



ELSEVIER

Available online at www.sciencedirect.com



Discrete Mathematics 270 (2003) 279–289

DISCRETE
MATHEMATICS

www.elsevier.com/locate/disc

Congruences for degenerate number sequences

Paul Thomas Young

Department of Mathematics, University of Charleston, Charleston, SC 29424, USA

Received 13 March 2001; received in revised form 11 February 2002; accepted 11 March 2002

Abstract

The degenerate Stirling numbers and degenerate Eulerian polynomials are intimately connected to the arithmetic of generalized factorials. In this article, we show that these numbers and similar sequences may in fact be expressed as p -adic integrals of generalized factorials. As an application of this identification we deduce systems of congruences which are analogues and generalizations of the Kummer congruences for the ordinary Bernoulli numbers.

© 2003 Elsevier B.V. All rights reserved.

Keywords: Degenerate weighted Stirling numbers; Degenerate Eulerian polynomials; Partial Stirling numbers; Kummer congruences; p -adic integration

1. Introduction

The interpretation of Stirling numbers as connection coefficients of linear transformations between generalized factorials has given rise to many useful generalizations of these numbers. For example, the *degenerate Stirling numbers* $S(n, k|\lambda)$ of the *second kind*, which were introduced by Carlitz, satisfy the relations ([2], Eq. (2.12))

$$(x|\lambda)_n = \sum_{k=0}^n S(n, k|\lambda)(x|1)_k, \tag{1.1}$$

where $(x|\lambda)_n$ is the generalized falling factorial defined in Section 2. This type of relation has permitted a unified treatment [12] of the various kinds of generalizations of Stirling numbers. The *noncentral Stirling numbers* and *noncentral Lah numbers* have been interpreted as higher-order differences of generalized factorials in [5]. Combinatorial properties of degenerate Bernoulli and Stirling polynomials have also been effectively studied by considering them as divided differences of binomial coefficients in [1]. The approach of this article will be to instead realize all these generalizations of

E-mail address: paul@math.cofc.edu (P.T. Young).

Stirling numbers, and other combinatorially important numbers, as p -adic integrals of generalized factorials $(x|\lambda)_n$. This realization implies general systems of congruences among these numbers, which are natural analogues of the Kummer congruences for the ordinary Bernoulli numbers (cf. [19], Cor. 5.14, and [20], Eq. (4.9)).

Carlitz [2] defined the second kind of degenerate Stirling numbers $S(n, k|\lambda)$ for $\lambda \neq 0$ by means of the exponential generating function

$$((1 + \lambda t)^\mu - 1)^k = k! \sum_{n=k}^{\infty} S(n, k|\lambda) \frac{t^n}{n!}, \quad (1.2)$$

where $\lambda\mu = 1$. Since $(1 + \lambda t)^\mu \rightarrow e^t$ as $\lambda \rightarrow 0$ this is evidently an extension of the generating function

$$(e^t - 1)^k = k! \sum_{n=k}^{\infty} S(n, k) \frac{t^n}{n!} \quad (1.3)$$

for the usual Stirling numbers of the second kind $S(n, k) = S(n, k|0)$. The method of this paper applies to any sequence $\alpha_n(\lambda)$ whose exponential generating function is of the form $h((1 + \lambda t)^\mu)$ with $h(T)$ a formal power series in $\mathbb{Z}_p[[T - 1]]$, where \mathbb{Z}_p is the ring of p -adic integers; we call such a sequence $\alpha_n(\lambda)$ the “degenerate number sequence arising from h ” in recognition of the analogy between the generating functions (1.2) and (1.3) for the degenerate and usual Stirling numbers of the second kind. So in our terminology $k!S(n, k|\lambda)$ is the degenerate number sequence arising from $h(T) = (T - 1)^k$.

Carlitz also considered the *degenerate Eulerian polynomials* $A_n(\lambda, x)$, which are related to generalized factorials ([2, Eqs. (7.1) and (7.5)]) by the identity

$$\sum_{k=0}^{\infty} (k|\lambda)_n x^k = \frac{A_n(\lambda, x)}{(1 - x)^{n+1}} \quad (1.4)$$

for $x \neq 1$ (here we use $A_n(\lambda, x)$ for the polynomial Carlitz referred to as $n!A_n(\lambda, x)$). Carlitz showed that these polynomials may be defined for $\lambda \neq 0$ by

$$\frac{1 - x}{1 - x(1 + \lambda t(1 - x))^\mu} = \sum_{n=0}^{\infty} A_n(\lambda, x) \frac{t^n}{n!}, \quad (1.5)$$

where $\lambda\mu = 1$ ([2, Eqs. (1.25) and (7.9)]), which reveals that the values of $A_n(\lambda, x)$ may be regarded as degenerate number sequences, in our terminology, when $1 - x$ is a p -adic unit (see Section 4). Taking the limit as $\lambda \rightarrow 0$ yields the usual Eulerian polynomials $A_n(x) = A_n(0, x)$ defined by

$$\frac{1 - x}{1 - xe^{(1-x)t}} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!}. \quad (1.6)$$

In Section 4 we show that if p is an odd prime, $1 - x$ is a p -adic unit, and $\lambda \in p\mathbb{Z}_p$, then for $m \equiv n \pmod{(p - 1)p^a}$ we have the congruence

$$A_m(\lambda, x) \equiv A_n(\lambda, x) \pmod{p^A \mathbb{Z}_p}, \quad (1.7)$$

where $A = \min\{m, n, a + 1\}$.

An congruence similar to (1.7) for the degenerate Stirling numbers $S(n, k|\lambda)$ of the second kind, which we prove in Section 3, is

$$k!S(m, k|\lambda) \equiv k!S(n, k|\lambda) \pmod{p^A \mathbb{Z}_p}, \tag{1.8}$$

where $A = \min\{m, n, a + 1\}$; this congruence also holds when $m \equiv n \pmod{(p - 1)p^a}$ and $\lambda \in p\mathbb{Z}_p$. A natural feature of our method is that stronger congruences may be obtained for the related degenerate number sequences arising from the power series $\varphi h(T)$, where φ is the linear transformation on $\mathbb{Z}_p[[T - 1]]$ defined formally by

$$\varphi h(T) = h(T) - \frac{1}{p} \sum_{\zeta^{p=1}} h(\zeta T) \tag{1.9}$$

(cf. [20], Eq. (2.14)). For example, let $T_p(n, k|\lambda)$ denote the degenerate number sequence arising from the polynomial $\varphi h(T)$, where $h(T) = (T - 1)^k$. In Section 3 we prove congruences implying

$$T_p(m, k|\lambda) \equiv T_p(n, k|\lambda) \pmod{p^{a+1} \mathbb{Z}_p} \tag{1.10}$$

for $m \equiv n \pmod{(p - 1)p^a}$ and $\lambda \in p\mathbb{Z}_p$. In this case the values at $\lambda = 0$, $T_p(n, k) = T_p(n, k|0)$, are the “partial” Stirling numbers studied by Lundell [16], Davis [8], and Clarke [7].

While many treatments of degenerate number sequences allow for arbitrary complex parameters, the parameter values in many of the important examples are actually rational numbers or integers. For this reason it is also valuable to be able to represent the sequences p -adically. Our basic representation theorem concerning degenerate number sequences, whose proof appears in Section 2, is as follows:

Theorem 1.1. *Let $h \in \mathbb{Z}_p[[T - 1]]$ and define the polynomials $\alpha_n(\lambda)$, $\hat{\alpha}_n(\lambda)$ for $n \geq 0$ by $h((1 + \lambda t)^\mu) = \sum_{n=0}^\infty \alpha_n(\lambda)(t^n/n!)$ and $(\varphi h)((1 + \lambda t)^\mu) = \sum_{n=0}^\infty \hat{\alpha}_n(\lambda)(t^n/n!)$. Then there is a \mathbb{Z}_p -valued measure γ_h on \mathbb{Z}_p such that*

$$\alpha_n(\lambda) = \int_{\mathbb{Z}_p} (x|\lambda)_n \, d\gamma_h(x)$$

and

$$\hat{\alpha}_n(\lambda) = \int_{\mathbb{Z}_p^\times} (x|\lambda)_n \, d\gamma_h(x)$$

for all $\lambda \in \mathbb{Z}_p$ and all n . Consequently $\alpha_n(\lambda), \hat{\alpha}_n(\lambda) \in \mathbb{Z}_p$ for all $\lambda \in \mathbb{Z}_p$ and all n .

If the parameters are integers (meaning that $h \in \mathbb{Z}[[T - 1]]$ and $\lambda \in \mathbb{Z}$) then $\alpha_n(\lambda) \in \mathbb{Z}$ and the above representation for $\alpha_n(\lambda)$ is independent of the choice of the prime p . This cannot be said for $\hat{\alpha}_n(\lambda)$, since the definition of φ itself is dependent on p . These two representations form the basis for our congruences. If p is a prime number we define the associated quantity q by

$$q = \begin{cases} p & \text{if } p > 2, \\ 4 & \text{if } p = 2. \end{cases} \tag{1.11}$$

Therefore $\phi(q) = p - 1$ if p is odd and $\phi(q) = 2$ if $p = 2$. Congruences (1.5), (1.8), (1.10) etc. are derived from the following theorem, which follows from Theorem 1.1.

Theorem 1.2. *With notation as above, if $\lambda \in p\mathbb{Z}_p$ and $m \equiv n \pmod{\phi(q)p^a}$ with $a \geq 0$ then*

$$\alpha_m(\lambda) \equiv \alpha_n(\lambda) \pmod{p^A \mathbb{Z}_p} \quad \text{and} \quad \hat{\alpha}_m(\lambda) \equiv \hat{\alpha}_n(\lambda) \pmod{p^{a+1} \mathbb{Z}_p},$$

where $A = \min\{m, n, a + 1\}$, whereas if $\lambda \in \mathbb{Z}_p^\times$ then for all $m \geq 0$ we have

$$\alpha_m(\lambda) \equiv 0 \pmod{m! \mathbb{Z}_p} \quad \text{and} \quad \hat{\alpha}_m(\lambda) \equiv 0 \pmod{m! \mathbb{Z}_p}.$$

In Section 3 we apply this theorem to the various generalizations of Stirling numbers, and in Section 4 we apply it to the Eulerian polynomials. Although we do not give the details here, the theorem also applies to the degenerate Bell numbers introduced by Carlitz [2], as well as other combinatorial sequences.

2. Proof of main results

Throughout this paper p will denote a prime number, \mathbb{Z}_p the ring of p -adic integers, \mathbb{Z}_p^\times the multiplicative group of units in \mathbb{Z}_p , and \mathbb{Q}_p the field of p -adic numbers. We use $\mathbb{Z}_p[T-1]$ and $\mathbb{Z}_p[[T-1]]$ to denote, respectively, the ring of polynomials and of formal power series in the indeterminate $(T-1)$ over \mathbb{Z}_p . The p -adic valuation “ ord_p ” is defined by setting $\text{ord}_p(x) = k$ if $x = p^k y$ with $y \in \mathbb{Z}_p^\times$. A congruence $x \equiv y \pmod{m\mathbb{Z}_p}$ is equivalent to $\text{ord}_p(x-y) \geq \text{ord}_p m$, and if x and y are rational numbers this congruence for all primes p is equivalent to the definition of congruence $x \equiv y \pmod{m}$ given in [11] (Section 2). The symbols λ and μ will always represent elements of \mathbb{Q}_p satisfying $\lambda\mu=1$. The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{i=0}^{n-1} (x - i\lambda) \tag{2.1}$$

for positive integers n , with the convention $(x|\lambda)_0 = 1$. Note that if $\lambda \neq 0$ then $(x|\lambda)_n = \lambda^n (\lambda^{-1}x|1)_n$.

Proof of Theorem 1.1. Let \mathcal{A} denote the set of all \mathbb{Z}_p -valued measures on \mathbb{Z}_p . As is well known (cf. [19,18]), there is a one-to-one correspondence

$$\mathcal{A} \leftrightarrow \mathbb{Z}_p[[T-1]], \tag{2.2}$$

under which each measure $\gamma \in \mathcal{A}$ corresponds to the formal power series

$$h_\gamma(T) = \int_{\mathbb{Z}_p} T^x d\gamma(x). \tag{2.3}$$

The integral in (2.3) represents a formal power series in $\mathbb{Z}_p[[T-1]]$ via the binomial expansion

$$\int_{\mathbb{Z}_p} T^x d\gamma(x) = \sum_{m=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{m} d\gamma(x) \right) (T-1)^m, \tag{2.4}$$

which is convergent for $T \in 1 + p\mathbb{Z}_p$. Let $\gamma = \gamma_h$ be the measure which corresponds to our power series $h(T)$ under this identification. Now if $t \in p\mathbb{Z}_p$ then $T = (1 + \lambda t)^\mu$ converges to an element of $1 + p\mathbb{Z}_p$ for any $\lambda \in \mathbb{Z}_p$. Substituting $T = (1 + \lambda t)^\mu$ in (2.3) yields a formal power series identity

$$\sum_{n=0}^{\infty} \alpha_n(\lambda) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1 + \lambda t)^{\mu x} d\gamma_h(x). \tag{2.5}$$

Therefore since

$$\left. \frac{d^n}{dt^n} ((1 + \lambda t)^{\mu x}) \right|_{t=0} = (x|\lambda)_n \tag{2.6}$$

and

$$\left. \frac{d^n}{dt^n} \left(\sum_{m=0}^{\infty} \alpha_m(\lambda) \frac{t^m}{m!} \right) \right|_{t=0} = \alpha_n(\lambda), \tag{2.7}$$

we have

$$\alpha_n(\lambda) = \int_{\mathbb{Z}_p} (x|\lambda)_n d\gamma_h(x). \tag{2.8}$$

From the definition of φ we compute

$$\begin{aligned} \varphi h(T) &= h(T) - \frac{1}{p} \sum_{\zeta^p=1} h(\zeta T) \\ &= \int_{\mathbb{Z}_p} \left(1 - \frac{1}{p} \sum_{\zeta^p=1} \zeta^x \right) T^x d\gamma_h(x). \end{aligned} \tag{2.9}$$

Since

$$\frac{1}{p} \sum_{\zeta^p=1} \zeta^x = \begin{cases} 0 & \text{if } x \in \mathbb{Z}_p^\times, \\ 1 & \text{if } x \in p\mathbb{Z}_p, \end{cases} \tag{2.10}$$

we then have

$$\varphi h(T) = \int_{\mathbb{Z}_p^\times} T^x d\gamma_h(x), \tag{2.11}$$

convergent for $T \in 1 + p\mathbb{Z}_p$, and valid as a power series identity in $\mathbb{Z}_p[[T - 1]]$. Substituting $T = (1 + \lambda t)^\mu$ in (2.11) with $t \in p\mathbb{Z}_p$ and evaluating the n th derivative (with respect to t) at $t = 0$ yields

$$\hat{\alpha}_n(\lambda) = \int_{\mathbb{Z}_p^\times} (x|\lambda)_n d\gamma_h(x). \tag{2.12}$$

As γ_h is a \mathbb{Z}_p -valued measure, both $\alpha_n(\lambda)$ and $\hat{\alpha}_n(\lambda)$ lie in \mathbb{Z}_p for all $\lambda \in \mathbb{Z}_p$ by (2.8) and (2.12). \square

We now turn to the congruences. It is easily established by induction that if $A \equiv B \pmod{p^a \mathbb{Z}_p[x]}$ then $A^{p^b} \equiv B^{p^b} \pmod{p^{a+b} \mathbb{Z}_p[x]}$ for any nonnegative integer b

(cf. [17], Proposition VII.3.2.4, p. 407). This principle will be needed for the proof of Theorem 1.2 and of the following proposition, which may be of some independent interest.

Proposition 2.1. *If p is an odd prime and $\lambda \in p\mathbb{Z}_p$ then for all positive integers m and r we have*

$$(x|\lambda)_{mp^r} \equiv x^{mp^r} \pmod{p^{r+1}\mathbb{Z}_p[x]}.$$

For $p = 2$, if $\lambda \in 2\mathbb{Z}_2$ then for all positive integers m and r we have

$$(x|\lambda)_{m2^r} \equiv x^{m2^r} \pmod{2^r\mathbb{Z}_2[x]}.$$

Proof. In [9] (Lemma 3) it is shown that for odd primes p ,

$$(x|1)_{p^r} \equiv (x^p - x)^{p^{r-1}} \pmod{p^r\mathbb{Z}[x]}, \quad (2.13)$$

while for $p = 2$ we have

$$(x|1)_{2^{r+1}} \equiv (x^2 - x)^{2^r} \pmod{2^r\mathbb{Z}[x]}. \quad (2.14)$$

Begin by regarding $(x|\lambda)_n$ as a homogeneous polynomial in x and λ of degree n in x , so that $(x|\lambda)_n = \lambda^n(x\lambda^{-1}|1)$. Replacing x by $x\lambda^{-1}$ and multiplying both sides of (2.13) by λ^{p^r} yields

$$(x|\lambda)_{p^r} \equiv x^{p^{r-1}}(x^{p-1} - \lambda^{p-1})^{p^{r-1}} \pmod{p^r\lambda\mathbb{Z}[x, \lambda]}, \quad (2.15)$$

since both sides of (2.13) are monic polynomials of degree p^r . By means of an evaluation homomorphism $\mathbb{Z}[x, \lambda] \rightarrow \mathbb{Z}_p[x]$, we now regard λ as an element of $p\mathbb{Z}_p$. This implies that

$$(x|\lambda)_{p^r} \equiv x^{p^{r-1}}(x^{p-1} - \lambda^{p-1})^{p^{r-1}} \pmod{p^{r+1}\mathbb{Z}_p[x]}. \quad (2.16)$$

But since $\lambda \in p\mathbb{Z}_p$, by induction we have

$$(x^{p-1} - \lambda^{p-1})^{p^{r-1}} \equiv x^{p^{r-1}(p-1)} \pmod{p^{r+1}\mathbb{Z}_p[x]} \quad (2.17)$$

for all positive integers r . By (2.16) we then have

$$(x|\lambda)_{p^r} \equiv x^{p^r} \pmod{p^{r+1}\mathbb{Z}_p[x]}. \quad (2.18)$$

Then if m is any positive integer, we have

$$\begin{aligned} (x|\lambda)_{mp^r} &= \prod_{j=0}^{m-1} (x - jp^r\lambda|\lambda)_{p^r} \equiv \prod_{j=0}^{m-1} (x - jp^r\lambda)^{p^r} \\ &\equiv x^{mp^r} \pmod{p^{r+1}\mathbb{Z}_p[x]}, \end{aligned} \quad (2.19)$$

proving the proposition for odd primes p . The result for $p = 2$ follows in the same way beginning from (2.14). \square

Since the multiplicative group $(\mathbb{Z}_p/q p^a \mathbb{Z}_p)^\times$ has order $\phi(q)p^a$ for $a \geq 0$ (where q is defined by (1.11)), it follows that $x^{\phi(q)p^a} \equiv 1 \pmod{q p^a \mathbb{Z}_p}$ for all $x \in \mathbb{Z}_p^\times$. This fact may be combined with the above results to prove our main congruence results.

Proof of Theorem 1.2. Assume that $\lambda \in p\mathbb{Z}_p$ and that $m \geq n \geq 0$. Then by Theorem 1.1,

$$\alpha_m(\lambda) - \alpha_n(\lambda) = \int_{\mathbb{Z}_p} (x|\lambda)_n ((x - n\lambda|\lambda)_{m-n} - 1) d\gamma_h(x) \tag{2.20}$$

and

$$\hat{\alpha}_m(\lambda) - \hat{\alpha}_n(\lambda) = \int_{\mathbb{Z}_p^\times} (x|\lambda)_n ((x - n\lambda|\lambda)_{m-n} - 1) d\gamma_h(x). \tag{2.21}$$

If $m \equiv n \pmod{\phi(q)p^a}$ then by Proposition 2.1 we have

$$(x - n\lambda|\lambda)_{m-n} \equiv (x - n\lambda)^{m-n} \equiv x^{m-n} \equiv 1 \pmod{p^{a+1}\mathbb{Z}_p} \tag{2.22}$$

for all $x \in \mathbb{Z}_p^\times$. Since the integrand in (2.21) lies in $p^{a+1}\mathbb{Z}_p$ for all $x \in \mathbb{Z}_p^\times$ and γ_h is a \mathbb{Z}_p -valued measure, we have $\hat{\alpha}_m(\lambda) - \hat{\alpha}_n(\lambda) \in p^{a+1}\mathbb{Z}_p$. If $x \in p\mathbb{Z}_p$ then $(x|\lambda)_n \in p^n\mathbb{Z}_p$, so by (2.20) we have $\alpha_m(\lambda) - \alpha_n(\lambda) \in p^A\mathbb{Z}_p$ where $A = \min\{n, a + 1\}$, proving the theorem in the case $\lambda \in p\mathbb{Z}_p$.

If $\lambda \in \mathbb{Z}_p^\times$ then for any $x \in \mathbb{Z}_p$ we have $(x|\lambda)_m = \lambda^m \binom{\lambda^{-1}x}{m} m! \in m!\mathbb{Z}_p$. From (2.8) and (2.12) it follows directly that $\alpha_m(\lambda)$ and $\hat{\alpha}_m(\lambda)$ lie in $m!\mathbb{Z}_p$ for all m , completing the proof of the theorem. \square

Remarks. The congruence $\alpha_m(0) \equiv \alpha_n(0) \pmod{p^A\mathbb{Z}_p}$ for $\lambda = 0$ is stated in [18] (Lemma 1) in the case of rational functions h . In [20] we showed that for $\lambda = 0$ one has in fact the stronger result $\Delta_c^k \hat{\alpha}_m(0) \equiv 0 \pmod{p^{k(a+1)}\mathbb{Z}_p}$ for all positive integers k , where Δ_c is the forward difference operator with increment $c \equiv 0 \pmod{\phi(q)p^a}$. However, the corresponding result for general $\lambda \in p\mathbb{Z}_p$ does not hold for the higher powers of Δ_c .

The congruences of Theorem 1.2 do imply that the degenerate number sequences $\alpha_n(\lambda)$ and $\hat{\alpha}_n(\lambda)$ are p -Honda sequences (cf. [17]) for all odd primes p and all $\lambda \in p\mathbb{Z}_p$; that is, they satisfy

$$\alpha_{mp^r}(\lambda) \equiv \alpha_{mp^{r-1}}(\lambda) \pmod{p^r\mathbb{Z}_p} \quad \text{and} \quad \hat{\alpha}_{mp^r}(\lambda) \equiv \hat{\alpha}_{mp^{r-1}}(\lambda) \pmod{p^r\mathbb{Z}_p} \tag{2.23}$$

for all positive integers m and r . This means that for $\lambda \in p\mathbb{Z}_p$ the differential forms

$$\sum_{n=1}^{\infty} \alpha_n(\lambda) T^n \frac{dT}{T} \quad \text{and} \quad \sum_{n=1}^{\infty} \hat{\alpha}_n(\lambda) T^n \frac{dT}{T}, \tag{2.24}$$

which are essentially formal Laplace transforms of $h((1 + \lambda t)^\mu)$ and $(\phi h)((1 + \lambda t)^\mu)$, are invariant differentials for formal group laws over \mathbb{Z}_p which are isomorphic over \mathbb{Z}_p to the formal multiplicative group law $F(X, Y) = X + Y - XY$.

3. Generalized Stirling numbers

In [12] a very general class $S(n, k; \alpha, \beta, r)$ of sequences which generalize the Stirling numbers was defined by the relations

$$(x|\alpha)_n = \sum_{k=0}^n S(n, k; \alpha, \beta, r) (x - r|\beta)_k. \tag{3.1}$$

These sequences satisfy inverse relations in pairs related by $(\alpha, \beta, r) \rightarrow (\beta, \alpha, -r)$; that is, (3.1) is inverse to

$$(x|\beta)_n = \sum_{k=0}^n S(n, k; \beta, \alpha, -r)(x+r|\alpha)_k. \quad (3.2)$$

An exponential generating function for the sequence $S(n, k; \alpha, \beta, r)$ is

$$(1 + \alpha t)^{r/\alpha} \left(\frac{(1 + \alpha t)^{\beta/\alpha} - 1}{\beta} \right)^k = k! \sum_{n=0}^{\infty} S(n, k; \alpha, \beta, r) \frac{t^n}{n!} \quad (3.3)$$

[12] (Theorem 2), which indicates that $k!S(n, k; \alpha, \beta, r)$ is the degenerate number sequence arising from $h(T) = T^r(T^\beta - 1)^k/\beta^k$ (with $\lambda = \alpha$), if this power series lies in $\mathbb{Z}_p[T-1]$. This will always be the case whenever $r \in \mathbb{Z}_p$ and $\beta \in \mathbb{Z}_p^\times$. If $\beta \in p\mathbb{Z}_p$ then Theorems 1.1 and 1.2 still apply to the series $\beta^k h(T)$, which lies in $\mathbb{Z}_p[T-1]$, but the congruences resulting from Theorem 1.2 for $S(n, k; \alpha, \beta, r)$ will be significantly weaker than those for $\beta \in \mathbb{Z}_p^\times$, which we record here.

Theorem 3.1. *Suppose that $r \in \mathbb{Z}_p$ and $\beta \in \mathbb{Z}_p^\times$. Then for all $\alpha \in p\mathbb{Z}_p$ we have*

$$k!S(m, k; \alpha, \beta, r) \equiv k!S(n, k; \alpha, \beta, r) \pmod{p^A \mathbb{Z}_p}$$

whenever $m \equiv n \pmod{\phi(q)p^a}$, where $A = \min\{m, n, a + 1\}$, and for all $\alpha \in \mathbb{Z}_p^\times$ we have

$$k!S(m, k; \alpha, \beta, r) \equiv 0 \pmod{m! \mathbb{Z}_p}$$

for all $m \geq 0$.

For odd primes p , putting $m = p$ and $n = 1$ in this theorem yields the congruence

$$S(p, k; \alpha, \beta, r) \equiv 0 \pmod{p\mathbb{Z}_p} \quad (3.4)$$

for $1 < k < p$ as long as $\beta \in \mathbb{Z}_p^\times$ and $\alpha, r \in \mathbb{Z}_p$; this congruence has been given for integers α, β, r in [12] (Theorem 3). We mention some other special cases to which this theorem applies. If $r = 0$, $\alpha = 0$, and $\beta = 1$ then $S(n, k; 0, 1, 0) = S(n, k)$ are the classical Stirling numbers of the second kind; in this case the result of Theorem 3.1 has appeared in [18] (Theorem A), but is stated incorrectly there (the terms $S(n, k)$, $S(m, k)$ should be replaced by $k!S(n, k)$ and $k!S(m, k)$). If $\alpha = 0$ and $\beta = 1$ we have the *weighted Stirling numbers of the second kind* $R(n, k, r)$ studied by Carlitz [3,4], and the congruences are valid when the weight r is a p -adic integer. This case includes the *noncentral Stirling numbers of the second kind* (cf. [14,6]) and the *noncentral Lah numbers* [6]. If $\beta = 1$ and $r = 0$ we have the degenerate Stirling numbers of the second kind and the congruences hold for $\alpha = \lambda \in \mathbb{Z}_p$. If $\beta = 1$ we have the *degenerate weighted Stirling numbers of the second kind* studied by Howard [10], and the congruences hold when $\alpha = \lambda \in \mathbb{Z}_p$ and the weight r is a p -adic integer.

For a prime p let $T_p(n, k; \alpha, \beta, r)$ be the degenerate number sequence arising from $\phi h(T)$, where $h(T) = T^r(T^\beta - 1)^k/\beta^k$ with $\lambda = \alpha$. We have the following congruences directly from Theorem 1.2.

Theorem 3.2. *Suppose that $r \in \mathbb{Z}_p$ and $\beta \in \mathbb{Z}_p^\times$. Then for all $\alpha \in p\mathbb{Z}_p$ we have*

$$T_p(m, k; \alpha, \beta, r) \equiv T_p(n, k; \alpha, \beta, r) \pmod{p^{a+1}\mathbb{Z}_p}$$

whenever $m \equiv n \pmod{\phi(q)p^a}$, and for all $\alpha \in \mathbb{Z}_p^\times$ we have

$$T_p(m, k; \alpha, \beta, r) \equiv 0 \pmod{m!\mathbb{Z}_p}$$

for all $m \geq 0$.

As mentioned in Section 1, the sequence $T_p(n, k) = T_p(n, k; 0, 1, 0)$ is the sequence of “partial” Stirling numbers studied in [16,7,8]; in general for $\alpha = \lambda = 0$ we may write

$$T_p(n, k; 0, \beta, r) = \beta^{-k} \sum_{\substack{0 \leq j \leq k \\ p \nmid \beta j + r}} (-1)^{k-j} \binom{k}{j} (\beta j + r)^n. \tag{3.5}$$

Clarke [7] conjectured that $T_p(n, k)$ and $k!S(n, k)$ always have the same p -adic valuation. This motivates the following more general question: For which $h \in \mathbb{Z}_p[[T - 1]]$ is it true that $\alpha_n(0)$ and $\hat{\alpha}_n(0)$ have the same p -adic valuation for all n ? Clarke’s conjecture is that $h(T) = (T - 1)^k$ always satisfies this condition.

4. Degenerate Eulerian polynomials

We define the *degenerate weighted Eulerian polynomials* $A_n(\lambda, r, x)$ by

$$\sum_{k=0}^{\infty} (k + r|\lambda)_n x^k = \frac{A_n(\lambda, r, x)}{(1 - x)^{n+1}} \tag{4.1}$$

for $\lambda \neq 0$ and $x \neq 1$. We remark that the polynomial $A_n(\lambda, r, x)$ was denoted by $n!A_n(\lambda, x, r)$ in [2] and by $A_n(x, r|\lambda)$ in [13]. By [15] (Proposition 2.1) we have the identities

$$A_{n,k}(\lambda, r) = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k + r - j|\lambda)_n \tag{4.2}$$

and

$$(t + r|\lambda)_n = \sum_{k=0}^n A_{n,k}(\lambda, r) \binom{t + n - k}{n}, \tag{4.3}$$

where

$$A_n(\lambda, r, x) = \sum_{k=0}^n A_{n,k}(\lambda, r) x^k. \tag{4.4}$$

These polynomials have been treated in [2] and [13], and may also be studied by the methods of Koutras [15]. When the weight r is taken to be zero we have the

degenerate Eulerian polynomials $A_n(\lambda, 0, x) = A_n(\lambda, x)$. Carlitz [2] showed that these polynomials have the generating function

$$\frac{(1-x)(1+\lambda t(1-x))^{\mu r}}{1-x(1+\lambda t(1-x))^{\mu}} = \sum_{n=0}^{\infty} A_n(\lambda, r, x) \frac{t^n}{n!} \quad (4.5)$$

and taking the limit as $\lambda \rightarrow 0$ we get

$$\frac{(1-x)e^{(1-x)rt}}{1-xe^{(1-x)t}} = \sum_{n=0}^{\infty} A_n(0, r, x) \frac{t^n}{n!}. \quad (4.6)$$

Taking $r = 0$ in (4.6) yields the usual Eulerian polynomials $A_n(x) = A_n(0, 0, x)$. The following congruences may be derived from Theorem 1.2.

Theorem 4.1. *Suppose that $1-x \in \mathbb{Z}_p^\times$ and $r \in \mathbb{Z}_p$. Then if $\lambda \in p\mathbb{Z}_p$ and $m \equiv n \pmod{\phi(q)p^a}$ we have*

$$A_m(\lambda, r, x) \equiv A_n(\lambda, r, x) \pmod{p^A \mathbb{Z}_p},$$

where $A = \min\{m, n, a+1\}$, and if $\lambda \in \mathbb{Z}_p^\times$ we have

$$A_m(\lambda, r, x) \equiv 0 \pmod{m! \mathbb{Z}_p}$$

for all $m \geq 0$.

Proof. If $1-x \in \mathbb{Z}_p^\times$ and $r \in \mathbb{Z}_p$, then the formal power series

$$h(T) = \frac{(1-x)T^{(1-x)r}}{1-xT^{1-x}} = T^{(1-x)r} \left(1 - \left(\frac{x}{1-x} \right) (T^{1-x} - 1) \right)^{-1} \quad (4.7)$$

lies in $\mathbb{Z}_p[[T-1]]$, because $x/(1-x) \in \mathbb{Z}_p$ and $T^{1-x} - 1$ is a power series in $\mathbb{Z}_p[[T-1]]$ with no constant term. If we put $\lambda' = (1-x)\lambda$ and $\lambda'\mu' = 1$ then $h((1+\lambda't)^{\mu'})$ is the generating function in (4.5) and the corresponding numbers $\alpha_n(\lambda') = A_n(\lambda, r, x)$ satisfy the congruences of Theorem 1.2. Since $\text{ord}_p \lambda' = \text{ord}_p \lambda$, the result follows. \square

References

- [1] A. Adelberg, A finite difference approach to degenerate Bernoulli and Stirling polynomials, *Discrete Math.* 140 (1995) 1–21.
- [2] L. Carlitz, Degenerate Stirling, Bernoulli, and Eulerian numbers, *Utilitas Math.* 15 (1979) 51–88.
- [3] L. Carlitz, Weighted Stirling numbers of the first and second kind—I, *Fibonacci Quart.* 18 (1980) 147–162.
- [4] L. Carlitz, Weighted Stirling numbers of the first and second kind—II, *Fibonacci Quart.* 18 (1980) 242–257.
- [5] C. Charalambides, M. Koutras, On the differences of the generalized factorials at an arbitrary point and their combinatorial applications, *Discrete Math.* 47 (1983) 183–201.
- [6] C. Charalambides, J. Singh, A review of the Stirling numbers, their generalizations and statistical applications, *Comm. Statist. Theory Methods* 17 (1988) 2533–2595.
- [7] F. Clarke, Hensel's lemma and the divisibility by primes of Stirling-like numbers, *J. Number Theory* 52 (1995) 69–84.
- [8] D.M. Davis, Divisibility by 2 of Stirling-like numbers, *Proc. Amer. Math. Soc.* 110 (1990) 597–600.

- [9] A. Gertsch, A. Robert, Some congruences concerning the Bell numbers, *Bull. Belg. Math. Soc.* 3 (1996) 467–475.
- [10] F.T. Howard, Degenerate weighted Stirling numbers, *Discrete Math.* 57 (1985) 45–58.
- [11] F.T. Howard, Explicit formulas for degenerate Bernoulli numbers, *Discrete Math.* 162 (1996) 175–185.
- [12] L. Hsu, P. Shiue, A unified approach to generalized Stirling numbers, *Adv. Appl. Math.* 20 (1998) 366–384.
- [13] L. Hsu, P. Shiue, On certain summation problems and generalizations of Eulerian polynomials and numbers, *Discrete Math.* 204 (1999) 237–247.
- [14] M. Koutras, Non-central Stirling numbers and some applications, *Discrete Math.* 42 (1982) 73–89.
- [15] M. Koutras, Eulerian numbers associated with sequences of polynomials, *Fibonacci Quart.* 32 (1994) 44–57.
- [16] A. Lundell, A divisibility property of Stirling numbers, *J. Number Theory* 10 (1978) 35–54.
- [17] A. Robert, *A Course in p -adic Analysis*, Springer, New York, 2000.
- [18] H. Tsumura, On some congruences for the Bell numbers and for the Stirling numbers, *J. Number Theory* 38 (1991) 206–211.
- [19] L. Washington, *Introduction to Cyclotomic Fields*, Springer, New York, 1982.
- [20] P.T. Young, Congruences for Bernoulli, Euler, and Stirling numbers, *J. Number Theory* 78 (1999) 204–227.