generating function of Laguerre polynomials*

pahio[†]

2013-03-22 3:16:14

We start from the definition of Laguerre polynomials via their Rodrigues formula

$$L_n(z)\;:=\;e^z\frac{d^n}{dz^n}e^{-z}z^n\qquad(n\;=\;0,\,1,\,2,\,\ldots). \tag{1}$$

The consequence

$$f^{(n)}(z)\;=\;\frac{n!}{2\pi i}\oint_C\frac{f(\zeta)}{(\zeta-z)^{n+1}}\;d\zeta \tag{2}$$

of Cauchy integral formula allows to write (1) as the complex integral

$$L_n(z)\;=\;\frac{n!}{2i\pi}\oint_C\frac{e^ze^{-\zeta}}{(\zeta-z)^{n+1}}\,d\zeta\;=\;\frac{n!}{2i\pi}\oint_C\frac{e^{z-\zeta}\,d\zeta}{(1-\frac{z}{\zeta})^n(\zeta-z)},$$

where C is any closed contour around the point z and the direction is anticlockwise. The substitution

$$\zeta-z\;:=\;\frac{zt}{1-t},\quad\zeta\;=\;\frac{z}{1-t},\quad t\;=\;1-\frac{z}{\zeta}\quad d\zeta\;=\;\frac{z\,dt}{(1-t)^2}$$

here yields

$$L_n(z)\;=\;\frac{n!}{2i\pi}\oint_{C'}\frac{e^{-\frac{zt}{1-t}}z\,dt}{(1-t)^2t^n\cdot\frac{zt}{1-t}}\;=\;\frac{n!}{2i\pi}\oint_{C'}\frac{e^{-\frac{zt}{1-t}}\,dt}{(1-t)t^{n+1}}$$

where the contour C' goes round the origin. Accordingly, by (2) we can infer that

$$L_n(z)\;=\;\left[\frac{d^n}{dt^n}\frac{e^{-\frac{zt}{1-t}}}{1-t}\right]_{t=0},$$

*⟨*GeneratingFunctionOfLaguerrePolynomials*⟩ created: ⟨*2013-03-2*⟩ by: ⟨*pahio*⟩ version: ⟨*42007*⟩ Privacy setting: ⟨*1*⟩ ⟨*Derivation*⟩ ⟨*33B99*⟩ ⟨*30B10*⟩ ⟨*26C05*⟩ ⟨*26A09*⟩ ⟨*33E30*⟩

whence we have found the generating function

$$\frac{e^{-\frac{zt}{1-t}}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(z)}{n!} t^n$$

of the Laguerre polynomials.