ , and  $1 + n_r - 2m_r \rightarrow \gamma = c/(p - 1)$  ( $p$ -adically), where  $c = p - 1 + 2a - b$ . Note that  $\alpha^{(i)} = \alpha$ ,  $\beta^{(i)} = \beta$ ,  $\gamma^{(i)} = \gamma$ , and  $\mu_x^{(i)} = 2a$ ,  $\mu_\beta^{(i)} = b$ ,  $\mu_\gamma^{(i)} = c$  for all  $i \geq 0$ . Since  $b < p - 1$ ,  $\mu_\gamma^{(i)} = c > 2a = \mu_\alpha^{(i)}$ ; and since  $2(b - a) < p - 1$ , we have  $\mu_\gamma^{(i)} = p - 1 + 2(a - b) + b > b = \mu_\beta^{(i)}$ . Therefore  $\mu_\gamma^{(i)} > \mu_\alpha^{(i)}, \mu_\beta^{(i)}$ , so the conditions of Theorem 2.3 are satisfied for  $F(X) = {}_2F_1(\alpha, \beta; \gamma; X)$ , and the ratio  $F(X)/F(X^p)$  extends to the region  $\mathfrak{D}$  where each  $|F_1^{(i)}(x)| = 1$ .

By taking  $F(X) = {}_2F_1(-2m_r, -n_r; 1 + n_r - 2m_r; X)$  and  $s = 0$  in Theorem 2.3 (i), we find that, since each  $|F^{(i)}(-1)| = 1$  and  $F^{(i)} = F_r^{(i)}$ , we have  $|F_1^{(i)}(-1)| = 1$  for all  $i, r \geq 0$ ; because of the continuous dependence of the polynomial  $F_1^{(i)}$  on its parameters, this is also true for  $F(X) = {}_2F_1(\alpha, \beta; \gamma; X)$ . Therefore the ratio can be prolonged to the value  $x = -1$ , and the value is given by

$$\begin{aligned} {}_2\mathfrak{F}_1 \left( \begin{matrix} \alpha, \beta \\ \gamma \end{matrix}; -1 \right) &= \lim_{r \rightarrow \infty} \frac{{}_2F_1 \left( \begin{matrix} -2m_r, -n_r \\ 1+n_r-2m_r \end{matrix}; -1 \right)}{{}_2F_1 \left( \begin{matrix} -2m_{r-1}, -n_{r-1} \\ 1+n_{r-1}-m_{r-1} \end{matrix}; (-1)^p \right)} \\ &= \lim_{r \rightarrow \infty} (-1)^{m_r - m_{r-1}} \frac{\binom{n_r}{m_r} \binom{n_{r-1}}{2m_{r-1}}}{\binom{n_{r-1}}{m_{r-1}} \binom{n_r}{2m_r}} \end{aligned}$$



$$\begin{aligned}
 &= (-1)^a \frac{J(\omega^{-a}, \omega^{a-b})}{J(\omega^{-2a}, \omega^{2a-b})} \\
 &= (-1)^a \frac{\Gamma_p\left(\frac{a}{p-1}\right) \Gamma_p\left(\frac{b-a}{p-1}\right)}{\Gamma_p\left(\frac{2a}{p-1}\right) \Gamma_p\left(\frac{b-2a}{p-1}\right)}, \tag{3.10}
 \end{aligned}$$

as asserted, using (3.8), Corollary 2.2, and the Gross-Koblitz formula (2.6).

A cohomological interpretation of this result can be seen as follows: One has the classical Euler integral formula

$$\frac{\Gamma(\alpha) \Gamma(\gamma - \alpha)}{\Gamma(\gamma)} {}_2F_1\left(\begin{matrix} \beta, \alpha \\ \gamma \end{matrix}; \lambda\right) = \int_0^1 \omega_{\beta, \alpha, \gamma} \quad (\text{Re } \gamma > \text{Re } \alpha > 0) \tag{3.11}$$

[9, p. 175] expressing  ${}_2F_1(\beta, \alpha; \gamma; \lambda)$  as a period of the differential form

$$\omega_{\beta, \alpha, \gamma} = x^{\alpha-1} (1-x)^{\gamma-\alpha-1} (1-\lambda x)^{-\beta} dx, \tag{3.12}$$

and giving a relation of the  ${}_2F_1$  to the cohomology associated with the algebraic function

$$y = x^{\alpha-1} (1-x)^{\gamma-\alpha-1} (1-\lambda x)^{-\beta}, \tag{3.13}$$

when  $\alpha, \beta, \gamma \in \mathbb{Q}$ . (In [9],  $f = (1-x)y$  is the function being studied.) When  $\lambda = -1$  and  $1 + \alpha = \gamma + \beta$ , Eq. (3.13) becomes

$$y = x^{\alpha-1} (1-x^2)^{-\beta}, \tag{3.14}$$

and therefore one has an automorphism  $(x, y) \mapsto (-x, (-1)^{\alpha-1}y)$ , which induces an action on cohomology, under which

$$\omega_{\beta, \alpha, \gamma} \mapsto (-1)^\alpha \omega_{\beta, \alpha, \gamma}. \tag{3.15}$$

For  $\alpha \in \mathbb{Q}$  this gives a representation of a group of roots of unity into the automorphism group of the two-dimensional cohomology, of which (3.15) is an isotypical component. The map  $(x, y) \mapsto (-x, (-1)^{\alpha-1}y)$  commutes with the Frobenius map  $(x, y) \mapsto (x^p, y^p)$  precisely when  $\alpha \in (2/(p-1))\mathbb{Z}$ ; in this case Frobenius is stable on the subspace of cohomology generated by  $\omega_{\beta, \alpha, \gamma}$ , and Theorem 3.1 may be viewed as arising from the isolation of this subspace. One may also note that the change of variable  $x^2 = t$  in (3.14) reduces the integral in (3.11) to a beta function, giving a derivation of the classical Kummer formula from the Euler formula.

A second *p*-adic analogue which is similarly obtainable from a combinatorial identity is that of Saalschütz' theorem [15], which gives the

value at  $x = 1$  of a Saalschützian  ${}_3F_2$  series. This and the remaining results of this section deal with the values of  ${}_3F_2$  functions at the singular point  $x = 1$  of Eq. (2.13). In addition to those described in [13], one also finds examples in of the number-theoretic use of such singular values of hypergeometric functions in parts (ii), (v) of Theorem 4.1 and Corollary 4.2 below.

**THEOREM 3.2.** *Suppose  $0 < a, b, c, d, e \leq p - 1$ , with  $a + b + c = d + e - (p - 1)$ . Then*

$$\begin{aligned}
 {}_3\mathfrak{F}_2 \left( \begin{matrix} a & b & c \\ p-1 & p-1 & p-1 \end{matrix} ; 1 \right) &= \frac{\Gamma_p \left( \frac{d}{p-1} \right) \Gamma_p \left( \frac{d-a-b}{p-1} \right) \Gamma_p \left( \frac{d-a-c}{p-1} \right) \Gamma_p \left( \frac{d-b-c}{p-1} \right)}{\Gamma_p \left( \frac{d-a}{p-1} \right) \Gamma_p \left( \frac{d-b}{p-1} \right) \Gamma_p \left( \frac{d-c}{p-1} \right) \Gamma_p \left( \frac{d-a-b-c}{p-1} \right)}.
 \end{aligned}$$

*Proof.* Suppose that  $s, m, n, q \in \mathbb{Z}^+$  satisfy  $s + q \leq n$  and  $m \leq n \leq m + q$ . Then we have the identity

$$\binom{m}{s} \binom{n}{q} = \sum_{j=0}^M \binom{n+j}{s+q} \binom{m+q-n}{j} \binom{n+s-m}{s-j}, \tag{3.16}$$

where  $M = \min(s, m + q - n)$  [14, p. 16, Eq. (12)]. Since

$$\begin{aligned}
 \binom{n+j}{s+q} &= \binom{n}{s+q} \frac{(n+1)_j}{(n+1-s-q)_j}, \\
 \binom{n+s-m}{s-j} &= (-1)^j \binom{n+s-m}{s} \frac{(-s)_j}{(n+1-m)_j},
 \end{aligned}$$

we may rewrite this as

$$\frac{\binom{m}{s} \binom{n}{q}}{\binom{n}{s+q} \binom{n+s-m}{s}} = {}_3F_2 \left( \begin{matrix} n+1, -s, n-m-q \\ n+1-s-q, n+1-m \end{matrix} ; 1 \right). \tag{3.17}$$

Now if  $a, b, c, d, e$  satisfy the conditions of the theorem, then for each  $r \geq 0$ , the integers  $s = s_r = a(p^r - 1)/(p - 1)$ ,  $m = m_r = (d - b)(p^r - 1)/(p - 1)$ ,  $n = n_r = (p - 1 - b)(p^r - 1)/(p - 1)$ ,  $q = q_r = (p - 1 + c - d)(p^r - 1)/(p - 1)$

satisfy the hypotheses of the above combinatorial identity. By Proposition 2.1,

$$\binom{m_r}{s_r}, \binom{n_r}{q_r}, \binom{n_r}{s_r + q_r}, \binom{n_r + s_r - m_r}{s_r} \in \mathbb{Z}_p^\times, \tag{3.18}$$

which as before implies that

$${}_3F_2 \left( \begin{matrix} n_r + 1, -s_r, n_r - m_r - q_r \\ n_r + 1 - s_r - q_r, n_r + 1 - m_r \end{matrix}; X \right) \in \mathbb{Z}_p[X] \tag{3.19}$$

and its value at  $x = 1$  is a  $p$ -adic unit, for each  $r \geq 0$ . As  $r \rightarrow \infty$ , the values of these parameters approach the limits  $b/(p-1)$ ,  $a/(p-1)$ ,  $c/(p-1)$ ,  $e/(p-1)$ , and  $d/(p-1)$ , respectively. Our hypotheses imply that  $a, b, c < d, e$ , so condition (ii) of Theorem 2.3 is satisfied. We may now complete the proof using (3.17) and a continuity argument as in Theorem 3.1, expressing the ratios of binomial coefficients in terms of  $\Gamma_p$  via Corollary 2.2 and the Gross-Koblitz formula.

We next give a  $p$ -adic analogue of Dixon's theorem [15], which gives the value at  $x = 1$  of a well-poised  ${}_3F_2$  series.

**THEOREM 3.3.** *Suppose  $0 < 2a \leq b \leq c < p - 1$ , with  $b + c - a \leq p - 1$ . Then*

$$\begin{aligned} & {}_3\mathfrak{F}_2 \left( \begin{matrix} \frac{2a}{p-1}, \frac{b}{p-1}, \frac{c}{p-1} \\ 1 + \frac{2a-b}{p-1}, 1 + \frac{2a-c}{p-1} \end{matrix}; 1 \right) \\ &= (-1)^a \frac{\Gamma_p \left( \frac{a}{p-1} \right) \Gamma_p \left( \frac{c-a}{p-1} \right) \Gamma_p \left( \frac{b-a}{p-1} \right) \Gamma_p \left( \frac{b+c-2a}{p-1} \right)}{\Gamma_p \left( \frac{2a}{p-1} \right) \Gamma_p \left( \frac{c-2a}{p-1} \right) \Gamma_p \left( \frac{b-2a}{p-1} \right) \Gamma_p \left( \frac{b+c-a}{p-1} \right)}. \end{aligned}$$

*Proof.* A terminating form of the classical Dixon's theorem [15, Eq. (III.9)] states that for  $n \in \mathbb{Z}^+$ , we have

$${}_3F_2 \left( \begin{matrix} \frac{2a}{p-1}, \frac{b}{p-1}, -n \\ 1 + \frac{2a-b}{p-1}, 1 + n + \frac{2a}{p-1} \end{matrix}; 1 \right) = \frac{\left(1 + \frac{2a}{p-1}\right)_n \left(1 + \frac{a-b}{p-1}\right)_n}{\left(1 + \frac{a}{p-1}\right)_n \left(1 + \frac{2a-b}{p-1}\right)_n}. \tag{3.20}$$

Being a finite summation theorem, this result is valid for  $p$ -adic parameters as well. For  $r \geq 0$  we set  $n = n_r = c(p^r - 1)/(p - 1)$ , and under our hypotheses on  $a, b, c$  we compute for  $r > 0$ ,

$$\frac{\left(1 + \frac{2a}{p-1}\right)_{n_r}}{\left(1 + \frac{2a}{p-1}\right)_{n_{r-1}}} = (-1)^{n_r} p^{n_{r-1}} \left(\frac{2ap}{p-1}\right) \frac{\Gamma_p\left(1 + n_r + \frac{2a}{p-1}\right)}{\Gamma_p\left(1 + \frac{2a}{p-1}\right)}; \tag{3.21a}$$

$$\frac{\left(1 + \frac{a}{p-1}\right)_{n_r}}{\left(1 + \frac{a}{p-1}\right)_{n_{r-1}}} = (-1)^{n_r} p^{n_{r-1}} \left(\frac{ap}{p-1}\right) \frac{\Gamma_p\left(1 + n_r + \frac{a}{p-1}\right)}{\Gamma_p\left(1 + \frac{a}{p-1}\right)}; \tag{3.21b}$$

$$\frac{\left(1 + \frac{2a-b}{p-1}\right)_{n_r}}{\left(1 + \frac{2a-b}{p-1}\right)_{n_{r-1}}} = (-1)^{n_r} p^{n_{r-1}} \frac{\Gamma_p\left(1 + n_r + \frac{2a-b}{p-1}\right)}{\Gamma_p\left(1 + \frac{2a-b}{p-1}\right)}; \tag{3.21c}$$

$$\frac{\left(1 + \frac{a-b}{p-1}\right)_{n_r}}{\left(1 + \frac{a-b}{p-1}\right)_{n_{r-1}}} = (-1)^{n_r} p^{n_{r-1}} \frac{\Gamma_p\left(1 + n_r + \frac{a-b}{p-1}\right)}{\Gamma_p\left(1 + \frac{a-b}{p-1}\right)}; \tag{3.21d}$$

which again shows that each  $|F_1^{(i)}(1)| = 1$ , with  $F$  as in (3.20), for all  $n = n_r$  with  $i, r \geq 0$ . Since  $n_r \rightarrow -c/(p - 1)$  as  $r \rightarrow \infty$ , the same is true for the  ${}_3F_2$  in the statement of the theorem. Condition (ii) of Theorem 2,3 is satisfied when  $b + c - 2a < p - 1$ ; however, we need  $b + c - a \leq p - 1$  in order for (3.21d) to hold, ensuring that  $1 \in \mathfrak{D}$ . Therefore under our assumptions there is a continuation of the associated ratio to the value  $x = 1$ . Letting  $r \rightarrow \infty$ , we obtain the value

$$\begin{aligned} & {}_3\mathfrak{F}_2 \left( \begin{matrix} \frac{2a}{p-1}, \frac{b}{p-1}, \frac{c}{p-1} \\ 1 + \frac{2a-b}{p-1}, 1 + \frac{2a-c}{p-1} \end{matrix}; 1 \right) \\ &= 2 \frac{\Gamma_p\left(1 + \frac{a}{p-1}\right) \Gamma_p\left(1 + \frac{2a-c}{p-1}\right) \Gamma_p\left(1 + \frac{2a-b}{p-1}\right) \Gamma_p\left(1 + \frac{a-b-c}{p-1}\right)}{\Gamma_p\left(1 + \frac{2a}{p-1}\right) \Gamma_p\left(1 + \frac{a-c}{p-1}\right) \Gamma_p\left(1 + \frac{2a-b-c}{p-1}\right) \Gamma_p\left(1 + \frac{a-b}{p-1}\right)}. \end{aligned} \tag{3.22}$$

By applying the translation formula to the first term in the numerator and denominator, and the reflection formula to the remaining terms, one obtains the statement of the theorem.

We remark that in the special case where  $2a = b = c$ , this result may be obtained from the combinatorial identity

$$(-1)^m \frac{(3m)!}{(m!)^3} = \sum_{k=0}^{2m} (-1)^k \binom{2m}{k}^3 = {}_3F_2 \left( \begin{matrix} -2m, -2m, -2m \\ 1, 1 \end{matrix}; 1 \right) \quad (3.23)$$

[14, p. 42], in a manner similar to that of the previous two theorems, and in this case yields the formal-group congruence

$$\begin{aligned} & \binom{3m_r}{2m_r} \binom{2m_r}{m_r} \\ & \equiv (-1)^a {}_3\mathfrak{F}_2 \left( \begin{matrix} \frac{2a}{p-1}, \frac{2a}{p-1}, \frac{2a}{p-1} \\ 1, 1 \end{matrix}; 1 \right) \binom{3m_r - 1}{2m_r - 1} \binom{2m_r - 1}{m_r - 1} \pmod{p^r \mathbb{Z}_p} \end{aligned} \quad (3.24)$$

for  $0 < a \leq (p-1)/3$ , where  $m_r = a(p^r - 1)/(p-1)$ .

We conclude these results with a *p*-adic version of Watson's theorem [15], which gives a result for a special class of  ${}_3F_2(1)$  which are neither well-poised nor Saalschützian.

**THEOREM 3.4.** *Suppose  $2a \leq 2b < 2a + 2b \leq p - 1$ , and  $2a < 2c \leq p - 1$ . Then*

$$\begin{aligned} & {}_3\mathfrak{F}_2 \left( \begin{matrix} \frac{2a}{p-1}, \frac{2b}{p-1}, \frac{c}{p-1} \\ \frac{1}{2} + \frac{a+b}{p-1}, \frac{2c}{p-1} \end{matrix}; 1 \right) \\ & = \frac{\Gamma_p \left( \frac{1}{2} \right) \Gamma_p \left( \frac{1}{2} + \frac{c}{p-1} \right) \Gamma_p \left( \frac{1}{2} + \frac{a+b}{p-1} \right) \Gamma_p \left( \frac{1}{2} + \frac{c-a-b}{p-1} \right)}{\Gamma_p \left( \frac{1}{2} + \frac{a}{p-1} \right) \Gamma_p \left( \frac{1}{2} + \frac{b}{p-1} \right) \Gamma_p \left( \frac{1}{2} + \frac{c-a}{p-1} \right) \Gamma_p \left( \frac{1}{2} + \frac{c-b}{p-1} \right)}. \end{aligned}$$

*Proof.* For  $r \geq 0$  we set  $n_r = a(p^r - 1)/(p-1)$  and  $m_r = 1/2 + d(p^r - 1)/(p-1)$ , where  $2b + 2d = p - 1$ ; we also set  $\gamma = c/(p-1)$ . The classical Watson's theorem [15, Eq. (III.23)] in a terminating form yields

$${}_3F_2 \left( \begin{matrix} -2n_r, 2m_r, \gamma \\ \frac{1}{2} + m_r - n_r, 2\gamma \end{matrix}; 1 \right) = \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2} + \gamma) \Gamma(\frac{1}{2} + m_r - n_r) \Gamma(\frac{1}{2} + \gamma + n_r - m_r)}{\Gamma(\frac{1}{2} - n_r) \Gamma(\frac{1}{2} + m_r) \Gamma(\frac{1}{2} + \gamma + n_r) \Gamma(\frac{1}{2} + \gamma - m_r)}. \quad (3.25)$$

Under our hypotheses on  $a, b, c$ , we compute, for  $r > 0$ ,

$$\frac{\Gamma(\frac{1}{2})/\Gamma(\frac{1}{2}-n_r)}{\Gamma(\frac{1}{2})/\Gamma(\frac{1}{2}-n_{r-1})} = (-1)^{n_r} p^{n_r-1} \frac{\Gamma_p(\frac{1}{2})}{\Gamma_p(\frac{1}{2}-n_r)}; \tag{3.26a}$$

$$\frac{\Gamma(\frac{1}{2}+m_r)/\Gamma(\frac{1}{2}+m_r-n_r)}{\Gamma(\frac{1}{2}+m_{r-1})/\Gamma(\frac{1}{2}+m_{r-1}-n_{r-1})} = (-1)^{n_r} p^{n_r-1} \frac{\Gamma_p(\frac{1}{2}+m_r)}{\Gamma_p(\frac{1}{2}+m_r-n_r)}; \tag{3.26b}$$

$$\frac{\Gamma(\frac{1}{2}+\gamma+n_r)/\Gamma(\frac{1}{2}+\gamma)}{\Gamma(\frac{1}{2}+\gamma+n_{r-1})/\Gamma(\frac{1}{2}+\gamma)} = (-1)^{n_r} p^{n_r-1} \frac{\Gamma_p(\frac{1}{2}+\gamma+n_r)}{\Gamma_p(\frac{1}{2}+\gamma)}; \tag{3.26c}$$

$$\frac{\Gamma(\frac{1}{2}+\gamma+n_r-m_r)/\Gamma(\frac{1}{2}+\gamma-m_r)}{\Gamma(\frac{1}{2}+\gamma+n_{r-1}-m_{r-1})/\Gamma(\frac{1}{2}+\gamma-m_{r-1})} = (-1)^{n_r} p^{n_r-1} \frac{\Gamma_p(\frac{1}{2}+\gamma+n_r-m_r)}{\Gamma_p(\frac{1}{2}+\gamma-m_r)}; \tag{3.26d}$$

which as before shows that each  $|F_1^{(i)}(1)| = 1$  for the  ${}_3F_2$  associated to the statement of the theorem. Since the hypotheses of Theorem 2.3 are satisfied under our conditions on  $a, b, c$  the stated value may be obtained precisely as in the previous theorems.

*Remark.* One may observe many relationships between these formulae. For example, by comparing Koblitz' result with Theorem 3.2, one finds that if  $a, b, c > 0$  and  $a + b + c = p - 1$ , then

$${}_3\mathfrak{F}_2 \left( \begin{matrix} a & b & c \\ p-1 & p-1 & p-1 \\ & 1 & 1 \end{matrix}; 1 \right) = {}_2\mathfrak{F}_1 \left( \begin{matrix} a & b \\ p-1 & p-1 \\ & 1 \end{matrix}; 1 \right)^2, \tag{3.27}$$

which follows essentially from the relationship between the two combinatorial identities, but otherwise seems to have no analogue in classical analysis. A comparison between Theorems 3.2 and 3.4 yields the more unusual result that if  $a + b + (p - 1)/2 = d \leq p - 1$ , then

$${}_3\mathfrak{F}_2 \left( \begin{matrix} a & b & \frac{1}{2} \\ p-1 & p-1 & \frac{1}{2} \\ & d & 1 \end{matrix}; 1 \right) = {}_3\mathfrak{F}_2 \left( \begin{matrix} 2a & 2b & \frac{1}{2} \\ p-1 & p-1 & \frac{1}{2} \\ & d & 1 \end{matrix}; 1 \right). \tag{3.28}$$

**EXAMPLES.** The relation between the analytic-continuation of ratios of  $p$ -adic hypergeometric functions and congruence zeta-functions of varieties has been explained by Dwork [6, Sect. 6]. The prototypical example is that of the the Gauss hypergeometric function  ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \lambda)$ , which is

the holomorphic solution at  $\lambda=0$  to the Fuchs–Picard equation for the Legendre family of elliptic curves with affine equation

$$E_\lambda: y^2 = x(x-1)(x-\lambda) \quad (\lambda \neq 0, 1). \tag{3.29}$$

If  $\lambda^p = \lambda$  and  $E_\lambda$  has non-supersingular reduction at  $p$  then the reciprocal unit root  $\alpha(\lambda)$  of the zeta-function of the reduced curve is given by

$$\alpha(\lambda) = (-1)^{(p-1)/2} {}_2\mathfrak{F}_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; \lambda\right) \tag{3.30}$$

[6, Eq. (6.29)]. From Theorem 3.1 we find that for  $p \equiv 1 \pmod{4}$ ,

$$\begin{aligned} \alpha(-1) &= {}_2\mathfrak{F}_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; -1\right) = (-1)^{(p-1)/4} \frac{\Gamma_p(\frac{1}{4})^2}{\Gamma_p(\frac{1}{2})} \\ &= (-1)^{(p-1)/4} J(\omega^{(1-p)/4}, \omega^{(1-p)/4}). \end{aligned} \tag{3.31}$$

In this case it is easy to check directly using character sums that the elliptic curve  $y^2 = x^3 - x$  has non-supersingular reduction precisely when  $p \equiv 1 \pmod{4}$  and in that case (3.31) does indeed give the reciprocal unit root of the zeta-function. Thus a unit-valued continuation for  ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \lambda)$  to  $\lambda = -1$  exists only under the hypothesis  $p \equiv 1 \pmod{4}$  implied in Theorem 3.1. In this example the action on cohomology (cf. Eq. (3.15)) is induced by the complex multiplication  $(x, y) \mapsto (-x, \sqrt{-1}y)$  on  $E_{-1}$  by the fourth roots of unity.

The results in this section do in some cases hold under hypotheses weaker than those stated here. We consider the  ${}_3F_2$  functions which arise from the family of  $K3$  surfaces with projective equation

$$X_1^4 + X_2^4 + X_3^4 + X_4^4 - 4\lambda X_1 X_2 X_3 X_4 = 0 \tag{3.32}$$

studied in [6, Sect. 6(j)]. A solution to the corresponding differential equation for this family at  $\lambda=0$  is given by

$${}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \\ \frac{1}{2}, \frac{3}{4} \end{matrix}; \lambda^4\right), \tag{3.33}$$

and at  $\infty$  a solution is given by

$$\lambda^{-1} {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1 \end{matrix}; \lambda^{-4}\right). \tag{3.34}$$

In that paper Dwork analyzed the solution at  $\infty$  and determined that the reciprocal unit root  $\alpha(\lambda)$  of the nontrivial factor of the zeta-function of the

reduced equation is given for  $\lambda$  lying in the region  $\mathfrak{D}$  associated to (3.34) and  $\lambda^4 \neq 1$  by

$$\alpha(\lambda) = {}_3\mathfrak{F}_2 \left( \begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1 \end{matrix}; \lambda^{-4} \right) \quad (3.35)$$

(the constant factor was determined [6, p. 76] to be 1). From Theorem 3.4 we find that the fourth roots of 1 also lie in  $\mathfrak{D}$  for  $p \equiv 1 \pmod{8}$ , and in this case the values corresponding to the singular fibres at  $\lambda^4 = 1$  are given by

$$\alpha(\lambda) = {}_3\mathfrak{F}_2 \left( \begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1 \end{matrix}; 1 \right) = \frac{\Gamma_p(\frac{1}{8})^2 \Gamma_p(\frac{3}{8})^2}{\Gamma_p(\frac{1}{2})^2} = J(\omega^{(1-p)/8}, \omega^{(3-3p)/8})^2. \quad (3.36)$$

Of course one may also use the solution at  $\lambda = 0$  and Dwork's methods in [6, 7] to conclude that for  $\lambda$  lying in the region  $\mathfrak{D}$  associated to (3.33) and  $\lambda^4 \neq 1$ ,

$$\alpha(\lambda) = \varepsilon \cdot {}_3\mathfrak{F}_2 \left( \begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \\ \frac{1}{2}, \frac{3}{4} \end{matrix}; \lambda^4 \right), \quad (3.37)$$

where the constant  $\varepsilon$  may be determined via specialization of the family to the diagonal surface at  $\lambda = 0$ . For  $p \equiv 1 \pmod{4}$  we may use the tuple  $u = (1, 1, 1, 1)$  in [6, Eqs. (6.36), (6.37)] to obtain

$$\varepsilon = p^{-1} g(\omega^{(1-p)/4})^4 = -\Gamma_p(\frac{1}{4})^4 \quad (3.38)$$

(the factor of  $p$  had been suppressed in Dwork's equations as extraneous). From Theorem 3.4 we find that for  $p \equiv 1 \pmod{8}$ ,

$${}_3\mathfrak{F}_2 \left( \begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \\ \frac{1}{2}, \frac{3}{4} \end{matrix}; 1 \right) = -\Gamma_p \left( \frac{3}{8} \right)^4 \Gamma_p \left( \frac{3}{4} \right)^2. \quad (3.39)$$

Therefore for  $p \equiv 1 \pmod{8}$  and  $\lambda^4 = 1$  the singular values are also given by

$$\alpha(\lambda) = \Gamma_p(\frac{3}{8})^4 \Gamma_p(\frac{1}{4})^2 = J(\omega^{(3-3p)/8}, \omega^{(3-3p)/8})^2. \quad (3.40)$$

(The agreement between these two expressions for the singular values of the  $\alpha(\lambda)$  may be demonstrated by means of the Gauss multiplication formula with  $x = 1/4$ ,  $n = 2$ , and the reflection formula.)

In this example all the  $\alpha(\lambda)$  for  $\lambda \in \mathfrak{D}$  are squares, due to the formula

$${}_3F_2 \left( \begin{matrix} 2a, a+b, 2b \\ a+b+\frac{1}{2}, 2a+2b \end{matrix}; x \right) = {}_2F_1 \left( \begin{matrix} a, b \\ a+b+\frac{1}{2} \end{matrix}; x \right)^2 \quad (3.41)$$



of Clausen [1, p. 185]. Applied to (3.33), (3.34), this yields

$${}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \\ \frac{1}{2}, \frac{3}{4} \end{matrix}; 1\right) = {}_2F_1\left(\begin{matrix} \frac{1}{8}, \frac{1}{8} \\ \frac{3}{4} \end{matrix}; 1\right)^2, \tag{3.42}$$

$${}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1 \end{matrix}; 1\right) = {}_2F_1\left(\begin{matrix} \frac{1}{8}, \frac{3}{8} \\ 1 \end{matrix}; 1\right)^2. \tag{3.43}$$

From Koblitz' criterion (Eq. (3.2) above) one verifies that the analytic element associated to the right side of (3.43) has a unit-valued continuation to  $x = 1$  (i.e.,  $1 \in \mathfrak{D}$ ) if and only if  $p \equiv 1, 3 \pmod{8}$ . Therefore by (3.1), (3.43), and the uniqueness of *p*-adic prolongments the relation

$${}_3\mathfrak{F}_2\left(\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1 \end{matrix}; 1\right) = \frac{\Gamma_p(\frac{1}{8})^2 \Gamma_p(\frac{3}{8})^2}{\Gamma_p(\frac{1}{2})^2} \tag{3.44}$$

from (3.36) holds also when  $p \equiv 3 \pmod{8}$ , although this is not covered by Theorem 3.4. (Using the full statement of the Gross-Koblitz formula [10] one finds that the product of  $\Gamma_p$  values in (3.44) yields a Jacobi sum over  $\mathbb{F}_{p^2}$  rather than the square of a Jacobi sum over  $\mathbb{F}_p$ .) However, neither of the functions in (3.42) satisfies hypothesis (ii) of Theorem 2.3 when  $p \not\equiv 1 \pmod{8}$ , so for the  ${}_3F_2$  in (3.42) one does not have a continuation at all except under the hypotheses implied in Theorem 3.4.

#### 4. APÉRY NUMBERS, FORMAL-GROUP CONGRUENCES, AND *p*-ADIC HYPERGEOMETRIC FUNCTIONS

Consider the sequences defined for  $n \geq 0$  by

$$a(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2; \tag{4.1a}$$

$$b(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}; \tag{4.1b}$$

$$c(n) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k}; \tag{4.1c}$$

$$d(n) = \sum_{k=0}^n \binom{n}{k}^3; \tag{4.1d}$$

$$v(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}. \tag{4.1v}$$

The  $c(n)$ ,  $b(n)$ , and  $a(n)$  have occurred in proofs of irrationality measures for  $\log 2$ ,  $\zeta(2)$ , and  $\zeta(3)$  (cf. [19, 18]); the  $b(n)$ ,  $d(n)$ ,  $v(n)$ ,  $a(n)$  occur as expansion coefficients of solutions to the Apéry differential operators (cf. [8, 2]). For each of  $\alpha(n) = a(n)$ ,  $b(n)$ ,  $c(n)$ ,  $d(n)$ ,  $v(n)$ , it is known that  $\omega_\alpha = \sum_{n=0}^{\infty} \alpha(n) t^{2n} dt$  is the invariant differential for a one-parameter commutative formal group law (cf. [2, 17, 19]) and that the coefficients  $\alpha(n)$  satisfy congruences of the Atkin–Swinnerton–Dyer type. Here we show how these congruences may be expressed naturally in terms of  $p$ -adic analytic continuations of hypergeometric functions.

**THEOREM 4.1.** (i) *If  $p \equiv 1 \pmod{4}$ , then for each  $r > 0$ ,*

$$\frac{c((p^r - 1)/2)}{c((p^{r-1} - 1)/2)} \equiv {}_2\mathfrak{F}_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; -1\right) \pmod{p^r \mathbb{Z}_p}.$$

(ii) *If  $p \equiv 1 \pmod{4}$ , then for each  $r > 0$ ,*

$$\frac{b((p^r - 1)/2)}{b((p^{r-1} - 1)/2)} \equiv {}_3\mathfrak{F}_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; 1\right) \pmod{p^r \mathbb{Z}_p}.$$

(iii) *If  $p \equiv 1, 3 \pmod{8}$ , then for each  $r > 0$ ,*

$$\frac{d((p^r - 1)/2)}{d((p^{r-1} - 1)/2)} \equiv {}_3\mathfrak{F}_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; -1\right) \pmod{p^r \mathbb{Z}_p}.$$

(iv) *Let  $\hat{4}$  denote the Teichmüller representative of 4 in  $\mathbb{Z}_p$ , i.e.,  $\hat{4}^p = \hat{4}$  and  $\hat{4} \equiv 4 \pmod{p\mathbb{Z}_p}$ . If  $p \equiv 1 \pmod{6}$ , then for each  $r > 0$ ,*

$$\frac{v((p^r - 1)/2)}{v((p^{r-1} - 1)/2)} \equiv {}_3\mathfrak{F}_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; \hat{4}\right) \pmod{p^r \mathbb{Z}_p}.$$

(v) *Let  $\Psi(q) = \sum_{n=1}^{\infty} \gamma_n q^n = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4$  denote the unique cusp form of weight 4 for the congruence subgroup  $\Gamma_0(8)$  (with  $q = e^{2\pi i\tau}$ ). If the  $p$ th Hecke polynomial  $H_p(T) = 1 - \gamma_p T + p^3 T^2$  associated to  $\Psi$  has a  $p$ -adic unit root  $\alpha_p^{-1}$ , then for each  $r > 0$ ,*

$$\frac{a((p^r - 1)/2)}{a((p^{r-1} - 1)/2)} \equiv {}_4\mathfrak{F}_3\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix}; 1\right) \pmod{p^r \mathbb{Z}_p}.$$

*Proof.* It has been noted [19, 3] that  $\sum_{n=0}^{\infty} c(n)t^{2n} dt$  is the expansion of the invariant differential  $\omega_E$  on the elliptic curve

$$E: y^2 = x(x^2 - 6x + 1) \tag{4.2}$$

with respect to the local parameter  $t = 1/\sqrt{x}$  at infinity. The associated Atkin–Swinnerton–Dyer congruences read

$$\begin{aligned} c((mp^r - 1)/2) - a_{E,p} c((mp^{r-1} - 1)/2) + pc((mp^{r-2} - 1)/2) \\ \equiv 0 \pmod{p^r \mathbb{Z}} \end{aligned} \tag{4.3}$$

for  $m, r \in \mathbb{Z}^+$ ,  $m$  odd, where

$$Z(E/\mathbb{F}_p; T) = \frac{1 - a_{E,p} T + pT^2}{(1 - T)(1 - pT)} \tag{4.4}$$

is the zeta-function of  $E$  over  $\mathbb{F}_p$ . For  $p \equiv 1 \pmod{4}$ ,  $E$  has non-super-singular reduction mod  $p$ , and therefore  $a_{E,p}$  is a  $p$ -adic unit. It follows by induction on (4.3) that  $c((p^r - 1)/2)$  is a  $p$ -adic unit for all  $r \geq 0$ , and then that

$$\frac{c((p^r - 1)/2)}{c((p^{r-1} - 1)/2)} \equiv \alpha_{E,p} \pmod{p^r \mathbb{Z}_p}, \tag{4.5}$$

where  $\alpha_{E,p} \in \mathbb{Z}_p^\times$  is a reciprocal root of  $Z(E/\mathbb{F}_p; T)$  (cf. [17, Theorem A.8]).

There are similar congruences in each of the remaining cases. In [17], J. Stienstra and F. Beukers have shown that  $\sum_{n=0}^\infty b(n) t^{2n} dt$ ,  $\sum_{n=0}^\infty d(n) t^{2n} dt$ , and  $\sum_{n=0}^\infty v(n) t^{2n} dt$  are expansions of invariant differentials of formal Brauer groups for certain K3 surfaces, from which they obtained the congruences

$$\begin{aligned} b((mp^r - 1)/2) - (4a^2 - 2p) b((mp^{r-1} - 1)/2) + p^2 b((mp^{r-2} - 1)/2) \\ \equiv 0 \pmod{p^r \mathbb{Z}} \end{aligned} \tag{4.6}$$

for odd  $m \in \mathbb{Z}^+$ , where  $p \equiv 1 \pmod{4}$ ,  $p = a^2 + 4b^2$ ;

$$\begin{aligned} d((mp^r - 1)/2) - (-1)^{(p-1)/2} (4a^2 - 2p) d((mp^{r-1} - 1)/2) \\ + p^2 d((mp^{r-2} - 1)/2) \equiv 0 \pmod{p^r \mathbb{Z}} \end{aligned} \tag{4.7}$$

for odd  $m \in \mathbb{Z}^+$ , where  $p \equiv 1, 3 \pmod{8}$ ,  $p = a^2 + 2b^2$  and

$$\begin{aligned} v((mp^r - 1)/2) - (4a^2 - 2p) v((mp^{r-1} - 1)/2) \\ + p^2 v((mp^{r-2} - 1)/2) \equiv 0 \pmod{p^r \mathbb{Z}} \end{aligned} \tag{4.8}$$

for odd  $m \in \mathbb{Z}^+$ , where  $p \equiv 1 \pmod{6}$ ,  $p = a^2 + 3b^2$ . In each of the cases (4.6), (4.7), (4.8), it is observed [17, Sect. 14] that the polynomial  $1 - (4a^2 - 2p) T + p^2 T^2$  is the  $p$ th Hecke polynomial for a certain cusp form of weight 3 and level 16 (resp. 8; resp. 12). One therefore has congruences

similar to (4.5), with  $\alpha_{E,p}$  replaced by reciprocal unit roots of these Hecke polynomials. Beukers [2] has also demonstrated the congruence

$$\begin{aligned} a((mp^r - 1)/2) - \gamma_p a((mp^{r-1} - 1)/2) + p^3 a((mp^{r-2} - 1)/2) \\ \equiv 0 \pmod{p^r \mathbb{Z}} \end{aligned} \tag{4.9}$$

for the  $a(n)$ , from which it follows that

$$\frac{a((p^r - 1)/2)}{a((p^{r-1} - 1)/2)} \equiv \alpha_p \pmod{p^r \mathbb{Z}_p}, \tag{4.10}$$

whenever  $\alpha_p^{-1}$  is a  $p$ -adic unit root of  $H_p(T) = 1 - \gamma_p T + p^3 T^2$ .

We now complete the proof of (iv). Let  $p$  be a prime,  $p \equiv 1 \pmod{6}$ , let  $n_r = (p^r - 1)/2$ , and let  $\mathfrak{D}$  be the domain of support of the analytic element  ${}_3\mathfrak{F}_2(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; x)$  described in Theorem 2.3 (ii). Since  $(-n_r)^{(i)} = -n_{r-i} \equiv \frac{1}{2} \pmod{p\mathbb{Z}_p}$  for  $0 \leq i < r$  and  $(-n_r)^{(i)} = 0$  for  $i \geq r$ , we see that  ${}_3\mathfrak{F}_2(-n_r, -n_r, \frac{1}{2}; 1, 1; x)$  also has support  $\mathfrak{D}$  for each  $r > 0$ . From the continuity with respect to the parameters of the  ${}_3\mathfrak{F}_2$  at  $(\alpha_1, \alpha_2, \alpha_3) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ , and the uniform convergence of the limit in Theorem 2.3 (ii), we find that for each  $N \in \mathbb{Z}^+$  there exists  $r_N \geq N$  such that

$$\begin{aligned} \frac{{}_3F_2\left(\begin{matrix} -n_{r+1}, -n_{r+1}, \frac{1}{2} \\ 1, 1 \end{matrix}; x\right)}{{}_3F_2\left(\begin{matrix} -n_r, -n_r, \frac{1}{2} \\ 1, 1 \end{matrix}; x^p\right)} \\ \equiv \frac{{}_3F_2\left(\begin{matrix} -n_r, -n_r, \frac{1}{2} \\ 1, 1 \end{matrix}; x\right)}{{}_3F_2\left(\begin{matrix} -n_{r-1}, -n_{r-1}, \frac{1}{2} \\ 1, 1 \end{matrix}; x^p\right)} \pmod{p^N \mathfrak{D}} \end{aligned} \tag{4.11}$$

for all  $x \in \mathfrak{D}$  and all  $r > r_N$ , which implies

$$\begin{aligned} \frac{{}_3F_2\left(\begin{matrix} -n_{r+1}, -n_{r+1}, \frac{1}{2} \\ 1, 1 \end{matrix}; x\right)}{{}_3F_2\left(\begin{matrix} -n_r, -n_r, \frac{1}{2} \\ 1, 1 \end{matrix}; x\right)} \\ \equiv \frac{{}_3F_2\left(\begin{matrix} -n_r, -n_r, \frac{1}{2} \\ 1, 1 \end{matrix}; x^p\right)}{{}_3F_2\left(\begin{matrix} -n_{r-1}, -n_{r-1}, \frac{1}{2} \\ 1, 1 \end{matrix}; x^p\right)} \pmod{p^N \mathfrak{D}}, \end{aligned} \tag{4.12}$$

since each factor is a  $p$ -adic unit. For  $x \in \mathbb{Z}_p \cap \mathfrak{D}$  we have also  $x^p \in \mathbb{Z}_p \cap \mathfrak{D}$ , so by  $s$ -fold iteration we obtain

$$\begin{aligned} & \frac{{}_3F_2\left(\begin{matrix} -n_{r+s}, -n_{r+s}, \frac{1}{2} \\ 1, 1 \end{matrix}; x\right)}{{}_3F_2\left(\begin{matrix} -n_{r+s-1}, -n_{r+s-1}, \frac{1}{2} \\ 1, 1 \end{matrix}; x\right)} \\ & \equiv \frac{{}_3F_2\left(\begin{matrix} -n_r, -n_r, \frac{1}{2} \\ 1, 1 \end{matrix}; x^{p^s}\right)}{{}_3F_2\left(\begin{matrix} -n_{r-1}, -n_{r-1}, \frac{1}{2} \\ 1, 1 \end{matrix}; x^{p^s}\right)} \pmod{p^N \mathbb{Z}_p} \end{aligned} \quad (4.13)$$

for all  $x \in \mathbb{Z}_p \cap \mathfrak{D}$ ,  $s \geq 0$ , and  $r > r_N$ . Upon setting  $x = 4$  we note that the left side of (4.13) becomes simply  $v(n_{r+s})/v(n_{r+s-1})$ . From (4.8) with  $m = r = 1$  we see that  $4 \in \mathfrak{D}$ , so by [17, A.8] there exists  $\beta_{3,p} \in \mathbb{Z}_p^\times$  such that

$$\frac{v(n_t)}{v(n_{t-1})} \equiv \beta_{3,p} \pmod{p^t \mathbb{Z}_p} \quad (4.14)$$

for all  $t > 0$ . Fixing  $r > r_N$  in and letting  $s \rightarrow \infty$ , we find therefore that the limit on the left side of (4.13) exists; as the right side is a rational function of  $x$ , we note that  $\lim_{s \rightarrow \infty} 4^{p^s} = \hat{4}$  and obtain

$$\begin{aligned} \beta_{3,p} &= \lim_{t \rightarrow \infty} \frac{{}_3F_2\left(\begin{matrix} -n_t, -n_t, \frac{1}{2} \\ 1, 1 \end{matrix}; 4\right)}{{}_3F_2\left(\begin{matrix} -n_{t-1}, -n_{t-1}, \frac{1}{2} \\ 1, 1 \end{matrix}; 4\right)} \\ & \equiv \frac{{}_3F_2\left(\begin{matrix} -n_r, -n_r, \frac{1}{2} \\ 1, 1 \end{matrix}; \hat{4}\right)}{{}_3F_2\left(\begin{matrix} -n_{r-1}, -n_{r-1}, \frac{1}{2} \\ 1, 1 \end{matrix}; \hat{4}\right)} \pmod{p^N \mathbb{Z}_p}. \end{aligned} \quad (4.15)$$

Letting  $r \rightarrow \infty$  and using the continuity at  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  then yields

$$\beta_{3,p} \equiv {}_3\mathfrak{F}_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; \hat{4}\right) \pmod{p^N \mathbb{Z}_p}; \quad (4.16)$$

as  $N \in \mathbb{Z}^+$  is arbitrary this implies

$$\beta_{3,p} = {}_3\mathfrak{F}_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; \hat{4}\right), \quad (4.17)$$

which together with (4.14) gives the result in (iv).

The proofs in the remaining cases are analogous but somewhat simpler; the fact that  $x = 1$  and  $x = -1$  are their own Teichmüller representatives makes the rearrangement of factors in (4.12) unnecessary for those cases.

The following corollary and ensuing remarks describe the  $p$ -adic integer  $\beta_{3,p}$  occurring in (4.17) and its companions from the other cases of Theorem 4.1.

**COROLLARY 4.2.** *For  $M = 2, 3, 4$  let  $\beta_{M,p}$  denote the reciprocal of the  $p$ -adic unit root of the polynomial  $P_{M,p}(T) = 1 - (4a^2 - 2p)T + p^2T^2$  whenever  $p = a^2 + Mb^2$ ,  $a, b \in \mathbb{Z}$ . With all other notations as in Theorem 4.1 and its proof, one has:*

(i) *If  $p \equiv 1 \pmod{4}$ , then*

$$\alpha_{E,p} = {}_2\mathfrak{F}_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; -1\right).$$

(ii) *If  $p \equiv 1 \pmod{4}$ , then*

$$\beta_{4,p} = {}_3\mathfrak{F}_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; 1\right).$$

(iii) *If  $p \equiv 1, 3 \pmod{8}$ , then*

$$\beta_{2,p} = (-1)^{(p-1)/2} {}_3\mathfrak{F}_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; -1\right).$$

(iv) *If  $p \equiv 1 \pmod{6}$ , then (as in (4.17))*

$$\beta_{3,p} = {}_3\mathfrak{F}_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; \hat{4}\right).$$

(v) *If  $p$  is a prime for which the  $p$ th Hecke polynomial  $H_p(T)$  associated to the cusp form  $\Psi(q)$  has a  $p$ -adic unit root  $\alpha_p^{-1}$ , then*

$$\alpha_p = {}_4\mathfrak{F}_3\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1 \end{matrix}; 1\right).$$

*Proof.* This is immediate from Theorem 4.1 and the congruences (4.5), (4.6), (4.7), (4.8), and (4.10).

*Remarks.* In [17, Sect. 14], it is shown that for  $M = 2, 3, 4$ ,  $P_{M,p}(T)$  is the  $p$ th Hecke polynomial associated to the cusp form  $\Phi_M(q)$ , where

$$\Phi_3(q) = q \prod_{n=1}^{\infty} (1 - q^{2n})^3 (1 - q^{6n})^3, \tag{4.18}$$

$$\Phi_4(q) = q \prod_{n=1}^{\infty} (1 - q^{4n})^6, \tag{4.19}$$

$$\Phi_2(q) = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{2n}) (1 - q^{4n}) (1 - q^{8n})^2, \tag{4.20}$$

and that  $\Phi_M(q)$  is the unique cusp form in  $S_3(\Gamma_0(4M), \varepsilon_M)$ , with the character  $\varepsilon_M$  on  $(\mathbb{Z}/4M\mathbb{Z})^\times$  defined by

$$\varepsilon_M(d \bmod 4M) = \left(\frac{M}{|d|}\right) (-1)^{(d-1)/2}, \quad (d, 4M) = 1. \tag{4.21}$$

Therefore the above corollary expresses unit roots of certain Hecke polynomials for weight 3 (resp. 4) in terms of *p*-adic  ${}_3\mathfrak{F}_2$  (resp.  ${}_4\mathfrak{F}_3$ ) functions.

In case (i) (resp. (ii)) of Theorem 4.1, one may also apply Theorem 3.1 (resp. 3.3) to conclude that the hypergeometric values given by the theorem may be expressed in terms of  $\Gamma_p$  as

$${}_2\mathfrak{F}_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; -1\right) = (-1)^{(p-1)/4} \frac{\Gamma_p(\frac{1}{4})^2}{\Gamma_p(\frac{1}{2})} \tag{4.22}$$

$$\left(\text{resp. } {}_3\mathfrak{F}_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; 1\right) = -\Gamma_p\left(\frac{1}{4}\right)^4\right). \tag{4.23}$$

In the first case, this value in the congruence for the  $c(n)$  has essentially been given by Coster [3, Sect. 6]; the second (for the  $b(n)$ ) may be obtained from the first together with a comparison with Lemma 2.8 (ii) and Theorem 3.1 (ii) of [4] and cases I and III of Theorem 13.1 of [17]. By comparison with Corollary 4.2 (ii) we see that (4.23) gives the  $\mathbb{F}_p$ -Jacobi sum formula

$$\beta_{4,p} = J(\omega^{(1-p)/4}, \omega^{(1-p)/4})^2 \tag{4.24}$$

for  $p \equiv 1 \pmod{4}$ .

We do not know of an expression similar to (4.22) or (4.23) for the values of the hypergeometric functions in (iii), (iv), (v) in terms of  $\Gamma_p$ . Since  $x = -1$  and  $x = 4$  are ordinary points of the differential equation (2.13) satisfied by the  ${}_3F_2$  of cases (iii), (iv), these expressions might not readily reduce to gamma values. In addition we note that in cases (i), (ii), (iii), (iv), the set of primes for which the continuation exists consists precisely of congruence classes (mod 24), whereas in case (v) the behavior is more irregular (for  $2 < p < 80,000$  the only primes for which the condition fails are  $p = 11$  and  $p = 3137$  [12]). It therefore seems likely that if there is such an expression in case (v), it is not of such a simple form.

In conclusion we remark that other types of formal-group congruences may be expressed in terms of *p*-adic hypergeometric functions.

M. Coster [4] has given mod  $p^{2r}$  determinations extending certain formal-group congruences for binomial coefficients; we have seen how such formal-group congruences may also be expressed in terms of well-poised hypergeometric functions via Theorem 3.1 (with  $2a = b$ ) and Theorem 3.3 (with  $2a = b = c$ ). J. Stienstra [16] has given a construction for logarithms of formal Picard groups and formal Brauer groups which possess integral representations resembling those of (hyper) elliptic integrals. The congruences associated to some of the examples (e.g., [16, pp. 908–909, Example 5.5, and the  $d = 2, 3$  cases of Example 4.13]) in that paper may also be expressed in terms of well-poised hypergeometric functions via Theorems 3.1 and 3.3 and the method of Section 4. We have obtained other unit-root formulae similar to those in [6] from Stienstra's construction, a topic we intend to address in a future article. Of interest also is the paper of T. Honda [11] concerning formal groups obtained by requiring that the *invariant differential* be given by a generalized hypergeometric function. Although this is a different notion than the present work, the expansion coefficients of these invariant differentials are products of generalized binomial coefficients, and therefore it may be that some of the associated congruences may be studied via the ideas of this paper.

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