Appendix
Curious and Exotic Identities for Bernoulli Numbers

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Bernoulli numbers, which are ubiquitous in mathematics, typically appear either as the Taylor coefficients of $x/\tan x$ or else, very closely related to this, as special values of the Riemann zeta function. But they also sometimes appear in other guises and in other combinations. In this appendix we want to describe some of the less standard properties of these fascinating numbers.

In Sect. A.1, which is the foundation for most of the rest, we show that, as well as the familiar (and convergent) *exponential* generating series[1]

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}\, x^n = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \cdots \tag{A.1}$$

defining the Bernoulli numbers, the less familiar (and divergent) *ordinary* generating series

$$\beta(x) = \sum_{n=0}^{\infty} B_n\, x^n = 1 - \frac{x}{2} + \frac{x^2}{6} - \frac{x^4}{30} + \frac{x^6}{42} - \cdots \tag{A.2}$$

also has many virtues and is often just as useful as, or even more useful than, its better-known counterpart (A.1). As a first application, in Sect. A.2 we discuss the "modified Bernoulli numbers"

$$B_n^* = \sum_{r=0}^{n} \binom{n+r}{2r} \frac{B_r}{n+r} \qquad \big(n \ge 1\big)\,. \tag{A.3}$$

[1]Here, and throughout this appendix, we use the convention $B_1 = -1/2$, rather than the convention $B_1 = 1/2$ used in the main text of the book.

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These numbers, which arose in connection with the trace formula for the Hecke operators acting on modular forms on $\mathrm{SL}(2,\mathbb{Z})$, have several unexpected properties, including the surprising periodicity

$$B_{n+12}^* = B_n^* \qquad (n \text{ odd}) \tag{A.4}$$

and a modified form of the classical von Staudt–Clausen formula for the value of B_n modulo 1. The following section is devoted to an identity discovered by Miki [A10] (and a generalization due to Gessel [A4]) which has the striking property of involving Bernoulli sums both of type $\sum B_r B_{n-r}$ and $\sum \binom{n}{r} B_r B_{n-r}$, i.e., sums related to both the generating functions (A.1) and (A.2). In Sect. A.4 we look at products of Bernoulli numbers and Bernoulli polynomials in more detail. In particular, we prove the result (discovered by Nielsen) that when a product of two Bernoulli polynomials is expressed as a linear combination of Bernoulli polynomials, then the coefficients are themselves multiples of Bernoulli numbers. This generalizes to a formula for the product of two Bernoulli polynomials in two different arguments, and leads to a further proof, due to I. Artamkin, of the Miki–Gessel identities. Finally, in Sect. A.5 we discuss the continued fraction expansions of various power series related to both (A.1) and (A.2) and, as an extra titbit, describe an unexpected appearance of one of these continued fraction expansions in connection with some recent and amazing discoveries of Yu. Matiyasevich concerning the non-trivial zeros of the Riemann zeta function.

This appendix can be read independently of the main text and we will recall all facts and notations needed. We should also add a warning: if you don't like generating functions, don't read this appendix!

A.1 The "Other" Generating Function(s) for the Bernoulli Numbers

Given a sequence of interesting numbers $\{a_n\}_{n\geq 0}$, one often tries to understand them by using the properties of the corresponding generating functions. The two most popular choices for these generating functions are $\sum_{n=0}^{\infty} a_n x^n$ ("ordinary generating function") and $\sum_{n=0}^{\infty} a_n x^n/n!$ ("exponential generating function"). Usually, of course, at most one of these turns out to have useful properties. For the Bernoulli numbers the standard choice is the exponential generating function (A.1) because it has an expression "in closed form." What is not so well known is that the ordinary generating function of the Bernoulli numbers, i.e., the power series (A.2), even though it is divergent for all non-zero complex values of x, also has extremely attractive properties and many nice applications. The key property that makes it useful, despite its being divergent and not being expressible as an elementary function, is the following functional equation:

Proposition A.1. *The power series* (A.2) *is the unique solution in* $\mathbb{Q}[[x]]$ *of the equation*

$$\frac{1}{1-x}\,\beta\Big(\frac{x}{1-x}\Big) - \beta(x) = x\,. \tag{A.5}$$

Proof. Let $\{B_n\}$ be unspecified numbers and define $\beta(x)$ by the first equality in (A.2). Then comparing the coefficients of x^m in both sides of (A.5) gives

$$\sum_{n=0}^{m-1} \binom{m}{n} B_n = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m > 1. \end{cases} \tag{A.6}$$

This is the same as the standard recursion for the Bernoulli numbers obtained by multiplying both sides of (A.1) by $e^x - 1$ and comparing the coefficients of $x^m/m!$ on both sides. □

The functional equation (A.5) can be rewritten in a slightly prettier form by setting

$$\beta_1(x) = x\,\beta(x) = \sum_{n=0}^{\infty} B_n\, x^{n+1}\,,$$

in which case it becomes simply

$$\beta_1\Big(\frac{x}{1-x}\Big) - \beta_1(x) = x^2\,. \tag{A.7}$$

A generalization of this is given by the following proposition.

Proposition A.2. *For each integer* $r \geq 1$, *the power series*

$$\beta_r(x) = \sum_{n=0}^{\infty} \binom{n+r-1}{n} B_n\, x^{n+r} \tag{A.8}$$

satisfies the functional equation

$$\beta_r\Big(\frac{x}{1-x}\Big) - \beta_r(x) = r\, x^{r+1} \tag{A.9}$$

and is the unique power series having this property.

Proof. Equation (A.9) for any fixed value of $r \geq 1$ is equivalent to the recursion (A.6), by the calculation

$$\beta_r\Big(\frac{x}{1-x}\Big) - \beta_r(x) = \sum_{n=0}^{\infty} \binom{n+r-1}{n} B_n \sum_{\ell=n+r}^{\infty} \binom{\ell}{n+r-1} x^{\ell+1}$$
$$= \sum_{\ell=r}^{\infty} \binom{\ell}{r-1} x^{\ell+1} \Bigg(\sum_{n=0}^{\ell-r} \binom{\ell-r+1}{n} B_n \Bigg) = r\, x^{r+1}\,.$$

Alternatively, we can deduce (A.9) from (A.7) by induction on r by using the easily checked identity

$$x^2\, \beta_r'(x) = r\, \beta_{r+1}(x) \qquad (r \geq 1) \tag{A.10}$$

and the fact that

$$x^2 \frac{d}{dx} F\Big(\frac{x}{1-x}\Big) = \Big(\frac{x}{1-x}\Big)^2 F'\Big(\frac{x}{1-x}\Big) \tag{A.11}$$

for any power series $F(x)$. $\square$

We observe next that the definition (A.8) makes sense for any r in $\mathbb{Z}$,[2] and that the properties (A.9) and (A.10) still hold. But this extension is not particularly interesting since $\beta_{-k}(x)$ for $k \in \mathbb{Z}_{\geq 0}$ is just a known polynomial in $1/x$:

$$\beta_{-k}(x) = \sum_{n=0}^{\infty} \binom{n-k-1}{n} B_n\, x^{n-k} = \sum_{n=0}^{k} (-1)^n \binom{k}{n} \frac{B_n}{x^{k-n}}$$
$$= B_k\Big(\frac{1}{x}\Big) + \frac{k}{x^{k-1}} = B_k\Big(\frac{1}{x}+1\Big) = (-1)^k\, B_k\Big(-\frac{1}{x}\Big),$$

where $B_k(X)$ is the kth Bernoulli polynomial. (One can also prove these identities by induction on k, using either (A.10) or else (A.9) together with the uniqueness statement in Proposition A.2 and the corresponding well-known functional equation for the Bernoulli polynomials.) However, there is a different and more interesting way to extend the definition of β_r to non-positive integral values of r. For $k \in \mathbb{Z}$, define

$$\gamma_k(x) = \sum_{n \geq \max(1,-k)} \frac{(n-1)!}{(n+k)!} B_{n+k}\, x^n \quad \in\ x\, \mathbb{Q}[[x]]\,.$$

Then one easily checks that $\gamma_{-r}(x) = (r-1)!\, \beta_r(x)$ for $r > 0$, so that the negative-index power series γ_k are just renormalized versions of the positive-index power series β_r. But now we do get interesting power series (rather than merely polynomials) when $k \geq 0$, e.g.

[2] Or even in $\mathbb{C}$ if we work formally in $x^r\, \mathbb{Q}[[x]]$.

$$\gamma_0(x) = \sum_{n=1}^{\infty} \frac{B_n\, x^n}{n}, \quad \gamma_1(x) = \sum_{n=1}^{\infty} \frac{B_{n+1}\, x^n}{n(n+1)}, \quad \gamma_2(x) = \sum_{n=1}^{\infty} \frac{B_{n+2}\, x^n}{n(n+1)(n+2)}. \tag{A.12}$$

The properties of these new functions corresponding to (A.10) and (A.9) are given by:

Proposition A.3. *The power series $\gamma_k(x)$ satisfy the differential recursion*

$$x^2\, \gamma_k'(x) = \gamma_{k-1}(x) - \frac{B_k}{k!}\, x \qquad (k \geq 0) \tag{A.13}$$

(*with* $\gamma_{-1}(x) = \beta_1(x)$) *as well as the functional equations*

$$\gamma_0\Big(\frac{x}{1-x}\Big) - \gamma_0(x) = \quad \log(1-x) + x\,, \tag{A.14}$$
$$\gamma_1\Big(\frac{x}{1-x}\Big) - \gamma_1(x) = -\Big(\frac{1}{x} - \frac{1}{2}\Big)\, \log(1-x) - 1\,,$$

and more generally for $k \geq 1$

$$\gamma_k\Big(\frac{x}{1-x}\Big) - \gamma_k(x) = \frac{(-1)^k}{k!}\left[B_k\Big(\frac{1}{x}\Big)\, \log(1-x) + P_{k-1}\Big(\frac{1}{x}\Big)\right], \tag{A.15}$$

where $P_{k-1}(X)$ is a polynomial of degree $k-1$, the first few values of which are $P_0(X) = 1$, $P_1(X) = X - \frac{1}{2}$, $P_2(X) = X^2 - X + \frac{1}{12}$, $P_3(X) = X^3 - \frac{3}{2}X^2 + \frac{1}{3}X + \frac{1}{12}$ and $P_4(X) = X^4 - 2X^3 + \frac{3}{4}X^2 + \frac{1}{4}X - \frac{13}{360}$.

Proof. Equation (A.13) follows directly from the definitions, and then Eqs. (A.14) and (A.15) (by induction over k) follow successively from (A.7) using the general identity (A.11). □

We end this section with the observation that, although $\beta(x)$ and the related power series $\beta_r(x)$ and $\gamma_k(x)$ that we have discussed are divergent and do not give the Taylor or Laurent expansion of any elementary functions, they are related to the *asymptotic* expansions of very familiar, "nearly elementary" functions. Indeed, Stirling's formula in its logarithmic form says that the logarithm of Euler's Gamma function has the asymptotic expansion

$$\log \Gamma(X) \sim \Big(X - \frac{1}{2}\Big) \log X - X + \frac{1}{2} \log(2\pi) + \sum_{n=2}^{\infty} \frac{B_n}{n(n-1)}\, X^{-n+1}$$

as $X \to \infty$, and hence that its derivative $\psi(X)$ ("digamma function") has the expansion

$$\psi(X) := \frac{\Gamma'(X)}{\Gamma(X)} \sim \log X - \frac{1}{2X} - \sum_{n=2}^{\infty} \frac{B_n}{n}\, X^{-n} = \log X - \gamma_0\Big(-\frac{1}{X}\Big)$$

as $X \to \infty$, with $\gamma_0(x)$ defined as in Eq. (A.12), and the functions $\beta_r(x)$ correspond similarly to the derivatives of $\psi(x)$ ("polygamma functions"). The transformation $x \mapsto x/(1-x)$ occurring in the functional equations (A.5), (A.9), (A.14) and (A.15) corresponds under the substitution $X = -1/x$ to the translation $X \mapsto X+1$, and the compatibility equation (A.11) simply to the fact that this translation commutes with the differential operator d/dX, while the functional equations themselves reflect the defining functional equation $\Gamma(X+1) = X\Gamma(X)$ of the Gamma function.

A.2 An Application: Periodicity of Modified Bernoulli Numbers

The "modified Bernoulli numbers" defined by (A.3) were introduced in [A14]. These numbers, as already mentioned in the introduction, occurred naturally in a certain elementary derivation of the formula for the traces of Hecke operators acting on modular forms for the full modular group [A15]. They have two surprising properties which are parallel to the two following well-known properties of the ordinary Bernoulli numbers:

$$n > 1 \text{ odd} \qquad \Rightarrow \qquad B_n = 0\,, \tag{A.16}$$

$$n > 0 \text{ even} \qquad \Rightarrow \qquad B_n \equiv -\sum_{\substack{p \text{ prime} \\ (p-1)|n}} \frac{1}{p} \pmod{1} \tag{A.17}$$

(von Staudt–Clausen theorem). These properties are given by:

Proposition A.4. *Let B_n^* $(n > 0)$ be the numbers defined by (A.3). Then for n odd we have*

$$B_n^* = \begin{cases} \pm 3/4 & \textit{if } n \equiv \pm 1 \pmod{12}, \\ \mp 1/4 & \textit{if } n \equiv \pm 3 \textit{ or } \pm 5 \pmod{12}, \end{cases} \tag{A.18}$$

and for n even we have the modified von Staudt–Clausen formula

$$2nB_n^* - B_n \equiv \sum_{\substack{p \textit{ prime} \\ (p+1)|n}} \frac{1}{p} \pmod{1}\,. \tag{A.19}$$

Remark. The modulo 12 periodicity in (A.18) is related, via the above-mentioned connection with modular forms on the full modular group $\mathrm{SL}(2,\mathbb{Z})$, with the well-known fact that the space of these modular forms of even weight $k > 2$ is the sum of $k/12$ and a number that depends only on $k \pmod{12}$.

Proof. The second assertion is an easy consequence of the corresponding property (A.17) of the ordinary Bernoulli numbers and we omit the proof. (It is given in [A15].) To prove the first, we use the generating functions for Bernoulli numbers introduced in Sect. A.1. Specifically, for $\lambda \in \mathbb{Q}$ we define a power series $g_\lambda(t) \in \mathbb{Q}[[t]]$ by the formula

$$g_\lambda(t) = \gamma_0\Big(\frac{t}{1-\lambda t+t^2}\Big) - \log(1-\lambda t+t^2)\,,$$

where $\gamma_0(x) = \sum_{n>0} B_n x^n/n$ is the power series defined in (A.12). For $\lambda = 2$ this specializes to

$$g_2(t) = \sum_{r=1}^{\infty} \frac{B_r}{r}\,\frac{t^r}{(1-t)^{2r}} - 2\log(1-t) = 2\sum_{n=1}^{\infty} B_n^* t^n\,. \tag{A.20}$$

with B_n^* as in (A.3). On the other hand, the functional equation (A.14) applied to $x = t/(1-\lambda t+t^2)$, together with the parity property $\gamma_0(x)+x = \gamma_0(-x)$, which is a restatement of (A.16), implies the two functional equations

$$g_{\lambda+1}(t) = g_\lambda(t) + \frac{t}{1-\lambda t+t^2} = g_{-\lambda}(-t)$$

for the power series g_λ. From this we deduce

$$\begin{aligned} g_2(t)-g_2(-t) &= \big(g_2(t)-g_1(t)\big) + \big(g_1(t)-g_0(t)\big) + \big(g_0(t)-g_{-1}(t)\big) \\ &= \frac{t}{1-t+t^2} + \frac{t}{1+t^2} + \frac{t}{1+t+t^2} = \frac{3t-t^3-t^5+t^7+t^9-3t^{11}}{1-t^{12}}\,, \end{aligned}$$

and comparing this with (A.20) immediately gives the desired formula (A.18) for B_n^*, n odd. ☐

We mention one further result about the modified Bernoulli numbers from [A15]. The ordinary Bernoulli numbers satisfy the asymptotic formula

$$B_n \sim (-1)^{(n-2)/2}\,\frac{2\,n!}{(2\pi)^n} \qquad (n\to\infty,\ n \text{ even}). \tag{A.21}$$

As one might expect, the modified ones have asymptotics given by a very similar formula:

$$B_n^* \sim (-1)^{(n-2)/2}\,\frac{(n-1)!}{(2\pi)^n} \qquad (n\to\infty,\ n \text{ even}). \tag{A.22}$$

The (small) surprise is that, while the asymptotic formula (A.21) holds to all orders in $1/n$ (because the ratio of the two sides equals $\zeta(n) = 1 + \mathrm{O}(2^{-n})$), this is not true of the new formula (A.22), which only acquires this property if the right-hand side is replaced by $(-1)^{n/2}\pi\, Y_n(4\pi)$, where $Y_n(x)$ is the nth Bessel function of the second kind.

Here is a small table of the numbers B_n^* and $\tilde{B}_n = 2n B_n^* - B_n$ for n even:

n	2	4	6	8	10	12	14	16	18	20	22
B_n^*	$\frac{1}{24}$	$-\frac{27}{80}$	$-\frac{29}{1260}$	$\frac{451}{1120}$	$-\frac{65}{264}$	$-\frac{6571}{12012}$	$\frac{571}{312}$	$-\frac{181613}{38080}$	$\frac{23663513}{1220940}$	$-\frac{10188203}{83600}$	$\frac{564133}{552}$
$\tilde{B}_n$	0	$-\frac{8}{3}$	$-\frac{3}{10}$	$\frac{136}{21}$	-5	$-\frac{4249}{330}$	$\frac{651}{13}$	$-\frac{3056}{21}$	$\frac{109269}{170}$	$-\frac{247700}{57}$	38775

A.3 Miki's Identity

The surprising identity described in this section was found and proved by Miki [A10] in an indirect and non-elementary way, using p-adic methods. In this section we describe two direct proofs of it, or rather, of it and of a very similar identity discovered by Faber and Pandharipande in connection with Chern numbers of moduli spaces of curves. The first, which is short but not very enlightening, is a variant of a proof I gave of the latter identity [A2] (but which with a slight modification works for Miki's original identity as well). The second one, which is more natural, is a slight reworking of the proof given by Gessel [A4] based on properties of Stirling numbers of the second kind. In fact, Gessel gives a more general one-parameter family of identities, provable by the same methods, of which both the Miki and the Faber–Pandharipande identities are special cases. In Sect. A.4 we will give yet a third proof of these identities, following I. Artamkin [A1].

Proposition A.5 (Miki). *Write $\mathcal{B}_n = (-1)^n B_n/n$ for $n > 0$. Then for all $n > 2$ we have*

$$\sum_{i=2}^{n-2} \mathcal{B}_i \mathcal{B}_{n-i} = \sum_{i=2}^{n-2} \binom{n}{i} \mathcal{B}_i \mathcal{B}_{n-i} + 2 H_n \mathcal{B}_n\,, \tag{A.23}$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the nth harmonic number.

(Faber–Pandharipande). *Write $b_g = (2 - 2^{2g})\,\dfrac{B_{2g}}{(2g)!}$ for $g \geq 0$. Then for all $g > 0$ we have*

$$\sum_{\substack{g_1+g_2=g\\ g_1,\,g_2>0}} \frac{(2g_1-1)!\,(2g_2-1)!}{2\,(2g-1)!}\, b_{g_1} b_{g_2} = \sum_{n=1}^{g} \frac{2^{2n} B_{2n}}{2n\,(2n)!}\, b_{g-n} + H_{2g-1}\, b_g\,. \tag{A.24}$$

First proof. We prove (A.24), following [A2]. Write the identity as $a(g) = b(g) + c(g)$ in the obvious way, and let $A(x) = \sum_{g=1}^{\infty} a(g)\, x^{2g-1}$, $B(x) = \sum_{g=1}^{\infty} b(g)\, x^{2g-1}$ and $C(x) = \sum_{g=1}^{\infty} c(g)\, x^{2g-1}$ be the corresponding odd generating functions. Using the identity $\sum_{g=0}^{\infty} b_g\, x^{2g-1} = \dfrac{1}{\sinh x}$, we obtain

$$\begin{aligned}
A(x) &= \frac{1}{2} \sum_{g_1,\, g_2 > 0} b_{g_1} b_{g_2} \int_0^x t^{2g_1-1}\,(x-t)^{2g_2-1}\, dt \qquad \text{(by Euler's beta integral)} \\
&= \frac{1}{2} \int_0^x \Big(\frac{1}{t} - \frac{1}{\sinh t}\Big)\Big(\frac{1}{x-t} - \frac{1}{\sinh(x-t)}\Big)\, dt\,, \\
B(x) &= \frac{1}{\sinh x} \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{2n\,(2n)!}\, x^{2n} = \frac{1}{\sinh x}\, \log\Big(\frac{\sinh x}{x}\Big)\,, \\
C(x) &= \sum_{g=1}^{\infty} b_g \int_0^x \frac{x^{2g-1} - t^{2g-1}}{x-t}\, dt = \int_0^x \Big[\frac{1}{x-t}\Big(\frac{1}{\sinh x} - \frac{1}{\sinh t}\Big) + \frac{1}{xt}\Big]\, dt\,,
\end{aligned}$$

and hence, symmetrizing the integral giving $C(x)$ with respect to $t \to x - t$,

$$\begin{aligned}
2\,A(x) - 2\,C(x) &= \int_0^x \bigg\{\Big(\frac{1}{t} - \frac{1}{\sinh t}\Big)\Big(\frac{1}{x-t} - \frac{1}{\sinh(x-t)}\Big) \\
&\qquad - \Big(\frac{1}{x-t} + \frac{1}{t}\Big)\Big(\frac{1}{\sinh x} + \frac{1}{x}\Big) \\
&\qquad + \frac{1}{x-t}\,\frac{1}{\sinh t} + \frac{1}{t}\,\frac{1}{\sinh(x-t)}\bigg\}\, dt \\
&= \int_0^x \Big(\frac{1}{\sinh(t)\,\sinh(x-t)} - \frac{x}{\sinh x}\,\frac{1}{t\,(x-t)}\Big)\, dt \\
&= \frac{1}{\sinh x}\, \log\Big(\frac{\sinh t}{t} \cdot \frac{x-t}{\sinh(x-t)}\Big)\Big|_{t=0}^{t=x} = 2\,B(x)\,.
\end{aligned}$$

A similar proof can be given for Miki's original identity (A.23), with "sinh" replaced by "tanh". ☐

Second proof. Now we prove (A.23), following the method in [A4]. Recall that the *Stirling number of the second kind* $S(k, m)$ is defined as the number of partitions of a set of k elements into m non-empty subsets or, equivalently, as $1/m!$ times the number of surjective maps from the set $\{1, 2, \ldots, k\}$ to the set $\{1, 2, \ldots, m\}$. It can be given either by the closed formula

$$S(k,m) = \frac{1}{m!} \sum_{\ell=0}^{m} (-1)^{m-\ell} \binom{m}{\ell} \ell^k \tag{A.25}$$

(this follows immediately from the second definition and the inclusion-exclusion principle, since ℓ^k is the number of maps from $\{1,2,\ldots,k\}$ to a given set of ℓ elements) or else by either of the two generating functions

$$\begin{aligned} \sum_{k=0}^{\infty} S(k,m)\, x^k &= \frac{x^m}{(1-x)(1-2x)\cdots(1-mx)}, \\ \sum_{k=0}^{\infty} S(k,m)\, \frac{x^k}{k!} &= \frac{(e^x-1)^m}{m!}, \end{aligned} \tag{A.26}$$

both of which can be deduced easily from (A.25). (Of course all of these formulas are standard and can be found in many books, including Chap. 2 of this one, where $S(k,m)$ is denoted using Knuth's notation $\left\{ {k \atop m} \right\}$.) From either generating function one finds easily that $S(k,m)$ vanishes for $k < m$, $S(m,m) = 1$, $S(m+1,m) = \frac{m^2+m}{2}$, and more generally that $S(m+n,m)$ for a fixed value of n is a polynomial in m (of degree $2n$, and without constant term if $n > 0$). Gessel's beautiful and very natural idea was to compute the first few coefficients of this polynomial using each of the generating functions in (A.26) and to equate the two expressions obtained. It turned out that this gives nothing for the coefficients of m^0 and m^1 (which are found from either point of view to be 0 and $\mathcal{B}_n$, respectively), but that the equality of the coefficients of m^2 obtained from the two generating functions coincides precisely with the identity that Miki had discovered!

More precisely, from the first formula in (A.26) we obtain

$$\begin{aligned} \log\Bigl(\sum_{n=0}^{\infty} S(m+n,m)\, x^n \Bigr) &= \sum_{j=1}^{m} \log\Bigl(\frac{1}{1-jx}\Bigr) = \sum_{r=1}^{\infty} \frac{1^r + 2^r + \cdots m^r}{r}\, x^r \\ &= \sum_{r=1}^{\infty} \Bigl(\frac{B_r}{r}\, m + \frac{(-1)^{r-1} B_{r-1}}{2}\, m^2 + \cdots \Bigr) x^r \end{aligned}$$

(the last line by the Bernoulli–Seki formula) and hence, exponentiating,

$$S(m+n,m) = \mathcal{B}_n\, m + \Bigl(n\mathcal{B}_{n-1} + \sum_{i=2}^{n-2} \mathcal{B}_i \mathcal{B}_{n-i} \Bigr) \frac{m^2}{2} + \cdots \quad (n \ge 3), \tag{A.27}$$

while from the second formula in (A.26) and the expansion $\log\bigl((e^x-1)/x\bigr) = \sum_{n>0} \mathcal{B}_n x^n/n!$ we get

$$\begin{aligned}
&S(m+n,m)\\
&\quad=\Big(1+\frac{m}{1}\Big)\Big(1+\frac{m}{2}\Big)\cdots\Big(1+\frac{m}{n}\Big)\times\text{Coefficient of }\frac{x^n}{n!}\text{ in }\Big(\frac{e^x-1}{x}\Big)^m\\
&\quad=\Big(1+H_nm+\cdots\Big)\bigg(\mathcal{B}_nm+\bigg(\sum_{i=1}^{n-1}\binom{n}{i}\mathcal{B}_i\mathcal{B}_{n-i}\bigg)\frac{m^2}{2}+\cdots\bigg)\\
&\quad=\mathcal{B}_n\,m+\bigg(2H_n\mathcal{B}_n+\sum_{i=1}^{n-1}\binom{n}{i}\mathcal{B}_i\mathcal{B}_{n-i}\bigg)\frac{m^2}{2}+\cdots\quad(n\ge 1)\,.
\end{aligned}\tag{A.28}$$

Comparing the coefficients of $m^2/2$ in (A.27) and (A.28) gives Eq. (A.23). □

Finally, we state the one-parameter generalization of (A.23) and (A.24) given in [A4]. For $n>0$ denote by $\mathcal{B}_n(x)$ the polynomial $B_n(x)/n$.

Proposition A.6 (Gessel). *For all $n>0$ one has*

$$\frac{n}{2}\bigg(\mathcal{B}_{n-1}(x)+\sum_{i=1}^{n-1}\mathcal{B}_i(x)\mathcal{B}_{n-i}(x)\bigg)=\sum_{i=1}^{n}\binom{n}{i}\mathcal{B}_i\,\mathcal{B}_{n-i}(x)+H_{n-1}\mathcal{B}_n(x)\,.\tag{A.29}$$

Gessel does not actually write out the proof of this identity, saying only that it can be obtained in the same way as his proof of (A.23) and pointing out that, because $\mathcal{B}_n(1)=\mathcal{B}_n$ and $2^{2g}\mathcal{B}_{2g}(1/2)=(2g-1)!\,b_g$, it implies (A.23) and (A.24) by specializing to $x=1$ and $x=1/2$, respectively.

A.4 Products and Scalar Products of Bernoulli Polynomials

If A is any algebra over $\mathbb{Q}$ and $e_0,\,e_1,\ldots$ is an additive basis of A, then each product e_ie_j can be written uniquely as a (finite) linear combination $\sum_k c_{ij}^k e_k$ for certain numbers $c_{ij}^k\in\mathbb{Q}$ and the algebra structure on A is completely determined by specifying the "structure constants" c_{ij}^k. If we apply this to the algebra $A=\mathbb{Q}[x]$ and the standard basis $e_i=x^i$, then the structure constants are completely trivial, being simply 1 if $i+j=k$ and 0 otherwise. But the Bernoulli polynomials also form a basis of $\mathbb{Q}[x]$, since there is one of every degree, and we can ask what the structure constants defined by $B_i(x)B_j(x)=\sum_k c_{ij}^k B_k(x)$ are. It is easy to see that c_{ij}^k can only be non-zero if the difference $r{:=}i+j-k$ is non-negative (because $B_i(x)B_j(x)$ is a polynomial of degree $i+j$) and even (because the nth Bernoulli polynomial is $(-1)^n$-symmetric with respect to $x\mapsto 1-x$). The surprise is that, up to an elementary factor, c_{ij}^k is equal simply to the kth Bernoulli number, except when $k=0$. This fact, which was discovered long ago by Nielsen [A11, p. 75] (although I was not aware of this reference at the time when Igor Artamkin and I had the discussions that led to the formulas and proofs described below), is stated in a precise form in the following proposition. The formula turns out to be somewhat simpler if we use the renormalized Bernoulli polynomials $\mathcal{B}_n(x) = \frac{B_n(x)}{n}$ rather than the $B_n(x)$ themselves when $n > 0$. (For $n = 0$ there is nothing to be calculated since the product of any $B_i(x)$ with $B_0(x) = 1$ is just $B_i(x)$.)

Proposition A.7. *Let i and j be strictly positive integers. Then*

$$\mathcal{B}_i(x)\,\mathcal{B}_j(x) = \sum_{0\le\ell<\frac{i+j}{2}} \left[\frac{1}{i}\binom{i}{2\ell} + \frac{1}{j}\binom{j}{2\ell}\right] B_{2\ell}\,\mathcal{B}_{i+j-2\ell}(x) + \frac{(-1)^{i-1}(i-1)!\,(j-1)!}{(i+j)!}\,B_{i+j}\;. \tag{A.30}$$

Note that, despite appearances, the (constant) second term in this formula is symmetric in i and j, because if $B_{i+j} \neq 0$ then i and j have the same parity.

Proof. Write $\mathcal{B}_{i,j}(x)$ for the right-hand side of (A.30). We first show that the difference between $\mathcal{B}_{i,j}(x)$ and $\mathcal{B}_i(x)\mathcal{B}_j(x)$ is constant. This can be done in two different ways. First of all, using $\mathcal{B}_n(x+1) - \mathcal{B}_n(x) = x^{n-1}$ we find

$$\begin{aligned}\mathcal{B}_{i,j}(x+1) - \mathcal{B}_{i,j}(x) &= \sum_{0\le\ell<\frac{i+j}{2}} \left[\frac{1}{i}\binom{i}{2\ell} + \frac{1}{j}\binom{j}{2\ell}\right] B_{2\ell}\,x^{i+j-2\ell-1} \\ &= x^{j-1}\left(\mathcal{B}_i(x) + \frac{1}{2}\,x^{i-1}\right) + x^{i-1}\left(\mathcal{B}_j(x) + \frac{1}{2}\,x^{j-1}\right) \\ &= \mathcal{B}_i(x+1)\mathcal{B}_j(x+1) - \mathcal{B}_i(x)\mathcal{B}_j(x)\,.\end{aligned}$$

It follows that the $\mathcal{B}_{i,j}(x) - \mathcal{B}_i(x)\mathcal{B}_j(x)$ is periodic and hence, since it is also polynomial, constant. Alternatively, we can use that $\mathcal{B}_n'(x)$ equals 1 for $n = 1$ and $(n-1)\mathcal{B}_{n-1}(x)$ for $n > 1$ to show by induction on $i+j$ that $\mathcal{B}_{i,j}(x)$ and $\mathcal{B}_i(x)\mathcal{B}_j(x)$ have the same derivative (we omit the easy computation) and hence again that their difference is constant. To show that this constant vanishes, it suffices to show that the integrals of the two sides of (A.30) over the interval [0,1] agree. Since the integral of $\mathcal{B}_n(x)$ over this interval vanishes for any $n > 0$, this reduces to the following statement, in which to avoid confusion with $i = \sqrt{-1}$ we have changed i and j to r and s. □

Proposition A.8. *Let r and s be positive integers. Then*

$$\int_0^1 B_r(x)\,B_s(x)\,dx = (-1)^{r-1}\,\frac{r!\,s!}{(r+s)!}\,B_{r+s}\;. \tag{A.31}$$

Proof. Here again we give two proofs. The first uses the Fourier development

$$B_k(x) = -\frac{k!}{(2\pi i)^k}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{e^{2\pi i n x}}{n^k} \qquad (0 < x < 1,\ k \ge 1) \tag{A.32}$$

discussed in Chap. 4, Theorem 4.11 of this book. (For $k = 1$ the sum converges only conditionally and one has to be a little careful.) Since the integral $\int_0^1 e^{2\pi ikx}\,dx$ equals $\delta_{k,0}$, this gives

$$\int_0^1 B_r(x)\,B_s(x)\,dx = (-1)^r\,\frac{r!\,s!}{(2\pi i)^{r+s}}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{1}{n^{r+s}} = (-1)^{r-1}\,\frac{r!\,s!}{(r+s)!}\,B_{r+s}$$

as desired. (The second equality, giving the well-known connection between Bernoulli numbers and the values at positive even integers of the Riemann zeta function, is just the case $k = r + s$, $x \to 0$ of (A.32).) The second proof, using generating functions, is just as short. Denote the left-hand side of (A.31), also for r or s equal to 0, by $I_{r,s}$. Then we have

$$\begin{aligned}\sum_{r,s\geq 0} I_{r,s}\,\frac{t^{r-1}}{r!}\,\frac{u^{s-1}}{s!} &= \int_0^1 \frac{e^{xt}}{e^t-1}\,\frac{e^{xu}}{e^u-1}\,dx = \frac{1}{e^t-1}\,\frac{1}{e^u-1}\,\frac{e^{t+u}-1}{t+u}\\ &= \frac{1}{t+u}\left[\frac{1}{e^t-1}-\frac{1}{e^{-u}-1}\right] = \sum_{k=0}^{\infty}\frac{B_k}{k!}\,\frac{t^{k-1}-(-u)^{k-1}}{t+u}\\ &= \frac{1}{tu}+\sum_{k\geq 2}\frac{B_k}{k!}\sum_{\substack{r,s\geq 1\\ r+s=k}} t^{r-1}\,(-u)^{s-1},\end{aligned}$$

and Eq. (A.31) follows by equating the coefficients of $t^{r-1}u^{s-1}$.

Before continuing, we show that Proposition A.7 immediately yields another proof of the identities of Miki and Gessel discussed in the preceding section. This method is due to I. Artamkin [A1] (whose proof, up to a few small modifications, we have followed here). Indeed, summing (A.30) over all $i,\ j \geq 1$ with $i + j = n$, and using the easy identities

$$\sum_{i=1}^{n-1}\frac{1}{i}\binom{i}{r} = \frac{1}{r}\binom{n-1}{r}\qquad (r>0)$$

and

$$\begin{aligned}\sum_{\substack{i,j\geq 1\\ i+j=n}} (-1)^{i-1}\,\frac{(i-1)!\,(j-1)!}{(n-1)!} &= \sum_{i=1}^{n-1}\int_0^1 (-x)^{i-1}(1-x)^{n-i-1}dx\\ &= \int_0^1 \left[(1-x)^{n-1}-(-x)^{n-1}\right]dx = \frac{1+(-1)^n}{n}\end{aligned}$$

(where the first equation is the beta integral again), we obtain

$$\frac{1}{2}\sum_{\substack{i,j\geq 1\\ i+j=n}} \mathcal{B}_i(x)\mathcal{B}_j(x) = H_{n-1}\,\mathcal{B}_n(x) + \sum_{r=2}^{n-1}\binom{n-1}{r}\mathcal{B}_r(0)\,\mathcal{B}_{n-r}(x) + \frac{\mathcal{B}_n(0)}{n}\,, \tag{A.33}$$

which is equivalent to Gessel's identity (A.29).

Proposition A.8 describes the scalar products among the Bernoulli polynomials with respect to the scalar product $(f, g) = \int_0^1 f(x)g(x)dx$. It is more natural to replace the Bernoulli polynomials $B_k(x)$ by their periodic versions $\overline{B}_k(x)$ (defined for $x \notin \mathbb{Z}$ as $B_k(x-[x])$ or by the right-hand side of (A.32), and for $x \in \mathbb{Z}$ by continuity if $k \neq 1$ and as zero if $k = 1$), since then the scalar product is simply the integral of $\overline{B}_r(x)\overline{B}_s(x)$ over the whole domain of definition $\mathbb{R}/\mathbb{Z}$. The first proof just given then carries over almost unchanged to give the following more general result:

Proposition A.9. *Let r and s be integers ≥ 1 and α, β two real numbers. Then*

$$\int_0^1 \overline{B}_r(x+\alpha)\,\overline{B}_s(x+\beta)\,dx = (-1)^{r-1}\,\frac{r!\,s!}{(r+s)!}\,\overline{B}_{r+s}(\alpha-\beta)\,. \tag{A.34}$$

Using this, one finds, with almost the same proof as before, the following generalization of Proposition A.7:

Proposition A.10. *Let i and j be positive integers. Then for any two variables x and y we have*

$$\begin{aligned}\mathcal{B}_i(x)\,\mathcal{B}_j(y) = &\sum_{m=0}^{\max(i,j)}\left[\frac{1}{i}\binom{i}{m}\mathcal{B}_{i+j-m}(y) + \frac{(-1)^m}{j}\binom{j}{m}\mathcal{B}_{i+j-m}(x)\right]B_m^+(x-y)\\ &+(-1)^{j-1}\,\frac{(i-1)!\,(j-1)!}{(i+j)!}\,B_{i+j}^+(x-y)\,,\end{aligned} \tag{A.35}$$

where $B_m^+(x)$ denotes the symmetrized Bernoulli polynomial

$$B_m^+(x) = \frac{B_m(x)+(-1)^m B_m(-x)}{2} = \frac{B_m(x+1)+B_m(x)}{2} = B_m(x) + \frac{m}{2}x^{m-1}.$$

The same calculation as was used above to deduce (A.33) from (A.30), but now applied to (A.35) instead of (A.30), gives the following generalization of Gessel's identity (A.29):

$$\sum_{\substack{i,j\geq 1\\ i+j=n}} \mathcal{B}_i(x)\mathcal{B}_j(y) - H_{n-1}\left(\mathcal{B}_n(x)+\mathcal{B}_n(y)\right)$$
$$= \sum_{m=1}^{n-1}\binom{n-1}{m}\left(\mathcal{B}_{n-m}(y)+(-1)^m\,\mathcal{B}_{n-m}(x)\right)\frac{B_m^+(x-y)}{m}$$
$$+\frac{1+(-1)^n}{n^2}\,B_n^+(x-y)\,. \quad \text{(A.36)}$$

We observe that Eq. (A.36) was also found by Hao Pan and Zhi-Wei Sun [A12] in a slightly different form, the right-hand side in their formula being

$$\sum_{m=1}^{n}\binom{n-1}{m-1}\left(B_{n-m}(y)\,\frac{B_m(x-y)}{m^2}+B_{n-m}(x)\,\frac{B_m(y-x)}{m^2}\right)$$
$$+\frac{1}{n}\,\frac{B_n(x)-B_n(y)}{x-y}\,, \quad \text{(A.37)}$$

which is easily checked to be equal to the right-hand side of (A.36); their formula has the advantage of being more visibly symmetric in x and y and of using only the Bernoulli polynomials $B_m(x)$ rather than the symmetrized Bernoulli polynomials $B_m^+(x)$, but the disadvantage of having a denominator $x-y$ (which of course disappears after division into the numerator $B_n(x)-B_n(y)$) rather than being written in an explicitly polynomial form.

We end this section by giving a beautifully symmetric version of the multiplication law for Bernoulli polynomials given by the same authors in [A13].

Proposition A.11 (Sun–Pan). *For each integer $n \geq 0$ define a polynomial $\begin{bmatrix} r & s\\ x & y\end{bmatrix}_n$ in four variables r, s, x and y by*

$$\begin{bmatrix} r & s\\ x & y\end{bmatrix}_n = \sum_{\substack{i,j\geq 0\\ i+j=n}} (-1)^i\binom{r}{i}\binom{s}{j}\,B_j(x)\;B_i(y)\,. \quad \text{(A.38)}$$

Then for any six variables r, s, t, x, y and z satisfying $r+s+t=n$ and $x+y+z=1$ we have

$$t\begin{bmatrix} r & s\\ x & y\end{bmatrix}_n + r\begin{bmatrix} s & t\\ y & z\end{bmatrix}_n + s\begin{bmatrix} t & r\\ z & x\end{bmatrix}_n = 0\,. \quad \text{(A.39)}$$

First proof (sketch). We can prove (A.39) in the same way as (A.36) was proved above, replacing the product $B_j(x)B_i(y)$ in (A.38) for i and j positive using formula (A.35) (with x and y replaced by $1-y$ and x) and then using elementary binomial coefficient identities to simplify the result. We do not give the full calculation, which is straightforward but tedious. □

Second proof. An alternative, and easier, approach is to notice that, since the left-hand side of (A.39) is a polynomial in the variables x, y and $z = 1 - x - y$, it is enough to prove the identity for $x,\ y,\ z > 0$ with $x + y + z = 1$. But for x and y between 0 and 1 we have from (A.32)

$$(2\pi i)^n \begin{bmatrix} r & s \\ x & y \end{bmatrix}_n = \sum_{a,\,b \in \mathbb{Z}} C_n(r, s; a, b)\, e^{2\pi i (bx - ay)}$$

with

$$C_n(r, s; a, b) = \begin{cases} \sum_{i,\,j \geq 1,\, i+j=n} (r)_i\, (s)_j\, a^{-i} b^{-j} & \text{if } a \neq 0, b \neq 0 \\ \quad -(r)_n a^{-n} & \text{if } a \neq 0, b = 0 \\ \quad -(s)_n b^{-n} & \text{if } a = 0, b \neq 0 \\ \quad 0 & \text{if } a = 0, b = 0 \end{cases}$$

where $(x)_m = x(x-1)\cdots(x-m+1)$ is the descending Pochhammer symbol. Equation (A.39) then follows from the identity

$$t\, C_n(r, s; a, b) + r\, C_n(s, t; b, c) + s\, C_n(t, r; c, a) = 0 \quad (a+b+c = 0,\ r+s+t = n)\,.$$

whose elementary proof (using partial fractions if $abc \neq 0$) we omit. □

We end by remarking on a certain formal similarity between the cyclic identity (A.39) and a reciprocity law for generalized Dedekind sums proved in [A5]. The classical Dedekind sums, introduced by Dedekind while posthumously editing some unpublished calculations of Riemann's, are defined by

$$s(b, c) = \sum_{h \pmod c} \overline{B}_1\Big(\frac{h}{c}\Big)\, \overline{B}_1\Big(\frac{bh}{c}\Big) \qquad (b,\ c \in \mathbb{N} \text{ coprime}),$$

where $\overline{B}_1(x)$ as usual is the periodic version of the first Bernoulli polynomial (equal to $x - \frac{1}{2}$ if $0 < x < 1$, to 0 if $x = 0$, and periodic with period 1), and satisfy the famous Dedekind reciprocity relation

$$s(b, c) + s(c, b) = \frac{b^2 + c^2 + 1}{12bc} - \frac{1}{4}\,.$$

This was generalized by Rademacher, who discovered that if a, b and c are pairwise coprime integers then the sum

$$s(a, b; c) = \sum_{h \pmod c} \overline{B}_1\Big(\frac{ah}{c}\Big)\, \overline{B}_1\Big(\frac{bh}{c}\Big) \tag{A.40}$$

which equals $s(a', c)$ for any a' with $aa' \equiv b \pmod{c}$ or $ba' \equiv a \pmod{c}$, satisfies the identity

$$s(a,b;c) + s(b,c;a) + s(c,a;b) = \frac{a^2+b^2+c^2}{12abc} - \frac{1}{4}\,. \tag{A.41}$$

A number of further generalizations, in which the functions $\overline{B}_1$ in (A.40) are replaced by periodic Bernoulli polynomials with other indices and/or the arguments of these polynomials are shifted by suitable rational numbers, were discovered later. The one given in [A5] concerns the sums

$$S_{m,n}\begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix} = \sum_{h \pmod{c}} \overline{B}_m\Big(a\,\frac{h+z}{c} - x\Big)\,\overline{B}_n\Big(b\,\frac{h+z}{c} - y\Big)\,, \tag{A.42}$$

where m and n are non-negative integers, a, b and c natural numbers with no common factor, and x, y and z elements of $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. (The hth summand in (A.42) depends on z modulo c, not just modulo 1, but the whole sum has period 1 in z.) For fixed m and n these sums do not satisfy any relation similar to the 3-term relation (A.41) for the case $m = n = 1$, but if we assemble all of the functions $S_{m,n}$ $(m, n \ge 0)$ into a single generating function

$$\mathfrak{S}\begin{pmatrix} a & b & c \\ x & y & z \\ X & Y & Z \end{pmatrix} = \sum_{m,n\ge 0} \frac{1}{m!\,n!} S_{m,n}\begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix}\Big(\frac{X}{a}\Big)^{m-1}\Big(\frac{Y}{b}\Big)^{n-1}\,, \tag{A.43}$$

in which X, Y and Z (which does not appear explicitly on the right) are formal variables satisfying $X + Y + Z = 0$, then we have the following relation:

Proposition A.12 ([A5]). *Let a, b, c be three natural numbers with no common factor, x, y, z three elements of $\mathbb{T}$, and X, Y, Z three formal variables satisfying $X + Y + Z = 0$. Then*

$$\mathfrak{S}\begin{pmatrix} a & b & c \\ x & y & z \\ X & Y & Z \end{pmatrix} + \mathfrak{S}\begin{pmatrix} b & c & a \\ y & z & x \\ Y & Z & X \end{pmatrix} + \mathfrak{S}\begin{pmatrix} c & a & b \\ z & x & y \\ Z & X & Y \end{pmatrix} = \begin{cases} 1/4 & \text{if } (x,y,z) \in (a,b,c)\mathbb{T}\,, \\ 0 & \text{otherwise.} \end{cases}$$

We do not give the proof of this relation, since three different proofs (all similar in spirit to various of the proofs that have been given in this appendix) are given in [A5], but we wanted to at least mention this generalized Dedekind–Rademacher reciprocity law because of its formal resemblance, and perhaps actual relationship, to the Sun–Pan reciprocity law (A.39).

A.5 Continued Fraction Expansions for Generating Functions of Bernoulli Numbers

There are several classical formulas expressing various versions of the standard (exponential) generating functions of the Bernoulli numbers as continued fractions. A simple example is

$$\tanh x \quad \left(= \sum_{n \geq 2} \frac{2^n (2^n - 1) B_n}{n!} x^{n-1} \right) \quad = \cfrac{x}{1 + \cfrac{x^2}{3 + \cfrac{x^2}{5 + \cfrac{x^2}{\ddots}}}}, \tag{A.44}$$

whose proof is recalled below, and a somewhat more complicated one, whose proof we omit, is

$$\frac{x/2}{\tanh x/2} \quad \left(= \sum_{n \geq 0} \frac{B_{2n}}{(2n)!} x^{2n} \right) \quad = \cfrac{1}{1 + \cfrac{a_1 x^2}{1 + \cfrac{a_2 x^2}{\ddots}}} \tag{A.45}$$

with a_n defined by

$$a_n = \begin{cases} -\dfrac{1}{12} & \text{if } n = 1\,, \\ \dfrac{(n+1)(n+2)}{(2n-2)(2n-1)(2n)(2n+1)} & \text{if } n \text{ is even}\,, \\ \dfrac{(n-2)(n-1)}{(2n-1)(2n)(2n+1)(2n+2)} & \text{if } n > 1 \text{ is odd}\,. \end{cases}$$

It was discovered by M. Kaneko that the convergents $P_n(x)/Q_n(x)$ of the continued fraction (A.45) could be given in a simple closed form, namely

$$P_n(x) = \sum_{i=0}^{n/2} \binom{n}{2i} \binom{2n+1}{2i}^{-1} \frac{x^i}{(2i+1)!}$$

$$Q_n(x) = \sum_{i=0}^{n/2} \binom{n+1}{2i} \binom{2n+2}{2i}^{-1} \frac{x^i}{(2i)!}$$

if n is even and a similar but slightly more complicated expression if n is odd. (It was in connection with this discovery that he found the short recursion formula for Bernoulli numbers discussed in Sect. 1.2 of the book.) Again we omit the proof, which is given in [A6].

What is perhaps more surprising is that there are also nice continued fraction expansions for certain non-standard (ordinary) generating functions of Bernoulli numbers of the type considered in Sect. A.1, and these are in some sense of even more interest because the continued fractions, unlike the power series themselves, converge for positive real values of the argument (and give the appropriate derivatives of $\psi(X)$ as discussed in the last paragraph of Sect. A.1). For instance, on the cover of the Russian original of Lando's beautiful book on generating functions [A7] one finds the pair of formulas[3]

$$1 \cdot x + 2 \cdot \frac{x^3}{3!} + 16 \cdot \frac{x^5}{5!} + 272 \cdot \frac{x^7}{7!} + \cdots = \tan x$$

$$1 \cdot x + 2 \cdot x^3 + 16 \cdot x^5 + 272 \cdot x^7 + \cdots = \cfrac{x}{1 - \cfrac{1 \cdot 2\, x^2}{1 - \cfrac{2 \cdot 3\, x^2}{1 - \cfrac{3 \cdot 4\, x^2}{1 - \cdots}}}}$$

The numbers 1, 2, 16, 272, ... defined by the first of these two formulas are just the numbers $(4^n - 2^n)|B_n|/n$, so the second formula gives a continued fraction expansion for the non-exponential generating function for essentially the Bernoulli numbers. Again we omit the proof, referring for this to the book cited, mentioning only the following alternative and in some ways prettier form of the formula:

$$\frac{1}{X} - \frac{2}{X^3} + \frac{16}{X^5} - \frac{272}{X^7} + \cdots = \cfrac{1}{X + \cfrac{1}{\cfrac{X}{2} + \cfrac{1}{\cfrac{X}{3} + \cdots}}} \tag{A.46}$$

in which the continued fraction is convergent and equal to $1 - \frac{X}{2}\big(\psi\big(\frac{X+4}{4}\big) - \psi\big(\frac{X+2}{4}\big)\big)$ for all $X > 0$.

Other continued fraction expansions for non-exponential Bernoulli number generating functions that can be found in the literature include the formulas

[3] In the English translation [A8] (which we highly recommend to the reader) this formula has been relegated to the exercises: Chapter 5, Problem 5.6, page 85.

$$\sum_{n=1}^{\infty} B_{2n}(4x)^n = \cfrac{x}{1+\frac{1}{2}+\cfrac{x}{\frac{1}{2}+\frac{1}{3}+\cfrac{x}{\frac{1}{3}+\frac{1}{4}+\cfrac{x}{\ddots}}}}\,,$$

or the equivalent but less appealing identity

$$\sum_{n=0}^{\infty} B_n x^n = \cfrac{1}{1+\cfrac{x}{\frac{2}{1}-\cfrac{x}{3+\cfrac{2x}{\frac{2}{2}-\cfrac{2x}{5+\cfrac{3x}{\frac{2}{3}-\cfrac{3x}{7+\cfrac{4x}{\frac{2}{4}-\cfrac{4x}{9+\cfrac{5x}{\ddots}}}}}}}}}}\,,$$

and

$$\sum_{n=1}^{\infty} (2n+1)\, B_{2n}\, x^n = \cfrac{x}{1+1+\cfrac{x}{1+\frac{1}{2}+\cfrac{x}{\frac{1}{2}+\frac{1}{2}+\cfrac{x}{\frac{1}{2}+\frac{1}{3}+\cfrac{x}{\frac{1}{3}+\frac{1}{3}+\cfrac{x}{\ddots}}}}}}$$

all given by J. Frame [A3] in connection with a statistical problem on curve fitting.

For good conscience's sake we give the proofs of one continued fraction of each of the two above types, choosing for this purpose the two simplest ones (A.44) and (A.46). We look at (A.44) first. Define functions $I_0, I_1, \ldots$ on $(0, \infty)$ by

$$I_n(a) = \int_0^a \frac{t^n(1-t/a)^n}{n!}\, e^t\, dt \quad \big(n \in \mathbb{Z}_{\geq 0}\,,\ a \in \mathbb{R}_{>0}\big)\,.$$

Integrating by parts twice, we find that

$$\begin{aligned} I_{n+1}(a) &= \int_0^a e^t \, \frac{d^2}{dt^2}\left[\frac{t^{n+1}(1-t/a)^{n+1}}{(n+1)!}\right] dt \\ &= \int_0^a e^t \left[\frac{t^{n-1}(1-t/a)^{n-1}}{(n-1)!} - \frac{4n+2}{a}\,\frac{t^n(1-t/a)^n}{n!}\right] dt \\ &= I_{n-1}(a) - \frac{4n+2}{a}\, I_n(a) \end{aligned}$$

for $n > 0$. Rewriting this as $\dfrac{I_{n-1}(a)}{I_n(a)} = \dfrac{4n+2}{a} + \dfrac{I_{n+1}(a)}{I_n(a)}$ and noting that

$$I_0(a) = e^a - 1\,, \quad I_1(a) = e^a\left(1 - \frac{2}{a}\right) + \left(1 + \frac{2}{a}\right)$$

by direct calculation, we obtain

$$\frac{1}{\tanh x} = \frac{e^{2x}+1}{e^{2x}-1} = \frac{1}{x} + \frac{I_1(2x)}{I_0(2x)} = \frac{1}{x} + \cfrac{1}{\cfrac{3}{x} + \cfrac{1}{\cfrac{5}{x} + \cfrac{1}{\ddots}}}\,,$$

which is equivalent to (A.44). Similarly, for (A.46), we define functions $J_0, J_1, \ldots$ on $(0, \infty)$ by

$$J_n(X) = \int_0^\infty \big(\tanh(t/X)\big)^n \, e^{-t}\, dt \quad \big(n \in \mathbb{Z}_{\geq 0}\,,\ X \in \mathbb{R}_{>0}\big).$$

This time $J_0(X)$ is simply the constant function 1, while $J_1(X)$ has the exact evaluation

$$J_1(X) = 1 - \frac{X}{2}\,\psi\Big(\frac{X}{4} + 1\Big) + \frac{X}{2}\,\psi\Big(\frac{X}{4} + \frac{1}{2}\Big)\,, \tag{A.47}$$

as is easily deduced from Euler's integral representation

$$\psi(x) = -\gamma + \int_0^1 \frac{1 - t^{x-1}}{1-t}\, dt\,,$$

as well as the asymptotic expansion

$$\begin{aligned} J_1(X) &\sim \int_0^\infty \Big(\frac{1}{X}\, t - \frac{2}{X^3}\,\frac{t^3}{3!} + \frac{16}{X^5}\,\frac{t^5}{5!} - \frac{272}{X^7}\,\frac{t^7}{7!} + \cdots\Big)\, e^{-t}\, dt \\ &\sim \frac{1}{X} - \frac{2}{X^3} + \frac{16}{X^5} - \frac{272}{X^7} + \cdots \end{aligned}$$

as $X \to \infty$. (This last expression can be written as $1 - X\gamma_0(2/X) + X\gamma_0(4/X)$ with γ_0 as in (A.12), in accordance with (A.47) and the relationship between $\gamma_0(X)$ and $\psi(X)$ given at the end of Sect. A.1.) On the other hand, integrating by parts and using $\tanh(x)' = 1 - \tanh(x)^2$, we find

$$\begin{aligned} J_n(X) &= \int_0^\infty e^{-t}\,\frac{d}{dt}\Big(\big(\tanh(t/X)\big)^n\Big)\,dt \\ &= \frac{n}{X}\int_0^\infty e^{-t}\,\big(\tanh(t/X)\big)^{n-1}\Big(1 - \big(\tanh(t/X)\big)^2\Big)\,dt \\ &= \frac{n}{X}\big(J_{n-1}(X) - J_{n+1}(X)\big) \end{aligned}$$

for $n > 0$, and rewriting this as $\dfrac{J_{n-1}(X)}{J_n(X)} = \dfrac{X}{n} + \dfrac{J_{n+1}(X)}{J_n(X)}$ we obtain that $J_1(X) = \dfrac{J_1(X)}{J_0(X)}$ has the continued fraction expansion given by the right-hand side of (A.46), as claimed. □

We end this appendix by describing an appearance of the continued fraction (A.46) in connection with the fantastic discovery of Yuri Matiyasevich that "the zeros of the Riemann zeta function know about each other." Denote the zeros of $\zeta(s)$ on the critical line $\Re(s) = \frac{1}{2}$ by ρ_n and $\overline{\rho_n}$ with $0 < \Im(\rho_1) \le \Im(\rho_2) \le \cdots$ and for $M \ge 1$ consider the finite Dirichlet series $\Delta_M(s)$ defined as the $N \times N$ determinant[4]

$$\Delta_M(s) = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n^{-\rho_1} & n^{-\overline{\rho_1}} & \cdots & n^{-\rho_M} & n^{-\overline{\rho_M}} & n^{-s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ N^{-\rho_1} & N^{-\overline{\rho_1}} & \cdots & N^{-\rho_M} & N^{-\overline{\rho_M}} & N^{-s} \end{vmatrix},$$

where $N = 2M + 1$. This function clearly vanishes when $s = \rho_n$ or $\overline{\rho_n}$ for $1 \le n \le M$, but Matiyasevich's discovery (for which we refer to [A9] and the other papers and talks listed on his website) was that its subsequent zeros are incredibly close to the following zeros of the Riemann zeta function, e.g., the first zero of Δ_{50} on $\frac{1}{2} + \mathbb{R}_{>0}$ following ρ_{50} differs in absolute value from ρ_{51} by less than 4×10^{-15}, the first zero of Δ_{1500} after ρ_{1500} differs in absolute value from ρ_{1501} by less than 5×10^{-1113}, and even the 300th zero of Δ_{1500} after ρ_{1500} differs in absolute value from ρ_{1801} by less than 5×10^{-766}! Moreover, if we write the Dirichlet series $\Delta_M(s)$ as $c_M \sum_{n=1}^N a_{M,n} n^{-s}$ with the normalizing constant c_M chosen to make $a_{M,1} = 1$,

[4] We have changed Matiyasevich's notations slightly for convenience of exposition.

then it turns out that the function $c_M^{-1}\Delta_M(s)$ not only has almost the same zeros, but is itself a very close approximation to $(1-2^{1-s})\zeta(s)$ over a long interval of the critical line.

In studying this latter function, Matiyasevich was led to consider the real numbers ν_M defined by $\nu_M = 4M\sum_{n=1}^{2M}\mu_{M,n}/n$, where $\mu_{M,n}$ denotes the coefficient of n^{-s} in the Dirichlet series $c_M^{-1}\Delta_M(s)/\zeta(s)$. Since by the nature of his investigation he was working to very high precision, he obtained very precise decimal expansions of these numbers, and in an attempt to recognize them, he computed the beginning of their continued fraction expansions. (Recall that rational numbers and real quadratic irrationalities can be recognized numerically by the fact that they have terminating or periodic continued fraction expansions.) To his surprise, when M was highly composite these numbers had very exceptional continued fraction expansions. For instance, for $2M = \text{l.c.m.}\{1, 2, \ldots, 10\} = 2520$, the number ν_M has a decimal expansion beginning $0.9998015873172093\cdots$ and a continued fraction expansion beginning [0, 1, 5039, 2520, 1680, 1260, 1008, 840, 720, 630, 560, 504]. In view of the fact that nearly all real numbers (in a very precise metrical sense) have continued fraction expansions with almost all partial quotients very small, this is certainly not a coincidence, and it is even more obviously not one when we notice that the numbers 5040, 2520, ... 504 are $5040/n$ for $n = 1, 2, \ldots, 10$. This leads one immediately to the continued fraction (A.46) with $X = 4M$ and hence, in view of the evaluation of that continued fraction given above, to the (conjectural) approximation $\nu_M \approx \frac{1}{2}\psi\big(M+1\big) - \frac{1}{2}\psi\big(M+\frac{1}{2}\big)$, which turns out indeed to be a very good one for M large, the two numbers differing only by one part in 10^{108} in the above-named case $2M = 2520$. We take this somewhat unusual story as a fitting place to end our survey of curious and exotic identities connected with Bernoulli numbers.

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