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SOME IDENTITIES INVOLVING GENERARALIZED HARMONOIC POLYNOMIAL AND POWER

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Abstract: In this paper, we obtain several general identity involving generalized harmonic polynomials and the power. From these identities, we also deduce some particular identities involving interestingly the number of combinations.

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1. Introduction and Preliminaries

In Combinatorics, harmonic numbers not only form an important class of combinatorial objects, but also play an important role in many areas of mathematics, including combinatorial analysis, graph theory, number theory, statistics and probability and so on.

In this paper, we will give a basic definition of the generalized harmonic polynomials, and some identities involving generalized harmonic polynomials and generalized harmonic number, the identities of this type might not have been presented before.

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Definition 1. Let $n, k, r \in Z$, $\gamma \in R^{+}(\gamma \neq 1)$, α, β are complex numbers, then we have

$$\sum_{n=0}^{\infty} H_{n,k,r}(\alpha,\beta,\gamma)(x)t^n = \frac{\gamma^{xt}}{(1-\beta t)^k} \cdot (-\ln(1-\alpha t))^r \,,$$

where, $H_{n,k,r}(\alpha,\beta,\gamma)(x)$ is called generalized harmonic polynomial about x with γ parameters, when $x = 0$ is called generalized harmonic number (see [2]).

Definition 2. (see [1], [6]) Let k be a positive integer, then the expression of operator T_k is

$$f = \sum_{n\geq 0} a_n t^n \to T_k f = \sum_{n\geq 0} n^k a_n t^n$$

where, $T = tD(D$ is Differential operator), $\{a_n\}$ is a sequence.

2. Some Main Results

In this section, we will give some Theorems involving Generalized Harmonic polynomial $H_{n,k,r}(\alpha,\beta,\gamma)(x)$ and other famous number sequences. such as, Stirling number and noncentral Stirling number, and so on.

Lemma. (see [1], [6]) *Let k be a positive integer, then we have*

$$\sum_{n\geq 0} n^k a_n t^n = \sum_{h=1}^{k} S(k,h) t^h D^h f$$

where, $S(n,k)$ (see [1]) is the Stirling number of the second kind.

Theorem 1. *Let n, k be a nonnegative integer, and $\alpha \in C$, then we have*

$$\sum_{n\geq 0} n^k t^n = \sum_{h=1}^{k} S(k,h) \frac{t^h}{(1-t)^{h+1}} \tag{1}$$

$$\sum_{n\geq 0} n^k \frac{t^n}{n!} = e^t \sum_{h=1}^{k} S(k,h) t^h \tag{2}$$

$$\sum_{n\geq 0} n^k \langle\alpha\rangle_n \frac{t^n}{n!} = \sum_{h=1}^{k} S(k,h) \langle\alpha\rangle_h \frac{t^h}{(1-t)^{\alpha+h}} \tag{3}$$

$$\sum_{n\geq 0} n^k (\alpha)_n \frac{t^n}{n!} = \sum_{h=1}^{k} S(k,h)(\alpha)_h (1+t)^{\alpha-h} \tag{4}$$

where, S(n,k) is the Stirling number of the second kind.

Proof. Let $a_n = 1$, then $f = \sum_{n\geq 0} t^n = \frac{1}{1-t}$, by the definition(2) and the Lemma, we have

$$T_k f = \sum_{n\geq 0} n^k t^n = \sum_{h=1}^{k} S(k,h) t^h D^h \frac{1}{1-t}$$

hence

$$\sum_{n\geq 0} n^k t^n = \sum_{h=1}^{k} S(k,h) \frac{t^h}{(1-t)^{h+1}}$$

Similarly, we suppose $a_n = \frac{1}{n!}, \frac{\langle\alpha\rangle_n}{n!}, \frac{(\alpha)_n}{n!}$, then using the same method in the definition(2) and the Lemma, we can get (2),(3),(4) respectively. $\square$

Theorem 2. *Let n, k, r be a nonnegative integer, $\gamma \in R^+ (\gamma \neq 1)$ and $\alpha, \beta \in C$, then we have*

$$\sum_{i=0}^{n} H_{i,k,r}(\alpha,\beta,\gamma)(x)(n-i)^k \beta^{n-i} = \sum_{h=1}^{k} H_{n-h,k+h+1,r}(\alpha,\beta,\gamma)(x) S(k,h) h! \beta^h$$

where, $S(n,k)$ (see [4], [5]) is the second kind of Stirling number.

Proof. Let n, k, r be a nonnegative integer, $\gamma \in R^+ (\gamma \neq 1)$ and $\alpha, \beta \in C$, since

$$\sum_{n\geq 0} H_{n,k,r}(\alpha,\beta,\gamma)(x) t^n = \frac{\gamma^{xt}}{(1-\beta t)^k} \cdot (-\ln(1-\alpha t))^r$$

and

$$\sum_{n\geq 0} n^k t^n = \sum_{h=1}^{k} h! S(k,h) \frac{t^h}{(1-t)^{h+1}},$$

so,

$$\sum_{n\geq 0} n^k \beta^n t^n = \sum_{h=1}^{k} \beta^h S(k,h) \frac{h! t^h}{(1-\beta t)^{h+1}},$$

hence,

$$\sum_{n\geq 0} H_{n,k,r}(\alpha,\beta,\gamma)(x)t^n \cdot \sum_{n\geq 0} n^k\beta^n t^n = \sum_{n\geq 0}\left(\sum_{i=0}^{n} H_{i,k,r}(\alpha,\beta,\gamma)(x)(n-i)^k\beta^{n-i}\right)t^n$$
$$= \sum_{h=1}^{k} S(k,h)\beta^h \frac{h!\gamma^{xt}t^h}{(1-\beta t)^k}\cdot(-\ln(1-\alpha t))^r$$
$$= \sum_{n\geq 0}\left(\sum_{h=1}^{k} H_{n-h,k+h+1,r}(\alpha,\beta,\gamma)(x)h!S(k,h)\beta^h\right)t^n\,,$$

then by compared coefficient both sides of above the formula, we can got the desired result. □

Corollary 1.1. *Let $x=0$ in Theorem(1), we can obtain*

$$\sum_{m=0}^{n} H_{m,k,r}(\alpha,\beta)\frac{(n-m)^k}{\beta^m} = \sum_{h=1}^{k} h!H_{n-h,k+h+1,r}(\alpha,\beta)\frac{S(k,h)}{\beta^{n-h}}\,,$$

where, $H_{n,k,r}(\alpha,\beta,\gamma)(0)=H_{n,k,r}(\alpha,\beta)$ (see [2]).

When $\alpha=\beta=1$ in the above formulas, we have

$$\sum_{m=0}^{n} H_{m,k,r}(n-m)^k = \sum_{h=1}^{k} h!H_{n-h,k+h+1,r}S(k,h)\,,$$

and

$$H_{n,k,r} = (-1)^{n+r}\frac{k!}{n!}s(n,r;k)\,,$$

hence

$$\sum_{m=0}^{n}(-1)^{m+r}s(m,r;k)\frac{(n-m)^k}{n!} = \sum_{h=1}^{k} H_{n-h,k+h+1,r}\frac{h!S(k,h)}{k!}\,,$$

where, $s(n,r;k)$ (see [1]) is the noncentral Stirling number of the first kind, $H_{n,k,r}(1,1)=H_{n,k,r}$ (see [2]).

and $k=r=1$ in the above formulas, have

$$\sum_{m=0}^{n} H_m(n-m) = H_{n-1,3,1}\,,$$

where, $H_{n,1,1} = H_n$ (see [2]).

Corollary 1.2. *Let $\gamma = e, \alpha = \beta = 1$ in Theorem1, we can obtain*

$$\sum_{m=0}^{n} H_{m,k,r}(x)(n-m)^k = \sum_{h=1}^{k} H_{n-h,k+h+1,r}(x)S(k,h)h!\,,$$

when, $k = r = 1$ in the above formulas, we have

$$\sum_{m=0}^{n} H_m(x)(n-m) = H_{n-1,3,1}(x)\,,$$

Therefore, we get the desired result.

Theorem 2. *Let n, m, k, r be a nonnegative integer, $\gamma \in R^+(\gamma \neq 1)$ and $a, \alpha, \beta \in C$, then*

$$\sum_{m=0}^{n} H_{m,k,r}(\alpha,\beta,\gamma)(x)\frac{\langle a\rangle_{n-m}(n-m)^k}{(n-m)!}$$
$$= \sum_{h=1}^{k} H_{n-h,k+h+1,r}(\alpha,\beta,\gamma)(x)S(k,h)\langle a\rangle_h\beta^h\,.$$

Proof. Since

$$\sum_{n\geq 0} n^k\langle a\rangle_n\frac{t^n}{n!} = \sum_{h=1}^{k} S(k,h)\frac{\langle a\rangle_h t^h}{(1-t)^{\alpha+h}}\,,$$

so we have

$$\sum_{n\geq 0} H_{n,k,r}(\alpha,\beta,\gamma)(x)t^n \cdot \sum_{n\geq 0} n^k\frac{\langle a\rangle_n t^n}{n!}$$
$$= \sum_{n\geq 0}\left(\sum_{m=0}^{n} H_{m,k,r}(\alpha,\beta,\gamma)(x)\frac{\langle a\rangle_{n-m}(n-m)^k}{(n-m)!}\right)t^n$$
$$= \sum_{h=1}^{k} S(k,h)\beta^h\frac{\langle a\rangle_h\gamma^{xt}t^h}{(1-\beta t)^k}\cdot(-\ln(1-\alpha t))^r$$
$$= \sum_{n\geq 0}\left(\sum_{k=1}^{h} H_{n-h,k+a+h,r}(\alpha,\beta,\gamma)(x)\beta^h S(n,k)\langle a\rangle_h\right)t^n$$

by compared coefficient of t^n both sides of above the formula, we can get the desired result. $\square$

Corollary. *Let $x = 0$ in Theorem2, we obtain an identity involving generalized harmonic number,*

$$\sum_{m=0}^{n} H_{m,k,r}(\alpha,\beta)\frac{\langle a\rangle_{n-m}(n-m)^k}{(n-m)!} = \sum_{h=1}^{k} H_{n-h,k+h+1,r}(\alpha,\beta)S(k,h)\langle a\rangle_h \beta^h \,.$$

where, $H_{n,k,r}(\alpha,\beta,\gamma)(0) = H_{n,k,r}(\alpha,\beta)$ (see [3]).

Theorem 3. *Let n, m, k, r be a nonnegative integer, $\gamma \in R^+ (\gamma \neq 1)$ and $a, \alpha, \beta \in C$, then we have*

$$\sum_{m=0}^{n} (-1)^m H_{m,k,r}(\alpha,\beta,\gamma)(x)\binom{a}{n-m}(n-m)^k \beta^{n-m}$$
$$= \sum_{h=1}^{k} (-1)^n H_{n,k+h-a,r}(\alpha,\beta,\gamma)(x)S(k,h)(a)_h \,.$$

Proof. Since

$$\sum_{n\geq 0} n^k (a)_n \frac{t^n}{n!} = \sum_{h=1}^{k} S(k,h)(a)_h (1+t)^{a-h} \,,$$

so,

$$\sum_{n\geq 0} \binom{a}{n} n^k (-\beta)^n t^n = \sum_{h=1}^{k} S(k,h)(a)_h \frac{1}{(1-\beta t)^{h-a}}$$

hence,

$$\sum_{n\geq 0} H_{n,k,r}(\alpha,\beta,\gamma)(x)t^n \cdot \sum_{n\geq 0} n^k (a)_n \frac{t^n}{n!}$$
$$= \sum_{n\geq 0}\left(\sum_{m=0}^{n} H_{m,k,r}(\alpha,\beta,\gamma)(x)\binom{a}{n-m}(n-m)^k(-\beta)^{n-m}\right)t^n$$
$$= \sum_{n\geq 0}\sum_{h=1}^{k} S(k,h)\frac{(a)_h \gamma^{xt}}{(1-\beta t)^{k+h-a}} \cdot (-\ln(1-\alpha t))^r$$

$$= \sum_{n\geq 0}\left(\sum_{h=1}^{k} H_{n,k+h-a,r}(\alpha,\beta,\gamma)(x)S(k,h)(a)_h\right)t^n$$

by compared coefficient of t^n both sides of above the formula, we can got this result, the proof is complete. □

Corollary. *Let $x=0$ in Theorem3, we obtain an identity involving generalized harmonic number,*

$$\sum_{m=0}^{n} H_{m,k,r}(\alpha,\beta)(-1)^{n-m}\binom{a}{n-m}(n-m)^k\beta^{n-m}$$
$$= \sum_{h=1}^{k} H_{n,k+h-a,r}(\alpha,\beta)S(k,h)\langle a\rangle_h\ .$$

Theorem 4. *Let n,m,k,r be a nonnegative integer, $\gamma\in R^+(\gamma\neq 1)$ and $\alpha,\beta\in C$, then we have*

$$\sum_{m=0}^{n} H_{m,k,r}(\alpha,\beta,\gamma)(x)\frac{(n-m)^k}{(n-m)!} = \sum_{i=0}^{n}\sum_{h=1}^{k} H_{i-h,k,r}(\alpha,\beta,\gamma)(x)\frac{S(k,h)}{(n-i)!}$$

Proof. Since

$$\sum_{n\geq 0} n^k\frac{t^n}{n!} = e^t\sum_{h=1}^{k} S(k,h)t^h\ ,$$

hence

$$\sum_{n\geq 0} H_{n,k,r}(\alpha,\beta,\gamma)(x)t^n\cdot\sum_{n\geq} n^k\frac{t^n}{n!} = \sum_{n\geq 0}\left(\sum_{m=0}^{n} H_{m,k,r}(\alpha,\beta,\gamma)(x)\frac{(n-m)^k}{(n-m)!}\right)t^n$$
$$= e^t\sum_{h=1}^{k} S(k,h)t^h\frac{\gamma^{xt}}{(1-\beta t)^{k+h-a}}\cdot(-\ln(1-\alpha t))^r$$
$$= \sum_{n\geq 0}\left(\sum_{i=0}^{n}\sum_{h=1}^{k} H_{i-h,k,r}(\alpha,\beta,\gamma)(x)\frac{S(k,h)}{(n-i)!}\right)t^n\ ,$$

by compared coefficient of t^n both sides of above formula, we have got the result. □

Corollary. *Let $x = 0$ in Theorem4, we have following the result,*

$$\sum_{m=0}^{n} H_{m,k,r}(\alpha,\beta)\frac{(n-m)^k}{(n-m)!} = \sum_{i=0}^{n}\sum_{h=1}^{k} H_{i-h,k,r}(\alpha,\beta)\frac{S(k,h)}{(n-i)!}\,.$$

the proof is complete.

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